TEORIJSKA I PRIMENJENA MEHANIKA 12 pp. 89—106, 1986, UDK 539.301

INFLUENCE OF COUPLE-STRESSES ON STRESS DISTRIBUTION IN AN INFINITE PLATE WITH AN ELLIPTIC HOLE

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(Received, 16. 04. 1986.)

1. Introduction

It was obserwed that the Classical Theory of Elasticity, for some classes of materials (polycrystalline metals, granular materials, etc.) are not in agreement with the results achieved by experiments, and also can not describe certain phenomena (very short acoustic waves in polycrystalline metals or in high polymers, for instance). For that reason the internal structure of materials has become the subject of intensive study resulting in new theories which gave more accurate description of determined classes of materials.

One of the most important problems of the plane theory of elasticity, as from the theoretical so from the practical point of view, is the problem of determination of the stress distribution around a hole, i.e., the problem of determination of the stress-concentration factor. This problem has been the object of studies in the classical theory of elasticity. These results have not always been in accordance with the results obtained by experiments, so it appeared necessary to consider this problem from the aspect of more adequate theories.

One of the first was the work of R. D. Mindlin, (1), who showed that in the case of stretching of an infinite plate with a circular hole, the stress-concentration factor varied in the range from 2,39 to 2.64, while the value of that factor in the classical theory of elasticity was 3, for the same class of materials. Based on the same model of elastic materials (elastic materials grade two), Savin, (3), considered the influence of couple-stresses on stress distribution in the surroundings of curvilinear holes using the method of small parameter in solving the problem. The small parameter represents the departure of a curvilinear boundary from a circular one.

We shall consider the problem of stress distribution in the case of the stretching of an infinite plate with an elliptic hole, also from the aspect of Mindlin's theory. We shall restrict ourselves to the case when the stretching force acts in the direction of the shorter axis of the ellipse, because this case is of the most important when determining the stress-concentration factor. We shall start from the basic equations of the theory (1), trying to solve the problem by the introduction of two stress functions, all the relations being expressed in the elliptic coordinate system. We shall try, differently from (3), to solve the problem in a direct way.

Finally, in accordance with the given boundary conditions, we shall form a system of algebraic equations, which allows under certain conditions, to determine the constants of integration.

2. Basic equations

The linear constitutive equations for isotropic elastic materials with couple-stresses (elastic materials grade two), in the case of infinitesimal deformations, have the form, (2),

$$t_{(ij)} = 2G\left(e_{ij} + \frac{\nu}{1-2\nu} I_e g_{ij}\right) \tag{2.1}$$

$$m_{ij} = 4\eta \varkappa_{ij} \tag{2.2}$$

where $t_{(ij)}$ is the symmetric part of the stress tensor, m_{ij} is the couple-stress tensor, e_{ij} is the relative deformation tensor, I_e is the first invariant of e_{ij} tensor, $\varkappa_{ij}$ is the curvature tensor, G is the shear modulus, ν is the Poisson's ratio and η is the new material constant.

The relations between the deformation tensor coordinates e_{ij} and $\varkappa_{ij}$ and the gradients of the displacement vector are of the form

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \quad (2.3) \qquad \varkappa_{,j} = \frac{1}{2}\varepsilon^{ikl} u_{k,l} \quad (2.4)$$

and,

$$\eta = c^2 G \tag{2.5}$$

where u_i is the displacement vector, and c is the characteristic lenght of the material.

The equations of equilibrium are of the form

$$t^{ij}{}_{,j} = 0 \quad (2.6) \qquad m^{ij}{}_{,j} + \varepsilon^{ijk} t_{jk} = 0. \quad (2.7)$$

In a state of plane strain, (6), in the x, y plane, the components of displacement $u_1 = u$ and $u_2 = v$, in the x and y directions, are the functions of both, x and y, and the third component of displacement is zero, $u_3 = w = 0$. Then, the relations (2.1) and (2.2) become

$$e_{xx} = \frac{1+\nu}{E}[t_{xx} - \nu(t_{xx} + t_{yy})], \quad e_{yy} = \frac{1+\nu}{E}[t_{yy} - \nu(t_{xx} + t_{yy})]$$

$$e_{xy} = \frac{1+\nu}{2E}(t_{xy} + t_{yx}); \quad t_{zz} = \nu(t_{xx} + t_{yy}) \tag{2.8}$$

$$m_{xz} = 4\eta\varkappa_{xz}, \quad m_{yz} = 4\eta\varkappa_{yz}$$

where E is Young's modulus.

The relations (2.6) and (2.7) are as follows

$$\frac{\partial t_{xx}}{\partial x} + \frac{\partial t_{xy}}{\partial y} = 0, \quad \frac{\partial t_{yx}}{\partial x} + \frac{\partial t_{yy}}{\partial y} = 0,$$
$$\frac{\partial m_{xz}}{\partial x} + \frac{\partial m_{yz}}{\partial y} + t_{xy} - t_{yx} = 0. \tag{2.9}$$

Equations of compatibility may be expressed in terms of stresses and couple-stresses in the form, (1),

$$\frac{\partial^2 t_{xx}}{\partial y^2} + \frac{\partial^2 t_{yy}}{\partial x^2} - \nu \Delta (t_{xx} + t_{yy}) = \frac{\partial^2}{\partial x \partial y} (t_{xy} + t_{yx})$$

$$\frac{\partial m_{xz}}{\partial y} = \frac{\partial m_{yz}}{\partial x}$$

$$m_{xz} = c^2 \frac{\partial}{\partial x} (t_{xy} + t_{yx}) - 2 c^2 \frac{\partial}{\partial y} [t_{xx} - \nu (t_{xx} + t_{yy})] \tag{2.10}$$

$$m_{yz} = - c^2 \frac{\partial}{\partial y} (t_{xy} + t_{yx}) + 2 c^2 \frac{\partial}{\partial x} [t_{yy} - \nu (t_{xx} + t_{yy})]$$

where

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \quad c^2 = \frac{\eta}{G}.$$

It easy to see that only three of the four equations of compatibility are independent.

3. Stress functions

Mindlin, (5), showed that stress forces and couple-stresses can be expressed in terms of two stress functions as follows

$$t_{xx} = \frac{\partial^2 U}{\partial y^2} - \frac{\partial^2 \psi}{\partial x \partial y}, \quad t_{yy} = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 \psi}{\partial x \partial y},$$
$$t_{xy} = - \frac{\partial^2 U}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial y^2}, \quad t_{yx} = - \frac{\partial^2 U}{\partial x \partial y} + \frac{\partial^2 \psi}{\partial x^2} \tag{3.1}$$

$$m_{xz} = \frac{\partial \psi}{\partial x}, \quad m_{yz} = \frac{\partial \psi}{\partial y} \tag{3.2}$$

where t_{xx}, t_{yy}, t_{xy} and t_{yx} are the components of the stress tensor and m_{xz} and m_{yz} are the components of the couple-stress tensor.

Function U satisfies the biharmonic differential equation

$$\Delta \Delta U = 0 \tag{3.3}$$

i.e., the Airy stress function must satisfy the same differential equation with or without couple-stresses.

Function ψ satisfies the differential equation

$$\Delta (\psi - c^2 \Delta \psi) = 0 \tag{3.4}$$

and, from the compatibility conditions, we have

$$\frac{\partial}{\partial x} (\psi - c^2 \Delta \psi) = - 2 (1 - \nu) c^2 \frac{\partial}{\partial y} (\Delta U)$$
$$\frac{\partial}{\partial y} (\psi - c^2 \Delta \psi) = 2 (1 - \nu) c^2 \frac{\partial}{\partial x} (\Delta U) \tag{3.5}$$

i.e., $\psi - c^2 \Delta \psi$ and $2(1-\nu) c^2 \Delta U$ must be conjugate harmonic functions. For that reason we coll the relations (3.5) the Cauchy-Riemann equations.

To summarize: in a state of plane strain without body forces or body couples, the stress functions U and ψ must be the solutions of equations (3.3) and (3.4) and must be mutually related in accordance with equations (3.5).

If we consider the problem in the complex plane $z = x + iy$, the equations (3.5) are the form

$$\begin{aligned} \frac{\partial}{\partial z}(\psi - c^2 \Delta \psi) &= -2(1-\nu) c^2 \cdot i \frac{\partial}{\partial z}(\Delta u) \\ \frac{\partial}{\partial \bar{z}}(\psi - c^2 \Delta \psi) &= 2(1-\nu) c^2 \cdot i \frac{\partial}{\partial \bar{z}}(\Delta u) \end{aligned} \tag{3.6}$$

where $\bar{z} = x - iy$ and $\Delta = 4 \dfrac{\partial^2}{\partial z \partial \bar{z}}$.

In the elliptic system defined by, (6),

$$z = l \operatorname{ch} \zeta, \quad \bar{z} = l \operatorname{ch} \bar{\zeta} \tag{3.7}$$

where $l^2 = a^2 - b^2$ (a i b are the semiaxes of the ellipse, $a < b$), equations (3.5) become

$$\begin{aligned} \frac{\partial}{\partial \zeta}(\psi - c^2 \Delta \psi) &= -2(1-\nu) c^2 i \frac{\partial}{\partial \zeta}(\Delta U) \\ \frac{\partial}{\partial \zeta}(\psi - c^2 \Delta \psi) &= 2(1-\nu) c^2 i \frac{\partial}{\partial \bar{\zeta}}(\Delta U) \end{aligned} \quad , \quad \Delta = \frac{4}{l^2 \, sh \, \zeta \, sh \, \bar{\zeta}} \cdot \frac{\partial^2}{\partial \zeta \partial \bar{\zeta}} \tag{3.8}$$

in wich U and ψ are the functions of variables ζ and $\bar{\zeta}$. If we express these variables in the form

$$\zeta = \mu + i\vartheta, \quad \bar{\zeta} = \mu - i\vartheta \tag{3.9}$$

where μ and ϑ are elliptic coordinates, and, accordingly functions U and ψ in terms of the same variables, we obtain equations (3.8) in the following form

$$\begin{aligned} \frac{\partial}{\partial \mu}(\psi - c^2 \Delta \psi) &= -2(1-\nu) c^2 \frac{\partial}{\partial \vartheta}(\Delta U) \\ \frac{\partial}{\partial \vartheta}(\psi - c^2 \Delta \psi) &= 2(1-\nu) c^2 \frac{\partial}{\partial \mu}(\Delta U), \end{aligned} \tag{3.10}$$

where

$$\Delta = \frac{2}{l^2 (ch \, 2\mu - \cos 2\vartheta)} \left(\frac{\partial^2}{\partial \mu^2} + \frac{\partial^2}{\partial \vartheta^2} \right),$$

and where we used relations

$$\frac{\partial}{\partial \zeta} = \frac{1}{2} \frac{\partial}{\partial \mu} - \frac{i}{2} \frac{\partial}{\partial \vartheta}, \quad \frac{\partial}{\partial \bar{\zeta}} = \frac{1}{2} \frac{\partial}{\partial \mu} + \frac{i}{2} \frac{\partial}{\partial \vartheta} \tag{3.11}$$

From the well-known relations

$$t_{ij} = t_{\alpha\beta} \frac{\partial x^{\alpha}}{\partial y^{i}} \frac{\partial x^{\beta}}{\partial y^{j}}, \qquad m_{iz} = m_{\alpha z} \frac{\partial x^{\alpha}}{\partial y^{i}}$$

where $x^1 = x$, $x^2 = y$, $y^1 = \mu$, $y^2 = \vartheta$, with respect to (3.1), (3.7) and (3.11), we obtain

$$\begin{aligned}
t_{\mu\mu} &= -\left(\frac{\partial^2 U}{\partial \zeta^2} - \frac{ch\,\zeta}{sh\,\zeta}\frac{\partial U}{\partial \zeta}\right) - \left(\frac{\partial^2 U}{\partial \bar{\zeta}^2} - \frac{ch\,\bar{\zeta}}{sh\,\bar{\zeta}}\frac{\partial U}{\partial \bar{y}}\right) + 2\frac{\partial^2 U}{\partial \zeta \partial \bar{\zeta}} - \\
&\quad - i\left(\frac{\partial^2 \psi}{\partial \zeta^2} - \frac{ch\,\zeta}{sh\,\zeta}\frac{\partial \psi}{\partial \zeta}\right) + i\left(\frac{\partial^2 \psi}{\partial \bar{\zeta}^2} - \frac{ch\,\bar{\zeta}}{sh\,\bar{\zeta}}\frac{\partial \psi}{\partial \bar{\zeta}}\right), \\
t_{\vartheta\vartheta} &= \left(\frac{\partial^2 U}{\partial \zeta^2} - \frac{ch\,\zeta}{sh\,\zeta}\frac{\partial U}{\partial \zeta}\right) + \left(\frac{\partial^2 U}{\partial \bar{\zeta}^2} - \frac{ch\,\bar{\zeta}}{sh\,\bar{\zeta}}\frac{\partial U}{\partial \bar{\zeta}}\right) + 2\frac{\partial^2 U}{\partial \zeta \partial \bar{\zeta}} + \\
&\quad + i\left(\frac{\partial^2 \psi}{\partial \zeta^2} - \frac{ch\,\zeta}{sh\,\zeta}\frac{\partial \psi}{\partial \zeta}\right) - i\left(\frac{\partial^2 \psi}{\partial \bar{\zeta}^2} - \frac{ch\,\bar{\zeta}}{sh\,\bar{\zeta}}\frac{\partial \psi}{\partial \bar{\zeta}}, \right. \\
t_{\mu\vartheta} &= -i\left(\frac{\partial^2 U}{\partial \zeta^2} - \frac{ch\,\zeta}{sh\,\zeta}\frac{\partial U}{\partial \zeta}\right) + i\left(\frac{\partial^2 U}{\partial \bar{\zeta}^2} - \frac{ch\,\bar{\zeta}}{sh\,\bar{\zeta}}\frac{\partial U}{\partial \bar{\zeta}}\right) + \\
&\quad + \left(\frac{\partial^2 \psi}{\partial \zeta^2} - \frac{ch\,\zeta}{sh\,\zeta}\frac{\partial \psi}{\partial \zeta}\right) + \left(\frac{\partial^2 \psi}{\partial \bar{\zeta}^2} - \frac{ch\,\bar{\zeta}}{sh\,\bar{\zeta}}\frac{\partial \psi}{\partial \bar{\zeta}}\right) - 2\frac{\partial^2 \psi}{\partial \zeta \partial \bar{\zeta}} \\
t_{\vartheta\mu} &= -i\left(\frac{\partial^2 U}{\partial \zeta^2} - \frac{ch\,\zeta}{sh\,\zeta}\frac{\partial U}{\partial \zeta}\right) + i\left(\frac{\partial^2 U}{\partial \bar{\zeta}^2} - \frac{ch\,\bar{\zeta}}{sh\,\bar{\zeta}}\frac{\partial U}{\partial \bar{\zeta}}\right) + \\
&\quad + \left(\frac{\partial^2 \psi}{\partial \zeta^2} - \frac{ch\,\zeta}{sh\,\zeta}\frac{\partial \psi}{\partial \zeta}\right) + \left(\frac{\partial^2 \psi}{\partial \bar{\zeta}^2} - \frac{ch\,\bar{\zeta}}{sh\,\bar{\zeta}}\frac{\partial \psi}{\partial \bar{\zeta}}\right) + 2\frac{\partial^2 \psi}{\partial \zeta \partial \bar{\zeta}} \\
m_{\mu z} &= \frac{\partial \psi}{\partial \zeta} + \frac{\partial \psi}{\partial \bar{\zeta}}, \qquad m_{\vartheta z} = i\left(\frac{\partial \psi}{\partial \zeta} - \frac{\partial \psi}{\partial \bar{\zeta}}\right)
\end{aligned} \tag{3.12}$$

where $t_{\mu\mu}$, $t_{\vartheta\vartheta}$, $t_{\mu\vartheta}$, $t_{\vartheta\mu}$, $m_{\mu z}$, $m_{\vartheta z}$ are the covariant components of the stress and couple-stress tensor in the elliptic system defined by (3.7).

Further, if we express functions U, ψ in term of variables μ, ϑ, we obtain

$$\begin{aligned}
t_{\mu\mu} &= \frac{\partial^2 U}{\partial \vartheta^2} + \frac{sh\,2\mu}{ch\,2\mu - \cos 2\vartheta}\frac{\partial U}{\partial \mu} - \frac{\sin 2\vartheta}{ch\,2\mu - \cos 2\vartheta}\frac{\partial U}{\partial \vartheta} - \frac{\partial^2 \psi}{\partial \mu \partial \vartheta} + \\
&\quad + \frac{\sin 2\vartheta}{ch\,2\mu - \cos 2\vartheta}\frac{\partial \psi}{\partial \mu} + \frac{sh\,2\mu}{ch\,2\mu - \cos 2\vartheta}\frac{\partial \psi}{\partial \vartheta}, \\
t_{\vartheta\vartheta} &= \frac{\partial^2 U}{\partial \mu^2} - \frac{sh\,2\mu}{ch\,2\mu - \cos 2\vartheta}\frac{\partial U}{\partial \mu} + \frac{\sin 2\vartheta}{ch\,2\mu - \cos 2\vartheta}\frac{\partial U}{\partial \vartheta} + \frac{\partial^2 \psi}{\partial \mu \partial \vartheta} - \\
&\quad - \frac{\sin 2\vartheta}{ch\,2\mu - \cos 2\vartheta}\frac{\partial \psi}{\partial \mu} - \frac{sh\,2\mu}{ch\,2\mu - \cos 2\vartheta}\frac{\partial \psi}{\partial \vartheta},
\end{aligned}$$

$$t_{\mu\vartheta} = -\frac{\partial^2 U}{\partial\mu\,\partial\vartheta} + \frac{\sin 2\vartheta}{ch\,2\mu - \cos 2\vartheta}\frac{\partial U}{\partial\mu} + \frac{sh\,2\mu}{ch\,2\mu - \cos 2\vartheta}\frac{\partial U}{\partial\vartheta} - \frac{\partial^2\psi}{\partial\vartheta^2} -$$

$$- \frac{sh\,2\mu}{ch\,2\mu - \cos 2\vartheta}\frac{\partial\psi}{\partial\mu} + \frac{\sin 2\vartheta}{ch\,2\mu - \cos 2\vartheta}\frac{\partial\psi}{\partial\vartheta}, \tag{3.13}$$

$$t_{\vartheta\mu} = -\frac{\partial^2 U}{\partial\mu\,\partial\vartheta} + \frac{\sin 2\vartheta}{ch\,2\mu - \cos 2\vartheta}\frac{\partial U}{\partial\mu} + \frac{sh\,2\mu}{ch\,2\mu - \cos 2\vartheta}\frac{\partial U}{\partial\vartheta} + \frac{\partial^2\psi}{\partial\mu^2} -$$

$$- \frac{sh\,2\mu}{ch\,2\mu - \cos 2\vartheta}\frac{\partial\psi}{\partial\mu} + \frac{\sin 2\vartheta}{ch\,2\mu - \cos 2\vartheta}\frac{\partial\psi}{\partial\vartheta},$$

$$m_{\mu z} = \frac{\partial\psi}{\partial\mu}, \quad m_{\vartheta z} = \frac{\partial\psi}{\partial\vartheta}.$$

The physical components of the stress and couple-stress tensor are of the form

$$t_{(\mu\mu)} = \frac{1}{g}t_{\mu\mu}, \quad t_{(\vartheta\vartheta)} = \frac{1}{g}t_{\vartheta\vartheta}, \quad t_{(\mu\vartheta)} = \frac{1}{g}t_{\mu\vartheta}, \quad t_{(\vartheta\mu)} = \frac{1}{g}t_{\vartheta\mu},$$

$$m_{(\mu z)} = \frac{1}{\sqrt{g}}m_{\mu z}, \quad m_{(\vartheta z)} = \frac{1}{\sqrt{g}}m_{\vartheta z}, \tag{3.14}$$

where

$$g = \frac{l^2}{2}(ch\,2\mu - \cos 2\vartheta). \tag{3.15}$$

4. Boundary conoitins

We consider an infinite plane with an elliptic hole, stretched at the infinity by forces p, the stretching force acting in the direction of the short axis of the ellipse. The boundary conditions are

$$t_{yy} = p, \quad t_{xx} = t_{xy} = t_{yx} = m_{xz} = m_{yz} = 0 \quad \text{when} \quad y \to \infty, \tag{4.1}$$

and, with respect to

$$t_{(\mu\mu)} + t_{(\vartheta\vartheta)} = t_{xx} + t_{yy}, \tag{4.2}$$

$$t_{(\mu\mu)} - t_{(\vartheta\vartheta)} + i\,(t_{(\mu\vartheta)} + t_{(\vartheta\mu)}) = [t_{xx} - t_{yy} + i\,(t_{xy} + t_{yx})]\,e^{-2i\alpha}, \tag{4.3}$$

$$m_{(\mu z)} + i\,m_{(\vartheta z)} = (m_{xz} + i\,m_{yz})\,e^{-i\alpha}, \tag{4.4}$$

where

$$e^{2i\alpha} = \frac{sh\,y}{sh\,\bar{y}}$$

we obtain, when $\mu \to \infty$,

$$t_{(\mu\mu)} + t_{(\vartheta\vartheta)} = p \tag{4.5}$$

$$t_{(\mu\mu)} - t_{(\vartheta\vartheta)} + i\,(t_{(\mu\vartheta)} + t_{(\vartheta\mu)}) = -\,p\,e^{-2i\alpha} \tag{4.6}$$

$$m_{(\mu z)} + i\, m_{(\vartheta z)} = 0. \tag{4.7}$$

For $\mu = \mu_0$, we have

$$t_{(\mu\mu)} = 0, \quad (4.8) \qquad t_{(\mu\vartheta)} = 0, \quad (4.9) \qquad m_{(\mu z)} = 0. \quad (4.10)$$

5. Solutions

We take the solutions of equation (3.3) in the form, (6),

$$U(\zeta, \bar{\zeta}) = \frac{l^2}{8} [2\, A\, ch\, \zeta\, ch\, \bar{\zeta} + B\, sh\, \zeta\, sh\, \bar{\zeta} + \bar{E}\, sh\, \bar{\zeta}\, ch\, \zeta + C(\zeta + \bar{\zeta}) + \\ + D\, ch\, 2\zeta + \bar{D}\, ch\, 2\bar{\zeta} + E\, sh\, 2\zeta + \bar{E}\, sh\, 2\bar{\zeta}] \tag{5.1}$$

i.e., in the equivalent form

$$U(\mu, \vartheta) = \frac{Al^2}{8}(ch\, 2\mu + \cos 2\vartheta) + \frac{Cl^2}{4}\mu + \frac{l^2}{2}\left(\frac{B}{8} + \frac{\bar{B}}{8}\right) sh\, 2\mu + \\ + l^2\left(\frac{D}{8} + \frac{\bar{D}}{8}\right) ch\, 2\mu \cos 2\vartheta + l^2\left(\frac{E}{8} + \frac{\bar{E}}{8}\right) sh\, 2\mu \cos 2\vartheta + \\ + i\left[\frac{l^2}{2}\left(\frac{B}{8} - \frac{\bar{B}}{8}\right)\sin 2\vartheta + l^2\left(\frac{D}{8} - \frac{\bar{D}}{8}\right) sh\, 2\mu \sin 2\vartheta + \\ + l^2\left(\frac{E}{8} - \frac{\bar{E}}{8}\right) ch\, 2\mu \sin 2\vartheta\right], \tag{5.2}$$

and taking in account

$$B = B_1 + i\, B_2, \quad \bar{B} = B_1 - i\, B_2, \quad D = D_1 + i\, D_2, \quad \bar{D} = D_1 - i\, D_2, \\ E = E_1 + i\, E_2, \quad \bar{E} = E_1 - i\, E_2. \tag{5.3}$$

we obtain finally

$$U(\mu, \vartheta) = \frac{A\, l^2}{8} ch\, 2\mu + \frac{B_1\, l^2}{8} sh\, 2\mu + \frac{C\, l^2}{4}\mu + l^2\left(\frac{A}{8} + \frac{D_1}{4} ch\, 2\mu + \right. \\ \left. + \frac{E_1}{4} sh\, 2\mu\right)\cos 2\vartheta - l^2\left(\frac{B_2}{8} + \frac{D_2}{4} sh\, 2\mu + \frac{E_2}{4} ch\, 2\mu\right)\sin 2\vartheta. \tag{5,4}$$

To determine ψ we use

$$\psi = \psi_p + \psi_h \tag{5.5}$$

where

$$\Delta\, \psi_p = 0, \quad (5.6)_1 \qquad \psi_h - c^2\, \Delta\, \psi_h = 0. \quad (5.6)_2$$

From equation (3.5), we obtain

$$\psi_p(\zeta, \bar{\zeta}) = 2(1 - \nu)\, c^2\, i\left[\frac{\bar{B}}{2}\frac{ch\, \bar{\zeta}}{sh\, \bar{\zeta}} - \frac{B}{2}\frac{ch\, \zeta}{sh\, \zeta}\right] + \text{const.} \tag{5.7}$$

i.e.,

$$\psi_p(\mu, \vartheta) = -\theta B_1 \frac{\sin 2\vartheta}{ch\, 2\mu - \cos 2\vartheta} + \theta B_2 \frac{sh\, 2\mu}{ch\, 2\mu - \cos 2\vartheta} + \text{const.}, \tag{5.8}$$

where

$$\theta = 2(1-\nu)\, c^2. \tag{5.9}$$

With respect to

$$\Delta = \frac{2}{l^2 (ch\, 2\mu - \cos 2\vartheta)} \left(\frac{\partial^2}{\partial \mu^2} + \frac{\partial^2}{\partial \vartheta^2} \right)$$

equation (5.5) becomes

$$\frac{\partial^2 \psi(\mu, \vartheta)}{\partial \mu^2} + \frac{\partial^2 \psi(\mu, \vartheta)}{\partial \vartheta^2} - \frac{l^2}{c^2} (ch^2 \mu - \cos^2 \vartheta)\, \psi(\mu, \vartheta) = 0. \tag{5.10}$$

We suppouse that the solution of this equation may be written in the form

$$\psi(\mu, \vartheta) = M(\mu) \cdot T(\vartheta) \tag{5.11}$$

and, substituting (5.11) in (5.10), we obtain two differential equations in the form

$$\frac{d^2 T(\vartheta)}{d\vartheta^2} + (a + 2q \cos 2\vartheta)\, T(\vartheta) = 0 \tag{5.12}$$

$$\frac{d^2 M(\mu)}{d\mu^2} - (a + 2q\, ch\, 2\mu)\, M(\mu) = 0 \tag{5.13}$$

where

$$2q = \frac{l^2}{2c^2}.$$

Equation (5.12) is the Mathieu differential equation for q negative and equation (5.13) is the modified Mathieu equation for q negative, (4).

It is easy to see that in our problem the solutions of equation (5.12) are of the form

$$s_{0_{2k+2}}(\vartheta, -q) = (-1)^k \sum_{n=0}^{\infty} (-1)^n B_{2n+2}^{(2k+2)} \sin(2n+2)\vartheta \tag{5.14}$$

$$c_{e_{2k}}(\vartheta, -q) = (-1)^k \sum_{n=0}^{\infty} (-1)^n A_{2n}^{(2k)} \cos 2n\vartheta, \tag{5.15}$$

while the solutions of equation (5.13) are of the form

$$S_{0_{2k+2}}(\mu, -q) =$$

$$= (-1)^k \frac{s'_{0_{2k+2}}(0, q)}{h^2 B_2^{(2k+2)}} \operatorname{tg} h\mu \sum_{n=0}^{\infty} (-1)^n (2n+2) B_{2n+2}^{(2k+2)} I_{2n+2}(2h\, ch\, \mu), \tag{5.16}$$

$$C_{e_{2k}}(\mu, -q) = (-1)^k \frac{c_{e_{2k}}(0, q)}{A_0^{(2k)}} \sum_{n=0}^{\infty} (-1)^n A_{2n}^{(2k)} I_{2n}(2 h\, ch\, \mu), \tag{5.17}$$

$$K_{o_{2k+2}}(\mu, -q) =$$

$$= (-1)^k \frac{s'_{o_{2k+2}}(0, q)}{\pi h^2 B_2^{(2k+2)}} \operatorname{tg} h\, \mu \sum_{n=0}^{\infty} (-1)^n (2n+2) B_{2n+2}^{(2k+2)} K_{2n+2}(2 h\, ch\, \mu), \tag{5.18}$$

$$K_{e_{2k}}(\mu, -q) = (-1)^k \frac{c_{e_{2k}}(0, q)}{\pi A_0^{(2k)}} \sum_{n=0}^{\infty} (-1)^n A_{2n}^{(2k)} K_{2n}(2 h\, ch\, \mu). \tag{5.19}$$

Functions (5.14) and (5.15) are the Mathieu functions for q negative with characteristic numbers (or separation constans) b_{2k+2} and a_{2k}. Functions (5.16)—(5.19) are the modified radial Mathieu functions of the first and second kind for q negative with the same characteristic numbers, (4).

Further, $h^2 = q$ and functions $I_{2n}(2 h\, ch\, \mu)$ and $K_{2n}(2 h\, ch\, \mu)$ are the modified Bessel functions of the first and second kind. The coefficients $A_{2n}^{(2k)}$ and $B_{2n+2}^{(2k+2)}$ are the functions of q and their form depends on the value of q.

From the boundary conditions (4.5), (4.6) and (4.7), taking into account that

$$S_{o_{2k+2}}(\mu, -q) \to \infty, \quad C_{e_{2k}}(\mu, -q) \to \infty \quad \text{when} \quad \mu \to \infty$$

we can see that this functions can not be the solution of our problem. For that reason, we take the solution of equation $(5.6)_2$ in the form

$$\psi_h(\mu, \vartheta) = \sum_{k=0}^{\infty} G_{2k+2}\, s_{o_{2k+2}}(\vartheta, -q)\, K_{o_{2k+2}}(\mu, -q) + \sum_{k=0}^{\infty} H_{2k}\, c_{e_{2k}}(\vartheta, -q)\, K_{e_{2k}}(\mu, -q) \tag{5.20}$$

where G_{2k+2} and H_{2k} are constants of integration.

From the relations (3.13), with respect to (5.4), (5.5), (5.8) and (5.20), we obtain

$$t_{\mu\mu} = \frac{A l^2}{4}(ch\, 2\mu - \cos 2\vartheta) + \frac{B_1 l^2}{4} \frac{sh\, 2\mu\, ch\, 2\mu}{ch\, 2\mu - \cos 2\vartheta} + \frac{C l^2}{4} \frac{sh\, 2\mu}{ch\, 2\mu - \cos 2\vartheta} +$$

$$+ \frac{D_1 l^2}{2}(1 - ch\, 2\mu \cos 2\vartheta) + \frac{E_1 l^2}{2}\left(\frac{1}{ch\, 2\mu - \cos 2\vartheta} - \cos 2\vartheta\right) sh\, 2\mu +$$

$$+ \frac{B_2 l^2 \sin 2\vartheta\,(2\, ch\, 2\mu - \cos 2\vartheta)}{4\,(ch\, 2\mu - \cos 2\vartheta)} + \frac{D_2 l^2}{2} sh\, 2\mu \sin 2\vartheta +$$

$$+\frac{E_2 l^2}{2}\left(ch\,2\,\mu+\frac{1}{ch\,2\,\mu-\cos 2\,\vartheta}\right)\sin 2\,\vartheta+$$

$$+6\,\theta\,B_1\,sh\,2\,\mu\,\frac{2-ch\,2\,\mu\cos 2\,\vartheta-\cos^2 2\,\vartheta}{(ch\,2\,\mu-\cos 2\,\vartheta)^3}+$$

$$+6\,\theta\,B_2\sin 2\,\vartheta\,\frac{2-ch\,2\,\mu\cos 2\,\vartheta-ch^2\,2\,\mu}{(ch\,2\,\mu-\cos 2\,\vartheta)^3}+$$

$$+\frac{\sin 2\,\vartheta}{ch\,2\,\mu-\cos 2\,\vartheta}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\left[G_{2k+2}\frac{\partial\,K_{0_{2k+2}}}{\partial\,\mu}B_{2n+2}^{(2k+2)}\sin(2\,n+2)\,\vartheta+\right.$$

$$\left.+H_{2k}\frac{\partial\,K_{e_{2k}}}{\partial\,\mu}A_{2n}^{(2k)}\cos s\,n\,\vartheta\right]+$$

$$+\frac{sh\,2\,\mu}{ch\,2\,\mu-\cos 2\,\vartheta}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\left[G_{2k+2}\,K_{0_{2k+2}}(2\,n+2)\,B_{2n+2}^{(2k+2)}\cos(2\,n+2)\,\vartheta-\right.$$

$$\left.-H_{2k}\,K_{e_{2k}}\cdot(2n)\,A_{2n}^{(2k)}\sin 2\,n\,\vartheta\right]-$$

$$-\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\left[G_{2k+2}\frac{\partial\,K_{0_{2k+2}}}{\partial\,\mu}(2\,n+2)\,B_{2n+2}^{(2k+2)}\cos(2\,n+2)\,\vartheta+\right.$$

$$\left.+H_{2k}\frac{\partial\,K_{e_{2k}}}{\partial\,\mu}(2\,n)\,A_{2n}^{(2k)}\sin 2\,n\,\vartheta\right],\tag{5.21}$$

$$t_{\mu\vartheta}=\frac{B_1 l^2}{4}\frac{ch\,2\,\mu\sin 2\,\vartheta}{ch\,2\,\mu-\cos 2\,\vartheta}+\frac{C\,l^2}{4}\frac{\sin 2\,\vartheta}{ch\,2\,\mu-\cos 2\vartheta}+\frac{D_1 l^2}{2}sh\,2\,\mu\sin 2\,\vartheta+$$

$$+\frac{E_1 l^2}{2}\sin 2\,\vartheta\left(ch\,2\,\mu+\frac{1}{ch\,2\,\mu-\cos 2\,\vartheta}-\frac{B_2 l^2}{4}\frac{sh\,2\,\mu\cos 2\,\vartheta}{ch\,2\,\mu-\cos 2\,\vartheta}-\right.$$

$$-\frac{D_2 l^2}{2}(1-ch\,2\,\mu\cos 2\,\vartheta)-\frac{E_2 l^2}{2}sh\,2\,\mu\left(\frac{1}{ch\,2\,\mu-\cos 2\,\vartheta}-\cos 2\,\vartheta\right)+$$

$$+6\,\theta\,B_1\sin 2\,\vartheta\,\frac{2-ch\,2\,\mu\cos 2\,\vartheta-ch^2\,2\,\mu}{(ch\,2\,\mu-\cos 2\,\vartheta)^3}-$$

$$-6\,\theta\,B_2\,sh\,2\,\mu\,\frac{2-ch\,2\,\mu\cos 2\,\vartheta-\cos^2 2\,\vartheta}{(ch\,2\,\mu-\cos 2\,\vartheta)^3}+$$

$$+\frac{\sin 2\,\vartheta}{ch\,2\,\mu-\cos 2\,\vartheta}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\left[G_{2k+2}\,K_{0_{2k+2}}(2\,n+2)\,B_{2n+2}^{(2k+2)}\cos(2\,n+2)\,\vartheta-\right.$$

$$\left.-H_{2k}\,K_{e_{2k}}(2\,n)\,A_{2n}^{(2k)}\sin 2\,n\,\vartheta\right]-$$

$$-\frac{sh\,2\,\mu}{ch\,2\,\mu-\cos 2\,\vartheta}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\Big[G_{2k+2}\frac{\partial K_{0_{2k+2}}}{\partial\mu}B_{2n+2}^{(2k+2)}\sin(2\,n+2)\,\vartheta+$$

$$+H_{2k}\frac{\partial K_{e_{2k}}}{\partial\mu}A_{2n}^{(2k)}\cos s\,n\,\vartheta\Big]+$$

$$+\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}[G_{2k+2}K_{0_{2k+2}}(2\,n+2)^2B_{2n+2}^{(2k+2)}\sin(2\,n+2)\,\vartheta+$$

$$+H_{2k}K_{e_{2k}}(2\,n)^2A_{2n}^{(2k)}\cos 2\,n\,\vartheta],$$

$$m_{\mu z}=2\,\theta\,B_1\frac{sh\,2\,\mu\sin 2\,\vartheta}{(ch\,2\,\mu-\cos 2\,\vartheta)^2}+2\,\theta\,B_2\frac{1-ch\,2\,\mu\cos 2\,\vartheta}{(ch\,2\,\mu-\cos 2\,\vartheta)^2}+$$

$$+\sum\sum\Big[G_{2k+2}\frac{\partial K_{0_{2k+2}}}{\partial\mu}B_{2n+2}^{(2k+2)}\sin(2\,n+2)\,\vartheta+H_{2k}\frac{\partial K_{e_{2k}}}{\partial\mu}A_{2n}^{(2k)}\cos s\,n\,\vartheta\Big].$$

Substituting (5.21) into (4.5), (4.6), (4.8), (4.9) and (4.10), taking in account (3.14), we have

$$A+B_1=p\quad(5.22)\qquad E_1+D_1=\frac{p}{2}\quad(5.23)\qquad B_2=D_2=E_2=0.\quad(5.24)$$

Relations (3.13) become

$$t_{(\mu\mu)}=\frac{A}{2}+\frac{B}{2}\frac{sh\,2\,\mu\,ch\,2\,\mu}{(ch\,2\,\mu-\cos 2\,\vartheta)^2}+\frac{C}{2}\frac{sh\,2\,\mu}{(ch\,2\,\mu-\cos 2\,\vartheta)}+$$

$$+D\frac{1-ch\,2\,\mu\cos 2\,\vartheta}{ch\,2\,\mu-\cos 2\,\vartheta}+E\Big[\frac{1}{(ch\,2\,\mu-\cos 2\,\vartheta)^2}-\frac{\cos 2\,\vartheta}{ch\,2\,\mu-\cos 2\,\vartheta}\Big]sh\,2\,\mu+$$

$$+\frac{12\,\theta\,B}{l^2}sh\,2\,\mu\frac{2-ch\,2\,\mu\cos 2\,\vartheta-\cos^2 2\,\vartheta}{(ch\,2\,\mu-\cos 2\,\vartheta)^4}+$$

$$+\frac{2\sin 2\,\vartheta}{l^2(ch\,2\,\mu-\cos 2\,\vartheta)^2}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}G_{2k+2}\frac{\partial K_{0_{2k+2}}}{\partial\mu}B_{2n+2}^{(2k+2)}\sin(2\,n+2)\,\vartheta+$$

$$+\frac{2\,sh\,2\,\mu}{l^2(ch\,2\,\mu-\cos 2\,\vartheta)^2}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}G_{2k+2}K_{0_{2k+2}}(2\,n+2)\,B_{2n+2}^{(2k+2)}\cos(2\,n+2)\,\vartheta-$$

$$-\frac{2}{(l^2(ch\,2\,\mu-\cos 2\,\vartheta)}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}G_{2k+2}\frac{\partial K_{0_{2k+2}}}{\partial\mu}(2\,n+2)\,B_{2n+2}^{(2k+2)}\cos(2\,n+2)\,\vartheta,$$

$$t_{(\vartheta\vartheta)}=A+B\frac{sh\,2\,\mu}{ch\,2\,\mu-\cos 2\,\vartheta}-t_{(\mu\mu)},\qquad(5.25)$$

$$t_{(\mu\vartheta)} = \frac{B}{2}\frac{ch\,2\,\mu\sin 2\,\vartheta}{(ch\,2\,\mu - \cos 2\,\vartheta)^2} + \frac{C}{2}\frac{\sin 2\,\vartheta}{ch\,2\,\mu - \cos\vartheta)^2} + D\frac{sh\,2\,\mu\sin 2\,\vartheta}{ch\,2\,\mu - \cos 2\,\vartheta} +$$

$$+ E\left[\frac{ch\,2\,\mu}{ch\,2\,\mu - \cos 2\,\vartheta} + \frac{1}{(ch\,2\,\mu - - \cos 2\,\vartheta)^2}\right]\sin 2\,\vartheta +$$

$$+ \frac{12\,\theta\,B}{l^2}\sin 2\,\vartheta\frac{2 - ch\,2\,\mu\cos 2\,\vartheta - ch^2\,2\,\mu}{(ch\,2\,\mu - \cos 2\,\vartheta)^4} +$$

$$+ \frac{2\sin 2\,\vartheta}{l^2\,(ch\,2\,\mu - \cos 2\,\vartheta)^2}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty} G_{2k+2}\,K_{0_{2k+2}}\,(2\,n+2)\,B_{2n+2}^{(2k+2)}\cos(2\,n+2)\,\vartheta +$$

$$+ \frac{2}{l^2\,(ch\,2\,\mu - \cos 2\,\vartheta)}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty} G_{2k+2}\,K_{0_{2k+2}}\,(2\,n+2)^2\,B_{2n+2}^{(2k+2)}\sin(2\,n+2)\,\vartheta -$$

$$- \frac{2\,sh\,2\,\mu}{l^2\,(ch\,2\,\mu - \cos 2\,\vartheta)^2}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty} G_{2k+2}\,\frac{\partial K_{0_{2k+2}}}{\partial\mu}\,B_{2n+2}^{(2k+2)}\sin(2\,n+2)\,\vartheta,$$

$$t_{(\mu\vartheta)} - t_{(\vartheta\mu)} = \frac{2}{l^2\,(ch\,2\,\mu - \cos 2\,\vartheta)}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty} G_{2k+2}\,B_{2n+2}^{(2k+2)}\left[(2\,n+2)^2\,K_{0_{2k+2}} - \right.$$

$$\left. - \frac{\partial^2 K_{0_{2k+2}}}{\partial\mu^2}\right]\sin(2\,n+2)\,\vartheta,$$

$$m_{(\mu z)} = \left[\frac{2}{l^2\,(ch\,2\,\mu - \cos 2\,\vartheta)}\right]^{1/2}\left[2\,\theta\,B\,\frac{sh\,2\,\mu\sin 2\,\vartheta}{ch\,2\,\mu - \cos 2\,\vartheta)^2} + \right.$$

$$\left. + \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} G_{2k+2}\,\frac{\partial K_{0_{2k+2}}}{\partial\mu}\,B_{2n+2}^{(2k+2)}\sin(2\,n+2)\,\vartheta\right]$$

$$m_{(\vartheta z)} = \left[\frac{2}{l^2\,(ch\,2\,\mu - \cos 2\,\vartheta)}\right]^{1/2}\left[2\,\theta\,B\,\frac{1 - ch\,2\,\mu\cos 2\,\vartheta}{(ch\,2\,\mu - \cos 2\,\vartheta)^2} + \right.$$

$$\left. + \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} G_{2k+2}\,K_{0_{2k+2}}\,(2\,n+2)\,B_{2n+2}^{(2k+2)}\cos(2\,n+2)\,\vartheta.\right.$$

The constants of integration, A, B, C, D, E, G_{2k+2} are the solutions of the system of algebraic equations

$$\frac{A\,l^2}{4}\left(ch^4\,2\,\mu + 3\,ch^2\,2\,\mu + \frac{3}{8}\right) + \frac{B\,l^2}{4}\,sh\,2\,\mu\,ch\,2\,\mu\left(ch^2\,2\,\mu + \frac{1}{2}\right) +$$

$$+ \frac{C\,l^2}{4}\,sh\,2\,\mu\left(ch^2\,2\,\mu + \frac{1}{2}\right) + \frac{D\,l^2}{2}\left(\frac{5}{2}\,ch^3\,2\,\mu + \frac{15}{8}\,ch\,2\,\mu\right) +$$

$$+\frac{E\,l^2}{2}\left(\frac{5}{2}\,ch^2\,2\,\mu+\frac{7}{8}\right)sh\,2\,\mu+9\,\theta\,B+$$

$$+\sum_k G_{2k+2}\frac{\partial\,K_{0_{2k+2}}}{\partial\,\mu}\left[B_2^{(2k+2)}\left(\frac{7}{2}\,ch^2\,2\,\mu+\frac{7}{8}\right)-B_4^{(2k+2)}\frac{7}{2}\,ch\,2\,\mu+B_6^{(2k+2)}\frac{7}{8}\right]+$$

$$+\sum_k G_{2k+2}\,K_{0_{2k+2}}\,sh\,2\,\mu\,[-\,2\,B_2^{(2k+2)}\,ch\,2\,\mu+B_4^{(2k+2)}]=0,$$

$$-\frac{A\,l^2}{4}(4\,ch^3\,2\,\mu+3\,ch\,2\,\mu)-\frac{B\,l^2}{4}\,2\,sh\,2\,\mu\,ch^2\,2\,\mu-\frac{C\,l^2}{4}\,2\,sh\,2\,\mu\,ch\,2\,\mu-$$

$$-\frac{D\,l^2}{2}\left(ch^4\,2\,\mu+\frac{21}{4}\,ch^2\,2\,\mu+\frac{3}{4}\right)-\frac{E\,l^2}{2}\,sh\,2\,\mu\left(ch^3\,2\,\mu+\frac{17}{4}\,ch\,2\,\mu\right)-$$

$$-\,6\,\theta\,B\,ch\,2\,\mu+\sum_k G_{2k+2}\,\frac{\partial\,K_{0_{2k+2}}}{\partial\,\mu}\Big[-\,B_2^{2k+2}\,(2\,ch^3\,2\,\mu+5\,ch\,2\,\mu)+$$

$$+\,B_4^{(2k+2)}\left(\frac{13}{2}\,ch^2\,2\,\mu+\frac{9}{4}\right)-B_6^{(2k+2)}\cdot 5\,ch\,2\,\mu+B_8^{(2k+2)}\cdot\frac{9}{8}\Big]+$$

$$+\sum_k G_{2k+2}\,K_{0_{2k+2}}\,sh\,2\,\mu\Big[B_2^{(2k+2)}\left(2\,(ch^2\,2\,\mu+\frac{3}{4}\right)-$$

$$-\,B_4^{(2k+2)}\cdot 4\,ch\,2\,\mu+B_6^{(2k+2)}\cdot\frac{3}{2}=0,$$

$$\frac{A\,l^2}{4}\left(3\,ch^2\,2\,\mu+\frac{1}{2}\right)+\frac{B\,l^2}{4}\cdot\frac{1}{2}\,sh\,2\,\mu\,ch\,2\,\mu+\frac{C\,l^2}{4}\cdot\frac{1}{2}\,sh\,2\,\mu+$$

$$+\frac{D\,l^2}{2}\left(\frac{3}{2}\,ch^3\,2\,\mu+2\,ch\,2\,\mu\right)+\frac{E\,l^2}{2}\left(\frac{3}{2}\,ch^2\,2\,\mu+1\right)sh\,2\,\mu-\frac{3}{2}\,\theta\,B+$$

$$+\sum_k G_{2k+2}\frac{\partial\,K_{0_{2k+2}}}{\partial\,\mu}\left[B_2^{(2k+2)}\left(\frac{5}{2}\,ch^2\,2\,\mu+1\right)-B_4^{(2k+2)}\,(4\,ch^3\,2\,\mu+6\,ch\,2\,\mu)+\right.$$

$$\left.+\,B_6^{(2k+2)}\left(\frac{19}{2}\,ch^2\,2\,\mu+\frac{19}{8}\right)-B_8^{(2k+2)}\cdot\frac{13}{2}\,ch\,2\,\mu+B_{10}^{(2k+2)}\,\frac{11}{8}\right]+\tag{5.26}$$

$$+\sum_k G_{2k+2}\,K_{0_{2k+2}}\,sh\,2\,\mu\Big[-\,2\,B_2^{(2k+2)}\,ch\,2\,\mu+B_4^{(2k+2)}\,(4\,ch^2\,2\,\mu+2)-$$

$$B_6^{(2k+2)}\cdot 6\,ch\,2\,\mu+B_8^{(2k+2)}\Big]=0,$$

$$-\frac{A\,l^2}{4}\,ch\,2\,\mu-\frac{D\,l^2}{2}\left(\frac{3}{4}\,ch^2\,2\,\mu+\frac{1}{4}\right)-\frac{E\,l^2}{2}\cdot\frac{3}{4}\,sh\,2\,\mu\,ch\,2\,\mu+$$

$$+\sum_k G_{2k+2}\frac{\partial K_{0_{2k+2}}}{\partial\mu}\Big[-B_2^{(2k+2)}\, ch\, 2\mu + B_4^{(2k+2)}\Big(\frac{11}{2}\, ch^2\, 2\mu + \frac{11}{8}\Big) -$$

$$- B_6^{(2k+2)}(6\, ch^3\, 2\mu + 9\, ch\, 2\mu) + B_8^{(2k+2)}\Big(\frac{25}{2}\, ch^2\, 2\mu + \frac{25}{8}\Big) - B_{10}^{(2k+2)}\cdot 8\, ch\, 2\mu +$$

$$+ B_{12}^{(2k+2)}\cdot\frac{13}{8}\Big] + \sum_k G_{2k+2} K_{0_{2k+2}}\, sh\, 2\mu\Big[B_2^{(2k+2)}\frac{1}{2} - B_4^{(2k+2)}\cdot 4\, ch\, 2\mu +$$

$$+ B_6^{(2k+2)}(6\, ch^2\, 2\mu + 3) - B_8^{(2k+2)}\cdot 8\, ch\, 2\mu + B_{10}^{(2k+2)}\frac{5}{2}\Big] = 0,$$

$$\frac{A\, l^2}{4}\cdot\frac{1}{8} + \frac{D\, l^2}{2}\cdot\frac{1}{8}\, ch\, 2\mu + \frac{E\, l^2}{2}\,\frac{1}{8}\, sh\, 2\mu + \sum_k G_{2k+2}\frac{\partial K_{0_{2k+2}}}{\partial\mu}\Big[B_2^{(2k+2)}\cdot\frac{1}{8} -$$

$$- B_4^{(2k+2)}\cdot\frac{5}{2}\, ch\, 2\mu + B_6^{(2k+2)}\Big(\frac{17}{2}\, ch^2\, 2\mu + \frac{17}{8}\Big) - B_8^{(2k+2)}(8\, ch^3\, 2\mu +$$

$$+ 12\, ch\, 2\mu) + B_{10}^{(2k+2)}\Big(\frac{31}{2}\, ch^2\, 2\mu + \frac{31}{8}\Big) - B_{12}^{(2k+2)}\cdot\frac{19}{2}\, ch^2\, 2\mu +$$

$$+ B_{14}^{(2k+2)}\cdot\frac{15}{8}\Big] + \sum_k G_{2k+2} K_{0_{2k+2}}\, sh\, 2\mu\Big[B_4^{(2k+2)} - B_6^{(2k+2)}\cdot 6\, ch\, 2\mu +$$

$$+ B_8^{(2k+2)}\cdot 8\Big(ch^2\, 2\mu + \frac{1}{2}\Big) - B_{10}^{(2k+2)}\, 10\, ch\, 2\mu + B_{12}^{(2k+2)}\cdot 3\Big] = 0,$$

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$$\frac{B\, l^2}{4} ch\, 2\mu\Big(ch^2\, 2\mu + \frac{1}{4}\Big) + \frac{C\, l^2}{4}\Big(ch^2\, 2\mu + \frac{1}{4}\Big) + \frac{D\, l^2}{2}\, sh\, 2\mu\Big(ch^3\, 2\mu +$$

$$+ \frac{3}{4}\, ch\, 2\mu\Big) + \frac{E\, l^2}{2}\Big(ch^4\, 2\mu + \frac{7}{2}\, ch^2\, 2\mu + \frac{1}{4}\Big) + 6\,\theta\, B\,(2 - ch^2\, 2\mu) +$$

$$+ \sum_k G_{2k+2} K_{0_{2k+2}}\Big[2\, B_2^{(2k+2)}(2\, ch^3\, 2\mu + 3\, ch\, 2\mu) - 4\, B_4^{(2k+2)}\Big(\frac{13}{2}\, ch^2\, 2\mu + \frac{7}{4}\Big) +$$

$$+ 6\, B_6^{(2k+2)}\cdot 5\, ch\, 2\mu - 8\, B_8^{(2k+2)}\cdot\frac{9}{8}\Big] -$$

$$- \sum_k G_{2k+2}\frac{\partial K_{0_{2k+2}}}{\partial\mu}\Big[B_2^{(2k+2)}\Big(ch^2\, 2\mu + \frac{1}{4}\Big) -$$

$$- B_4^{(2k+2)}\, ch\, 2\mu + B_6^{(2k+2)}\cdot\frac{1}{4}\Big]\, sh\, 2\mu = 0,$$

$$-\frac{B\,l^2}{4}\,ch^2\,2\,\mu-\frac{C\,l^2}{4}\,ch\,2\,\mu-\frac{D\,l^2}{2}\,sh\,2\,\mu\left(\frac{3}{2}\,ch^2\,2\,\mu+\frac{1}{4}\right)-$$

$$-\frac{E\,l^2}{2}\left(\frac{3}{2}\,ch^3\,2\,\mu+\frac{5}{4}\,ch\,2\,\mu\right)-3\,\theta\,B\,ch\,2\,\mu+$$

$$+\sum G_{2k+2}\,K_{0_{2k+2}}\left[-2\,B_2^{(2k+2)}\left(\frac{5}{2}\,ch^2\,2\,\mu+\frac{1}{4}\right)+\right.$$

$$+4\,B_4^{(2k+2)}\,(4\,ch^3\,2\,\mu+6\,ch\,2\,\mu)-6\,B_6^{(2k+2)}\left(\frac{19}{2}\,ch^2\,2\,\mu+\frac{19}{8}\right)+$$

$$\left.+8\,B_8^{(2k+2)}\cdot\frac{13}{2}\,ch\,2\,\mu-10\,B_{10}^{(2k+2)}\cdot\frac{11}{8}\right]-$$

$$-\sum_k G_{2k+2}\frac{\partial K_{0_{2k+2}}}{\partial\mu}\,sh\,2\,\mu\left[-B_2^{(2k+2)}\,ch\,2\,\mu+B_4^{(2k+2)}\left(ch^2\,2\,\mu+\frac{1}{2}\right)-\right.$$

$$\left.-B_6^{(2k+2)}\,ch\,2\,\mu+B_8^{(2k+2)}\cdot\frac{1}{4}\right]=0,$$

$$\frac{B\,l^2}{4}\cdot\frac{1}{4}\,ch\,2\,\mu+\frac{C\,l^2}{4}\cdot\frac{1}{4}+\frac{D\,l^2}{2}\,sh\,2\,\mu\cdot\frac{3}{4}\,ch\,2\,\mu+\frac{E\,l^2}{2}\left(\frac{3}{4}\,ch^2\,2\,\mu+\right)$$

$$+\sum_k G_{2k+2}\,K_{0_{2k+2}}\left[4\,B_2^{(2k+2)}\,ch\,2\,\mu-4\,B_4^{(2k+2)}\left(\frac{11}{2}\,ch^2\,2\,\mu+\frac{11}{8}\right)+\right.$$

$$+6\,B_6^{(2k+2)}\,(6\,ch^3\,2\,\mu+9\,ch\,2\,\mu)-8\,B_8^{(2k+2)}\left(\frac{25}{2}\,ch^2\,2\,\mu+\frac{25}{8}\right)+$$

$$\left.+10\,B_{10}^{(2k+2)}\cdot 8\,ch\,2\,\mu-12\,B_{12}^{(2k+2)}\cdot\frac{13}{8}\right]-$$

$$-\sum_k G_{2k+2}\frac{\partial K_{0_{2k+2}}}{\partial\mu}\,sh\,2\,\mu\left[B_2^{(2k+2)}\cdot\frac{1}{4}-B_4^{(2k+2)}\cdot ch\,2\,\mu+\right.$$

$$\left.+B_6^{(2k+2)}\cdot\left(ch^2\,2\,\mu+\frac{1}{2}\right)-B_8^{(2k+2)}\,ch\,2\,\mu+B_{10}^{(2k+2)}\cdot\frac{1}{4}\right]=0,$$

$$-\frac{D\,l^2}{2}\frac{1}{8}\,sh\,2\,\mu-\frac{E\,l^2}{2}\frac{1}{8}\,ch\,2\,\mu+\sum_k G_{2k+2}\,K_{0_{2k+2}}\left[-2\,B_2^{(2k+2)}\cdot\frac{1}{8}+\right.$$

$$+4\,B_4^{(2k+2)}\cdot\frac{5}{4}\,ch\,2\,\mu-6\,B_6^{(2k+2)}\frac{17}{2}\,ch^2\,2\,\mu+\left.\frac{17}{2}\right)+$$

$$+8\,B_8^{(2k+2)}\,(8\,ch^3\,2\,\mu+12\,ch\,2\,\mu)-10\,B_{10}^{(2k+2)}\left(\frac{31}{2}\,ch^2\,2\,\mu+\frac{31}{8}\right)+$$

$$\left.+12\,B_{12}^{(2k+2)}\frac{17}{2}\,ch\,2\,\mu-14\,B_{14}^{(2k+2)}\frac{15}{8}\right]-$$

$$- \sum_k G_{2k+2} \frac{\partial K_{0_{2k+2}}}{\partial \mu} sh\, 2\mu \Big[B_4^{(2k+2)} \frac{1}{4} - B_6^{(2k+2)} ch\, 2\mu +$$

$$+ B_8^{(2k+2)} \Big(ch^2\, 2\mu + \frac{1}{2} \Big) - B_{10}^{(2k+2)} ch\, 2\mu + B_{12}^{(2k+2)} \frac{1}{4} \Big] = 0,$$

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$$2\,\theta\, B\, sh\, 2\mu + \sum_k G_{2k+2} \frac{\partial K_{0_{2k+2}}}{\partial \mu} \Big[B_2^{(2k+2)} \Big(ch^2\, 2\mu + \frac{1}{4} \Big) -$$

$$- B_4^{(2k+2)} ch\, 2\mu + B_6^{(2k+2)} \frac{1}{4} \Big] = 0,$$

$$\sum_k G_{2k+2} \frac{\partial K_{0_{2k+2}}}{\partial \mu} \Big[- B_2^{(2k+2)} ch\, 2\mu + B_4^{(2k+2)} \Big(ch^2\, 2\mu + \frac{1}{2} \Big) -$$

$$- B_6^{(2k+2)} ch\, 2\mu + B_8^{(2k+2)} \frac{1}{4} \Big] = 0,$$

$$\sum_k G_{2k+2} \frac{\partial K_{0_{2k+2}}}{\partial \mu} \Big[B_2^{(2k+2)} \frac{1}{4} - B_4^{(2k+2)} ch\, 2\mu +$$

$$+ B_6^{(2k+2)} \Big(ch^2\, 2\mu + \frac{1}{2} \Big) - B_8^{(2k+2)} ch\, 2\mu + B_{10}^{(2k+2)} \frac{1}{4} \Big] = 0,$$

$$\sum_k G_{2k+2} \frac{\partial K_{0_{2k+2}}}{\partial \mu} \Big[B_4^{(2k+2)} \frac{1}{4} - B_6^{(2k+2)} ch\, 2\mu +$$

$$+ B_8^{(2k+2)} \Big(ch^2\, 2\mu + \frac{1}{2} \Big) - B_{10}^{(2k+2)} ch\, 2\mu + B_{12}^{(2k+2)} \frac{1}{4} \Big] = 0,$$

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$$A + B = p, \quad E + D = \frac{p}{2},$$

which is formed from the given boundary conditions (4.5)—(4.10).

This system could be solved by a method of approximations. We should obtain the value of stress $t_{(\vartheta\vartheta)}$ at the limiting points of the greater axis, for $\vartheta = 0, \pi$, after the determination of constants. That value is of particular interest when determining the stress-concentration factor.

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EINFLUSS DER MOMENTENSPANNUNGEN AUF DIE SPANNUNGSVERTEILUNG IN ENDLOSER EBENE MIT ELLIPTISCHEM LOCH

Es wurde hier, vom Standpunkt der Mindlins linearisierter Elastizitaetstheorie mit Momentenspannungen aus, das Problem der endlosen Ebene mit elliptischem Loch eroertert, die in Richtung der kuerzeren Ellipsenachse gleichmaessig gespannt ist. Dieses Problem wurde durch Einfuehrung der Spannungsfunktionen geloest. Es wurden die Formel fuer Spannungstensorkoordinaten und Momentenspannungskoordinaten im elliptischen Koordinatensystem ermittelt. In Uebereinstimmung mit gegebenen Grenzbedingungen wurde das System der algebraischen Gleichungen zur Bestimmung der Intergrationskonstanten formiert.

UTICAJ NAPONSKIH SPREGOVA NA RASPODELU NAPONA U BESKONAČNOJ OBLASTI SA ALIPTIČNIM OTVOROM

U radu je posmatran problem beskonačne oblasti sa eliptičnim otvorom koja je ravnomerno zategnuta u pravcu kraće ose elipse, sa stanovi ta Mindlinove linearne teorije elastičnosti sa naponskim spregovima. Problem je rešavan uvođenjem naponskih funkcija. Dobijeni su izrazi za koordinate tenzora napona i naponskog sprega u eliptičnom sistemu koordinata. U skladu sa zadatim graničnim uslovima, formiran je sistem algebarskih jednačina za određivanje konstanti integracije.

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