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A CONTRIBUTION TO THE APPLICATION OF COMPLEX ANALYSIS IN SOLVING OF BOUNDARY VALUE PROBLEMS OF PLANE LINEAR VISCOELASTOSTATICS

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1. Introduction

Let us denote by X the space of continous functions having continuous derivatives $\frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial x \partial y}, \frac{\partial^2 f}{\partial y^2}, \frac{\partial^2 f}{\partial t^2}$ in $\mathbf{R}^2 \times [0, \infty)$. For $t < 0$ these functions wanish identicaly. In this space let act two endomorphical operators M and L which apply only on the time variable t whith following properties:

(I) operators are linear
(II) operators are continous
(III) operators are real
(IV) there exists $T > 0$ in the sence that M, $L + aM$, $a = 1, 2, 3$ are regular operators for $\forall\ t \in [0, T]$.

The continouity of operators should be considered in the sence of [1]. The third property-reality of operators-should mean that from a real function by applying said operators aloways follows a real function.

Let the functions σ_x, σ_y, τ_{xy}, ε_x, ε_y and γ_{xy} be from X. Between them exsist the following operator relations

$$\sigma_x = L(\varepsilon_x + \varepsilon_y) + 2M\varepsilon_x \tag{1.1}$$

$$\sigma_y = L(\varepsilon_x + \varepsilon_y) + 2M\varepsilon_y \tag{1.1‘}$$

$$\tau_{xy} = M\gamma_{xy} \tag{1.1“}$$

$$\varepsilon_x = \frac{\partial y}{\partial x} \tag{1.2}$$

$$\varepsilon_y = \frac{\partial v}{\partial y} \tag{1.2‘}$$

$$\gamma_{xy} = \frac{\partial v}{\partial y} + \frac{\partial v}{\partial x} \tag{1.2“}$$

$$u = u(x, y, t) \tag{1.3}$$

$$v = v(x, y, t) \tag{1.3'}$$

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0 \tag{1.4}$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0 \tag{1.4'}$$

In the case when L and M are simply operators meaning multiplication with a constant, then σ_x, σ_y, τ_{xy} are the components of stress tensor, describing the usual state of linear elastostatics. In a similar maner we can also treat the case when M and L are operators, defining multiplication by timedependent factor. In the general case when u and v are arbitrarty functions including time t and M in L are operators with properties (i—iv) we shall denote the equations (i—iv) as a basic sistem of equations of linear visco-elastostatics. In [1] M. E. Gurtin and E. Sternberg gave the solution of this sistem for considerally broad class of operators L and M considering somehow different conditions as compared with (i—iv). In this paper we shall show a possibility of solving the sistem (1.1)—(1.4) at least in a formal sence in an analogical way if compared with the classical state.

2. General equations

The equations (1.1)—(1.4) we can satisfy introducing

$$\sigma_x = \frac{\partial^2 U}{\partial y^2}, \quad \tau_{xy} = -\frac{\partial^2 U}{\partial x \partial y}, \quad \sigma_y = \frac{\partial^2 U}{\partial x^2} \tag{2.1}$$

$$U = U(x, y, t)$$

As well as in classical state we find here

$$\sigma_x + \sigma_y = 2(L + M)(\varepsilon_x + \varepsilon_y) \tag{2.2}$$

$$2 M \varepsilon_x = -\sigma_y + \frac{1}{2}(L + 2M)(L + M)^{-1}(\sigma_x + \sigma_y) \tag{2.3}$$

$$2 M \varepsilon_y = -\sigma_x + \frac{1}{2}(L + 2M)(L + M)^{-1}(\sigma_x + \sigma_y) \tag{2.4}$$

Following the deduction [2] we must consider the consequence imposed by (ii). Thence it follows that the operators L and M commute with operators of partial derivation on space variables as well as with operators of integration over a subset of $\mathbf{R}^2$. When these properties are considered in the compatibility equation

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} \tag{2.5}$$

as well as equations (2.1)—2.4) it follows

$$(L + 2M)(L + M)^{-1}(\Delta\Delta U) = 0 \tag{2.6}$$

Finally, by observing (iv) we get at once

$$\Delta\Delta U = 0 \tag{2.7}$$

Thus our deduction also follows to

$$U = Re\,[z\,\varphi(z, t) + \chi(z, t)] \tag{2.8}$$

In (2.8) the $\varphi(z, t)$ and $\chi(z, t)$ are analytic functions of $z = x + iy$, while the time t appears as a parameter. Now we find the relation

$$-\left(\frac{\partial U}{\partial x} + i\frac{\partial U}{\partial x}\right) = \varphi(z, t) + z\,\overline{\varphi'(z, t)} + \overline{\psi(z, t)} \tag{2.9}$$

By a dash we denote derivatives of z while $\psi(z, t) = d\chi/dz$. The left side of this equation has an analog sence compared with classical state. Further from (2.1)

$$\sigma_x + \sigma_y = \Delta U \tag{2.10}$$

and then considernig (2.7) also

$$\sigma_x + \sigma_y = 4\,Re\,[\varphi'(z, t)] \tag{2.11}$$

and

$$2\,M\,\varepsilon_x = -\frac{\partial^2 U}{\partial x^2} + 2\,(L + 2\,M)\,(L + 2\,M)\,(L + M)^{-1}\frac{\partial\,Re\,\varphi(z, t)}{\partial x} \tag{2.12}$$

$$2\,M\,\varepsilon_y = -\frac{\partial^2 U}{\partial y^2} + 2\,(L + 2\,M)\,(L + M)^{-1}\frac{\partial\,Im\,\varphi(z, t)}{\partial y} \tag{2.13}$$

By integration and observing (1.2“) while adding we get

$$2\,M\,(u + iv) = -\left(\frac{\partial U}{\partial x} + i\frac{\partial U}{\partial y}\right) + 2\,(L + 2\,M)\,(L + M)^{-1}\,\varphi(z, t) + $$
$$+ i\,C(t)\,z + C_1(t) \tag{2.14}$$

where it should be noted

$$C(t) = \overline{C(t)} \tag{2.15}$$

This is a real function dependent on the time t only. $C_1(t)$ however is arbitrary complex function of time t only. Introducing a new operator

$$K = 2\,(L + 2\,M)\,(L + M)^{-1} - I = (L + 3\,M)\,(L + M)^{-1} \tag{2.16}$$

in which I is the unity operator, then by supposition $C(t) = C_1(t) = 0$ from (2.9) and (2.14) follows

$$2\,M\,(u + iv) = K\,\varphi(z, t) - z\,\overline{\varphi'(z, t)} - \overline{\psi(z, t)} \tag{2.17}$$

The elements of stress tensor are given by the same relation knownin the classical state. We already have (2.11) as one of these. The other two follow from (2.1) and (2.8).

By the same way known from classical state it can be proved in this case that there can be found for each given stress tensor a biharmonic function $U(x, y, t)$

which can be defined up to the additive term $a(t)\,x + b(t)\,y + c(t)$. $a(t)$, $b(t)$ and $c(t)$ are arbitrary real functions of the time t only. The functions $\varphi(z, t)$ and $\psi(z, t)$ however can be determined when the stress tensor is given up to the additive terms i $C(t)\,z + C_1(t)$ and $C_2(t)$. $C(t)$ is a arbitrary real function, while $C_1(t)$ and $C_2(t)$ are arbitrary complex cunction of time only. In the case, where the displacement $u + iv$ is given hold two additional relations

$$C(t) = 0 \tag{2.18}$$

$$K C_1(t) - \overline{C_2(t)} = 0 \tag{2.19}$$

All the proofs for these statements can be readily seen through the classical state and can be omitted.

3. Basic boundary-value problems and uniqueness theorems

The definition of the first and the second boundary value problem remain unchanged when compared with the classical state. Of cause in our case the former constants become functions of the time t only.

When defining the first boundary value problem in which on the boundary C the loads $f(z, t)$ [2] are given, the time t appears as a parameter. When denoting

$$F(z) = \varphi(z, t) + z\,\overline{\varphi'(z, t)} + \overline{\psi(z, t)} \tag{3.1}$$

and from the equation

$$\int_C [\varphi'(z, t)\,\overline{F(z)}\,dz - \overline{\varphi'(z, t)}\,F(z)\,dz] = -\,8\,i \iint_D \{Re\,[\varphi'(z, t)]\}^2\,d\,S \tag{3.2}$$

it follows when given $f(z, t) = 0$, $z \in C$:

$$\varphi(z, t) = i\,C(t)\,z + C_1(t) \tag{3.3}$$

$$\psi(z, t) = C_2(t) \tag{3.3'}$$

$$\rho_0(t) = \rho_1(t) = \rho_2(t) = \ldots = \rho_N(t) = C_1(t) + \overline{C_2(t)} \tag{3.3''}$$

$C(t)$ is a real arbitrary function, $C_1(t)$ and $C_2(t)$ however arbitrary complex functions of the time only. If the region is infinite becomes $C(t) = 0$.

When defining the second boundary value problem in which on the boundary C of region D displacements $g(z, t)$ are given there is the approache quite more difficult compared with the classical state.

We define

$$G(z) = -\,K\,\varphi(z, t) + z\,\overline{\varphi'(z, t)} + \overline{\psi(z, t)} \tag{3.4}$$

and apply the formula

$$\int_C [-\,\varphi'(z, t)\,\overline{G(z)}\,dz + \overline{\varphi'(z, t)}\,G(z)\,dz] =$$

$$= 4\,i \iint_D \{Re\,[\varphi'(z, t)]\,(K - I)\,Re\,[\varphi'(z, t)] + Im\,[\varphi'(z, t)]\,(K - I)\,zm\,[\varphi'(z, t)] +$$

$$+\,2\,Im^2\,[\varphi'(z, t)]\}\,d\pounds \tag{3.5}$$

When $g(z, t) = 0$ on C, then is no possibility to follow the classical approach. It is necessary to put additional conditions upon the operator K. Considering this restriction we now prove

Theorem I

If for every real function $f(z, t) \in X$ and $t \in [0, T]$ holds

$$\Phi_1(t) = \iint\limits_D f(z, t)(K - z) f(z, t)\, d\mathcal{S} \geqslant 0 \tag{3.6}$$

then from $g(z, t) = 0$, $z \in C$ and from the equation (3.5) it follows

$$\varphi'(z, t) = 0, \; z \in D < t \in [0, T] \tag{3.7}$$

Proof:

If (3.6) holds then from $g(z, t) = 0$, $z \in C$ from (3.5) follows

$$t \in [0, T] \Rightarrow Im[\varphi'(z, t)] = 0, \; z \in D$$

and further

$$\varphi'(z, t) = r(t) \tag{3.8}$$

where $r(t)$ is an arbitrary real function of time only. Also from (3.5) it follows for $t \in [0, T]$

$$r(t)(K - I) r(t) = 0 \tag{3.9}$$

From (3.8) we get

$$\varphi(z, t) = r(t) z + C_1(t) \tag{3.10}$$

where $C_1(t)$ is for the time being an arbitrary complex function of the time only. Further from $g(z, t) = G(z, t) = 0$ it follows

$$F(z) = G(z) + (K + I)\varphi(z, t) = (K + I)[r(t) z + C_1(t)] \tag{3.11}$$

This can present the first boundary-value problem, which can be solved by the same functions as in the case of the second one. General solution of the problem (1.10) however can be expressed explicitly:

$$\varphi(z, t) = \frac{1}{2}(K + I) r(t) z + i\, C(t) z + C_2(t) \tag{3.12}$$

$$\psi(z, t) = C_3(t) \tag{3.13}$$

$$C_2(t) + C_3(t) = (K + I) C_1(t) \tag{3.14}$$

By equalizing expressions for $\varphi(z, t)$ from (3.10) and (3.12) we get

$$t \in [0, T] \Rightarrow C(t) = 0 \wedge (K - z) r(t) = 0 \tag{3.15}$$

Also because of $K - I = 2M(L + M)^{-1}$ it follows

$$t \in [0, T] \Rightarrow r(t) = 0 \tag{3.16}$$

thus it holds (3.7).

Further from (3.10) and (3.12) we have

$$C_2(t) = C_1(t) \tag{3.17}$$

$$\varphi(z, t) = C_1(t) \tag{3.18}$$

$$\psi(z, t) = K\,\overline{C_1(t)} \tag{3.19}$$

In quite a similar way we can also prove the

Theorem II If there exsists a contious function $f_0(t, \tau)$ which is defined for $t \in [0, T]$ and $\tau \in [0, t]$ and has positive values everywhere exept at $\tau = t$ and which asures for each real function $f(z, t) \in X$ and $t \in [0, T]$

$$\Phi_2(t) = \int_O^t f_0(t, \tau)\,\Phi_1(\tau)\,d\tau \geqslant 0 \tag{3.20}$$

then from $g(z, t) = 0,\ z \in C$ and from (3.5) it follows

$$\varphi'(z, t) = 0,\ z \in D \wedge t \in [0, T]$$

Proof: If (3.20) and (3.5) hold then by analogy whith already shown

$$Im\,[\varphi'(z, t)] = 0, \quad z \in D \text{ and } t \in [0, T].$$

This proof is contimed from this point on as alreavy shown.

4. The existence of the solution of boundary value problems

Let be the boundary C of the region D smooth enough [2], the function $\omega(z, t)$ on it however should be with regard to arc length s moreover of such property which enables the use of Sherman method [2]. Further let us define

$$\varphi(z, t) = \frac{1}{2\pi i}\int_C \frac{\omega(\zeta, t)}{\zeta - z}\,d\zeta \tag{4.1}$$

$$\psi(z, t) = \frac{1}{2\pi i}\int_C \frac{-K\,\overline{\omega(\zeta, t)} - \bar{t}\,\omega'(\zeta, t)}{\zeta - z}\,d\zeta \tag{4.2}$$

and Sh.rman's operator

$$\mathcal{S}[K, \omega(z, t)] = \omega(z, t) + \frac{1}{2\pi i}\int_C \omega(\zeta, t)\,d\ln\frac{\zeta - z}{\bar{\zeta} - \bar{z}} +$$

$$+ \frac{K^{-1}}{2\pi i}\int_C \overline{\omega(\zeta, t)}\ d\,\frac{\zeta - z}{\bar{\zeta} - \bar{z}} \tag{4.3}$$

Sherman' equation for the function $\omega(z, t)$ for the first boundary-value problem in simply connected region D a.sumes the form

$$\mathcal{S}[-I, \omega(z, t)] + iz\,Re\left[\int_C \frac{\omega(\zeta, t)}{\zeta^2}\,d\zeta\right] = f(z, t),\ \ z \in C,\ o \in D \tag{4.4}$$

where it is still

$$Re\,[\int_C \overline{f(z, t)\,dz}] = 0 \tag{4.5}$$

In equations (4.4) and (4.5) the time t appears only as a parameter. Therefore all the proof for the existence of the solution (4.4) takes analogue way as in the classical case. The equation (4.4) has for each value of its right side exactly one solution. The generalization on the multiplyconnected region and on the infinite region can be done quite analogue as in the classical state with the only difference that the former constants appearing in the logarithmic terms become now functions of t.

When dealing with second boundary-value problem however Sherman's equation for a simply connected region D takes the form

$$\mathcal{S}[K, \omega(z, t)] = K^{-1}. 2M(u + iv) = f(z, t), \ z \in C \tag{4.6}$$

In this equation K is now the operator from (2.16). The generalization on multiply connected region takes again already known classical way. However the examination of the existence of the solution $\omega(z, t)$ does not take the same way. In the first place when proving that the homogenous equation (4.6) for $f(z, t) = 0$, $z \in C$ only the trivial solution $\omega(z, t) = 0$, if the conditions of the theorem I or II are satisfied. That means that the equation (4.6) having an arbitrary right side has at most one solution. If a further condition on the operator K^{-1} being compact in the given interval $[0, T]$ is added then for the equation (4.6) hold all the Fredholms' theorems. Then the Riesz theorem asures the existence of the solution $\omega(z, t)$ for an arbitrary value of the right side.

5. Exemples for the Operators L and M

The following integral operator, which is throughly shown in [1] fulfills all the properties (i)—(iv).

$$L(f) = \int_{-\infty}^{t} k_1(t - \xi) f_\xi(x, y, \xi)\, d\xi \tag{5.1}$$

where $k_1(t)$ is a real function of the time t only and at negative values of t becomes zero. Now we can the upper equation put in the form

$$L_1(f) = f(x, y + 0)\, k_1(t) + \int_{0}^{t} k_1(t - \xi) f_\xi(x, y, \xi)\, d\xi \tag{5.2}$$

and analogue

$$M_1(f) = f(x, y, + 0)\, k_2(t) + \int_{0}^{t} k_2(t - \xi) f_\xi(x, y, \xi)\, d\xi \tag{5.3}$$

If $k_1(t)$ and $k_2(t)$ are real, nonnegative and twice continously derivable and if $k_2(+0) > 0$ and $k_1(+0) + k_2(+0) > 0$, then also M^{-1} and $(L_1 + M_1)^{-1}$ take the forms (5.2) and (5.3) and so do all the other operators introduced so far. The functions $k_1(t)$ and $k_2(t)$ being nonnegative we can make us of the theorem I (or theorem II). The integral operator being also compact on an arbitrary given interval $[0, T]$ is seen that for such a pair of operators L_1 and M_1 holds also the theory show already at the arbitrary given T.

The operator [3]

$$L_2(f) = f(x, y, +0)\, k_1[\mu(t)] + \int_O^t k_1[\mu(t) - \mu(\xi)]\, f_\xi(x, y, \xi)\, d\xi \qquad (5.4)$$

where $\mu(t)$ is continous and increesing function with two properties

$$\mu(t) = 0 \quad t \leqslant 0 \qquad (5.5)$$

$$\lim_{t \to \infty} \mu(t) = \infty \qquad (5.6)$$

is now different from $L_1(f)$. Namely it has not the property of translation invariance except in the cases where $\mu(t)$ is a linear function. However we can transform it with regard to the new wariable to the form (5.2). If we write

$$\begin{aligned} U &= \mu(t);\; f(x, y, t) = F(x, y, U) \\ u &= \mu(\xi);\; f(x, y, \xi) = F(x, y, u) \\ &[0, t] \Rightarrow [0, U] \end{aligned} \qquad (5.7)$$

we get

$$L_2(f) = L_1(F) = F(x, y, +0)\, k_1(U) + \int_O^U k_1(U - u)\, F_u(x, y, u)\, du \qquad (5.8)$$

Hence all the allready shown theory also holds in this case for the operators L_2 and M_2 of the form (5.4).

Finally let it be mentioned that we can in the case of operators (5.2) and (5.3) successfully solve the second boundary-value problem also by the use of Laplace transforms. Let us examine the transformation of the term $K\varphi(z, t)$! Introducing $\pounds(f(t)) = \tilde{f}(s)$ and considering (2.16) and [1] we get

$$\pounds[K\varphi(z, t)] = \frac{\tilde{k_1}(s) + 3\,\tilde{k_2}(s)}{\tilde{k_1}(s) + \tilde{k_2}(s)} \cdot \varphi(z, s) \qquad (5.9)$$

If we write

$$\tilde{\varkappa}(s) = \frac{\tilde{k_1}(s) + 3\,\tilde{k_2}(s)}{\tilde{k_1}(s) + \tilde{k_2}(s)} \qquad (5.10)$$

then from (3.4) it follows

$$-\tilde{\varkappa}(s)\,\tilde{\varphi}(z, s) + z\,\overline{\tilde{\varphi}'(z, s)} + \overline{\tilde{\psi}(z, s)} = \tilde{g_1}(z, s), \quad z \in C \qquad (5.11)$$

where it means

$$\tilde{g_1}(z, s) = \pounds[-2\,M_1\, g(z, t))] = -2\, s\, \tilde{k_1}(s)\, g(z, s)$$

The last equation is actually entirely analogue to the equation of the second boundary-value problem when given in classical state. It is obvious that, for real and large values of s, because of nonnegativeness of $k_1(t)$ and $k_2(_2)$, always is $\tilde{\varkappa}(_1) > 1$ too.

REFERENCES

[1] M. E. Gurtin, E. Sternberg: *On the linear Theory of Viscoelasticity*, Arch. Rational Mech. Anal., Vol. 11, pp. 291—355, 1962

[2] I. Babuška, K. Rektorys, F. Vyčichlo: *Mathematische Elastizit·tstheorie der ebenen Probleme*, Akademie Verlag Berlin 1960

[3] W. G. Knauss, I. J. Emri: *Nonlinear viscoelasticity based on free volume consideration*; Computers & Structures, Vol 13, pp 123—128, 1981

A CONTRIBUTION TO THE APPLICATIONS OF COMPLEX ANALYSIS IN SOLVING OF BOUNDARY VALUE PROBLEMS OF PLANE LINEAR VISCOELASTOSTATICS

In this paper the general equations of the plane linear viscoelastostatics are deduced. The elements of the stress and strain tensors are given by means of a pair of analytic functions, which depend on the space variable z and time t, the later appearing as a parameter. The definitions of two basic boundary-value problems are given as well as the proofs of the uniqueness theorems. The existence of the solutions is prooved by the transformation of the problem to the solution of Fredholm's integral equation of the second order. When considering the second boundary-value problem it is necessary to introduce additional conditions of the operator K. Two examples of such operators are given, where the shown theory is entirely fulfilled.

DOPRINOS K UPOTREBI KOMPLEKSNE ANALIZE PRI REŠAVANJU DOPRINOS K UPORABI KOMPLEKSNE ANALIZE PRI REŠAVANJU ROBNIH PROBLEMOV RAVNINSKE LINEARNE VISKOELASTOSTATIKE

V tem sestavku so izvedene osnovne formule v linearni viskoelastostatiki. Elementi deformacijskega in napetostnega tenzorja se izražajo z dvema analitičnima funkcijama krajevne spremenljivke z in časa t, ki nastopa kot parameter. Izvedeni sta definiciji osnovnih dveh robnih problemov in dokazana izreka o enoličnosti rešitve. Eksistenca rešitev je dokazana s prevedbo problema na reševanje Fredholmove integralske enačbe drugega reda. Pri drugem robnem problemu je treba za operator K upoštevati dodatne zahteve. Dana sta dva zgleda operatorjev, kjer izvedena teorija v celoti velja.

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