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AN EXTREMUM VARIATIONAL PRINCIPLE FOR FÖPPL-HENCKY EQUATION

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1. Introduction

In 1907 Föppl [1] derived equilibrium equations describing the deformation of a thin circular plate under uniformly distributed normal pressure. Later Hencky [2] reduced the problem to a single second order non-linear equation, known as the Föppl-Hencky equation

$$f'' + \frac{3}{x} f' + \frac{2}{f^2} = 0, \quad 0 < x < 1 \quad (.)' = \frac{d}{dx}(.), \tag{1}$$

with the following boundary conditions

$$f'(0) = 0 \quad f(1) = a \tag{2}$$

In (1) f is the dimension'ess radial stress and x is a dimensionless radial coordinate.

The boundary value problem (1), (2) has been the subject of many investigations. In [3] it was shown that (1), (2) has a solution if a (which depends on the external load) is larger than 0.6473. Later in [4] it was shown that (1), (2) has a *unique* positive solution belonging to $C^2(0,1) \cap C^0[0,1]$ for any value of $a > 0$. A version of this proof may be found in [5] also.

Our intention in this note is to construct an extremum variational principle for (1), (2) along the lines presented in [6]. Then we shall use this principle to obtain approximate solutions to (1), (2) and to estimate the L_2 norm of the error of such approximate solutions.

We note that the same problem was treated in [7] by the complementary variational principles. However, no estimate of the L_2 norm of the error is given in [7].

1. A priori Bounds on the Solution

For later use, in connection with error estimating procedure, we need some bounds on the solution f of (1), (2). Multiplying (1) with x^3 and integrating we get

$$f'(x) = -\frac{1}{x^3} \int_0^x \frac{2u^3}{f^2(u)} du \tag{3}$$

From (3) it follows that f is a decreasing function, so that

$$\sup f(x) = f(0) = b; \qquad x \in (0, 1) \tag{4}$$

$$\inf f(x) = f(1) = a; \qquad x \in (0, 1) \tag{5}$$

To estimate b we integrate (3) once more to get

$$b - a = 2 \int_0^1 \frac{1}{x^3} \left(\int_0^x \frac{u^3}{f^2(u)} du \right) dx \tag{6}$$

Using (4) in (6) we have

$$b - a \geqslant \frac{1}{4\, b^2} \tag{7}$$

Inequality (7) is easy to solve. The only real solution reads

$$b \geqslant \frac{a}{3} + \sqrt[3]{\frac{1}{8} + \frac{a^3}{27} + \sqrt{\frac{1}{64} + \frac{a^3}{108}} + \frac{1}{8} + \frac{a^3}{27} - \sqrt{\frac{1}{64} + \frac{a^3}{108}}} \tag{8}$$

giving a computable lower bound to b.

Next we note that the boundary condition $(2)_1$ implies that

$$f(x) = b + \frac{1}{2} f''(0)\, x^2 + 0\,(x^3). \tag{9}$$

Substituting (9) into (1) and taking a limit when $x \to 0$, we get

$$f''(0) = - \frac{1}{2\, b^2} < 0. \tag{10}$$

We claim that (10) implies

$$f''(x) \leqslant 0, \qquad x \in (0, 1). \tag{11}$$

We prove now (11). Suppose it is not satisfied. Let $x_2 \in (0, 1)$ be a value of x such that $f''(x_2) > 0$. Then since $f''(0) < 0$ there is a $x^* < x_2$ such that $f''(x^*) = 0$. At $x = x^*$ $f'(x)$ has a local minimum. Let $x_1 < x^*$ be such that $f'(x_1) = f'(x_2)$. By this choice of x_1, we have $f''(x_1) < 0$. Since for $x = x_1$ eq. (1) is satisfied we have

$$f''(x_1) + \frac{3}{x_1} f'(x_1) + \frac{2}{f^2(x_1)} = 0 \tag{12}$$

We claim that if (12) holds than eq. (1) for $x = x_2$ can not be satisfied. To see this, note that

$$f''(x_2) > f''(x_1); \quad \frac{3}{x_2} f'(x_2) > \frac{3}{x_1} f'(x_1); \quad \frac{2}{f^2(x_2)} > \frac{2}{f^2(x_1)} \tag{13}$$

The first two inequalities follow from the choice of x_2 and the third is a consequence of the fact that $f(x)$ is a decreasing function.

Thus $f''(x) \leqslant 0$ i.e. $f(x)$ is a concave function. Then

$$f(x) \geqslant b - (b - a)\, x \tag{14}$$

The inequality (14) could be used in (6) to get an upper bound on b. This would lead to a rather complicated algebraic relation. Therefore we use the weaker bound $f(x) > a$ in (6). Then the following upper bound on b is obtained

$$b \leqslant a + \frac{1}{4\, a^3}. \tag{15}$$

The estimate (15) is in agreement with the results of [4].

2. Variational Principle and Error Estimate

For the analysis that we intend to execute it proves more convenient to transform eq. (1) in the form

$$\frac{d}{dt}\left[t^m \frac{df}{dt} \right] + \frac{2}{f^2} = 0. \tag{16}$$

It is easy to see that (16) is obtained from (1) if we introduce a new independent variable t by the relation

$$t = x^4 \tag{17}$$

Then (1) becomes

$$(t^{3/2} \dot{f})^{\cdot} + \frac{1}{8\, f^2} = 0 \quad (\dot{\ }) = \frac{d}{dt}(\,.\,), \tag{18}$$

while the boundary conditions (2) transform to

$$(t^{3/4} \dot{f})_{t=0} = 0 \quad f(1) = a \tag{19}$$

The Lagrangian for (18) reads

$$L = \frac{t^{3/2} \dot{f}^2}{2} + \frac{1}{8\, f}. \tag{20}$$

From (20) we get the following expressions for the „generalized momenta" and Hamiltonion.

$$p = t^{3/2} \dot{f}; \quad H = \frac{p^2}{2\, t^{3/2}} - \frac{1}{8\, f}. \tag{21}$$

Following the procedure presented in [6] we conclude that the functional

$$I(F) = \int_0^1 \left[t^{3/2} \dot{F}^2 + \frac{1}{8\, F} - \frac{\sqrt{2}}{2} \sqrt{-(t^{3/2} \dot{F})^{\cdot}} \right] dt - aF(1), \tag{22}$$

is stationary on the exact solution f of (18), (19). Moreover, the value of the functional for $F = f$ is equal to zero. That is

$$\delta I(f, \varepsilon) = 0; \quad I(f) = 0; \quad \varepsilon = F - f \tag{23}$$

In (22) F is an admissible trial function, that is $F \in W$, where the set W is defined as

$$W = \{F : F \in C^2(0, 1), (\dot{F} t^{3/4})_{t=0} = 0; \ F(1) = a, \ (t^{3/2} \dot{F})^{\cdot} < 0\} \tag{24}$$

In our application F will be an approximate solution to (18), (19). Thus the error of this approximate solution is ε, and we proceed now to estimate it.

First note that $I(F)$ is Frechét differentiable, so that

$$I(F) = I(f) + \delta I(f, \varepsilon) + \frac{1}{2} \delta^2 I(\varphi, \varepsilon), \tag{25}$$

where $\delta^2 I(\varphi, \varepsilon)$ is the second variation of I calculated on the function φ given by

$$\varphi = f + \eta\varepsilon \qquad 0 < \eta < 1 \tag{26}$$

Calculating the second variation of (22) and observing (23) we get from the relation (25)

$$2 I(F) = \int_0^1 \left\{ 2\, t^{3/2}\, \dot{\varepsilon}^2 + \frac{1}{4\, \varphi^3}\, \varepsilon^2 + \frac{\sqrt{2}}{8} \frac{[(t^{3/2}\, \dot{\varepsilon})^{\cdot}]^2}{[-(t^{3/2} \dot{\varphi})^{\cdot}]^{3/2}} \right\} dt \tag{27}$$

Let A and C be a constants such that, for given F, we have

$$A \leqslant \inf_{t \in (0, 1)} \frac{\sqrt{2}}{8\, [-(t_3/^2\, \dot{\varphi})^{\cdot}]^{3/2}} \tag{28}$$

$$C \leqslant \inf_{t \in (0, 1)} \frac{1}{4\, \varphi_3} \tag{29}$$

Using (28) and (29) equ. (27) becomes

$$2 I(F) \geqslant \int_0^1 \{A\, [(t^{3/2}\, \dot{\varepsilon})^{\cdot}]^2 + 2\, t^{3/2}\, \dot{\varepsilon}^2 + C\, \varepsilon^2\}\, dt \tag{30}$$

To determine A and C we use the estimates (8) and (15). Thus

$$A = \min \left\{ \inf_{t \in = (0, 1)} \frac{\sqrt{2}}{8} \frac{1}{[-(t^{3/2}\, \dot{F})^{\cdot}]^{3/2}}; \ \frac{\sqrt{2}}{8} \frac{1}{8\, [a + \frac{1}{4} a^2]^2} \right\} \tag{31}$$

$$C = \min \inf_{t \in (0, 1)} f \left\{ \frac{1}{4\, F^3}; \quad \frac{1}{4\, (a + \frac{1}{4} a^2)^3} \right\} \tag{32}$$

Fllowing now the procedure described in [6] we get the following estimate

$$||\varepsilon||_{L_2} \leqslant \left\{ \frac{2\, I(F)}{S_1} \right\}^{1/2} \tag{33}$$

where $||\varepsilon||_{L_2} = \left(\int_0^1 \varepsilon^2 \, dt \right)^{1/2}$ is the L_2 norm of ε and S_1 is given by

$$S_1 = A\, \Lambda_1^2 + 2\, \Lambda_1 + C \tag{34}$$

In (34) we denoted by Λ_1 the smallest eigenvalue of the following spectral problem

$$(t^{3/2}\, \dot{\Phi}_n)^{\cdot} + \Lambda_n\, \Phi_n = 0; \quad (t^{3/4}\, \dot{\Phi}_n)_{t=0} = 0; \quad \Phi_n\,(1) = 0 \tag{35}$$

It is easy to see that Λ_n' s are determined from the equation

$$\mathcal{J}_1\,(\Lambda_n^{1/2}) = 0 \tag{36}$$

where $\mathcal{J}_1$ is the Bessel function of the first kind of the order one. Therefore, $\Lambda_1 = 14.75$.

3. Numerical Results

To illustrate the theory we shall find an approximate solution to (18), (19), for two specific values of the load parameter a, namely $a = 1$ and $a = 1.5$

The approximate solution is chosen in the form

$$F = C_1 - \frac{1}{4\, C_1^2}\, t^{1/2} - C_2\, t \tag{37}$$

where C_1 and C_2 are constants.

The function F with the form (37) satisfies (10) for any value of C_1 and C_2. Since F must satisfy (35) we choose

$$C_2 = C_1 - \frac{1}{4\, C_1^2} - a, \tag{38}$$

so that finally

$$F = C_1\,[1 - t] + \frac{1}{4\, C_1^2}\,[t - t^{1/2}] + at \tag{39}$$

The constant C_1 is determined by substituting (39) into (22) and minimizing with respect to C_1. The results together with the error estimates are given in the table hereunder

a	C_1	I		A		$\lVert\varepsilon\rVert_{L_2}$	
1	1.197	7.59	10^{-6}	0.01414	0.128	6.8	10^{-4}
1.5	1.602	6.32	10^{-7}	0.008513	0.05978	2.80	10^{-4}

From the results presented in the table we conclude that, although simple, the approximate solution (39) has remarkable accuracy. The accuracy of the solution could be improved if more elaborate trial functions (with more constants) are used.

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EIN EXTREMALES VARIATIONSPRINZIP FÜR FÖPPL-HENCKY DIFFERENTIALGLEICHUNG

Ein extremales Variatiosprinzip und die Feller Abschätzungen Methode für Föppl-Hencky Differentialgleichung werden angegeben. Dabei werden Approximation (Näherung) lösungen für zwei Last parameter angegeben.

EKSTREMALNI VARIJACIONI PRINCIP ZA FÖPPL-HENCKY-jevu DIFERENCIJALNU JEDNAČINU

U radu je formulisan ekstremalni varijacioni princip za Föppl-Hencky-jevu diferencijalnu jednačinu. Dat je i metod ocene greške. Sem toga, za dve vrednosti parametra opterećenja, određeno je i približno rešenje problema.

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