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THE STABILITY OF THE ROTATING MOTION OF THE ROTOR WITH VARYABLE MASS

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1. Introduction

In this paper the stability of the rotating motion of the rotor with varyable mass with zero deflection of the mass center is analysed.

The rotor is assumed as a shaft-disc system. The shaft has a constant circular cross — section. The shaft is supported in two rigid bearings. In the middle of the shaft a disc is settled. The mass of the disc is varying during the time. The mathematical model of this rotor is given in the paper [1] in the form:

$$m(\tau)\frac{d^2 z}{dT^2} = Z \tag{1}$$

where:

$m(\tau)$ — the varyable dimensionless mass

$\tau = \mu T$ — slow varyable time

μ — a small positive parameter

Z — the complex force

$z = x + iy$ — complex deflection of mass center of the rotor

$i = \sqrt{-1}$ — imaginary unit

x, y — coordinates of the mass center of the rotor

T — dimensionless time

For the case when the elastic force in the shaft is linear, it is

$$m(\tau)\frac{d^2 z}{dT^2} + z = Z \tag{2}$$

Let us analyse the stabilty of the trivial solution of the differential equation (1) or (2), i.e. the stability of the rotating motion of the rotor with zero deflection of the mass center. Let us apply the methods of Lyapunov [2].

2. Rotor with the clearance in the bearing

In the system rotor — bearing there is a nonlinearity which is the result of the celarance in the bearing. The clearance has a great influence on the motion of the rotor. Because of that there is tendency of decreasing of the clearance in the bearing. Hence, in the bearing there is always a clearance which make possible the axial motion and deflection of the rotor in accordance to the housing and the varying of the dimensions due to temperature varying. The radial force in the bearing due to the clearance is a function of the deflection of the mass center and after paper [3] is

$$F_R = \mu K (|z| - \delta) \quad \text{for } |z| \geqslant \delta \tag{3}$$

where:

δ — the radial clearance between the external ring and housing

$|z| = (x^2 + y^2)^{1/2}$ — radial deflection of the center of the rotor

$K = \text{const}$ — the coefficient of rigidity

and

$$Z_R = -K\mu\left(1 - \frac{\delta}{|z|}\right) z \tag{4}$$

Substituting the eq. (4) in the eq. (2) it is

$$\frac{d^2 z}{dT^2} m(\tau) + z = -K\mu\left(1 - \frac{\delta}{|z|}\right) z \tag{5}$$

or

$$\begin{aligned} \frac{d^2 x}{dT^2} m(\tau) + x &= -K\mu\left(1 - \frac{\delta}{(x^2 + y^2)^{1/2})}\right) x \\ \frac{d^2 y}{dT^2} m(\tau) + y &= -K\mu\left(1 - \frac{\delta}{(x^2 + y^2)^{1/2})}\right) y \end{aligned} \tag{6}$$

Let us assume a Lyapunov function in the form

$$V = m(\tau)\left[\left(\frac{dx}{dT}\right)^2 + \left(\frac{dy}{dT}\right)^2\right] + 2\left(\sqrt{x^2 + y^2}\sqrt{\frac{\mu K + 1}{2}} - \frac{\delta\mu K\sqrt{2}}{2\sqrt{K\mu + 1}}\right)^2 + m(\tau)(x^2 + y^2) \tag{7}$$

The function (7) is positive for all values of x, y, dx/dT and dy/dT

$$V > 0 \tag{8}$$

for all $T \geqslant T_0$ and where

$m(\tau) \geqslant 1$ is a continuous function.

The first derivative of the function (7) along the integrating line of differential equations (6) is

$$\frac{dV}{dT} = \mu\frac{dm}{d\tau}\left[\left(\frac{dx}{dT}\right)^2 + \left(\frac{dy}{dT}\right)^2\right] + 2m(\tau)\left(x\frac{dx}{dT} + y\frac{dy}{dT}\right) + \mu\frac{dm}{d\tau}(x^2 + y^2) \tag{9}$$

where

$\frac{dm}{d\tau}$ is a positive function for all values of τ

Let us assume a function W

$$W = \mu \left[\left(\frac{dx}{dT}\right)^2 + \left(\frac{dy}{dT}\right)^2\right] + 2\left(x\frac{dx}{dT} + y\frac{dy}{dT}\right) + \mu (x^2 + y^2) \tag{10}$$

The function (10) can be separated into two quadratic forms

$$\begin{aligned} &\mu \left(\frac{dx}{dT}\right)^2 + 2x\frac{dx}{dT} + \mu x^2 \\ &\mu \left(\frac{dy}{dT}\right)^2 + 2y\frac{dy}{dT} + \mu y^2 \end{aligned} \tag{11}$$

The function W is a continuous function and positive definite, for

$$0 < \mu < 1 \tag{12}$$

Hence,

$$\begin{aligned} &\frac{dV}{dT} \geqslant W > 0 \quad \text{for} \quad x, y, \frac{dx}{dT}, \frac{dy}{dT} \neq 0 \text{ and } m(\tau) \geqslant 1 \\ &\frac{dV}{dT} = W = 0 \quad \text{for} \quad x, y, \frac{dx}{dT}, \frac{dy}{dT} = 0 \end{aligned} \tag{13}$$

and the function dV/dT is a positive definite function in the district $V > 0$.

So, it can be concluded that the function (7) satisfies the conditions of the Lyapunov's theorem of unstability (see [2]), and the rotating motion of this rotor with zero deflection of the mass center is unstable.

3. Rotor with the nonlinear elastic force

The elastic force in the shaft can be nonlinear and after the paper [4] is

$$Z_e = -z - \mu b z |z|^2 \tag{14}$$

where

b is a positive coefficient of the nonlinear elastic force. Mathematical model for this kind of rotor is

$$m(\tau)\frac{d^2 z}{dT^2} + z = -\mu K\left(1 - \frac{\delta}{|z|}\right) z - \mu b z |z|^2 \tag{15}$$

or

$$\begin{aligned} &m(\tau)\frac{d^2 x}{dT^2} + x = -\mu K\left(1 - \frac{\delta}{(x^2 + y^2)^{1/2}}\right) x - \mu b x (x^2 + y^2) \\ &m(\tau)\frac{d^2 y}{dT^2} + y = -\mu K\left(1 - \frac{\delta}{(x^2 + y^2)^{1/2}}\right) y - \mu b y (x^2 + y^2) \end{aligned} \tag{16}$$

Let us assume the Lyapunov function in the form

$$V = m(\tau)\left[\left(\frac{dx}{dT}\right)^2 + \left(\frac{dy}{dT}\right)^2\right] + 2\left(\sqrt{x^2+y^2}\sqrt{\frac{\mu K+1}{2}} - \frac{a\,\delta\mu K\sqrt{2}}{2\sqrt{\mu K+1}}\right)^2 +$$

$$+ m(\tau)(x^2+y^2) + \mu b\,\frac{(x^2+y^2)^2}{2} \tag{17}$$

which sastisfies for all $T \geqslant T_0$ and for all small absolute values of x, y, dx/dT and dy/dT the relacion (8).

The first derivative of the function (17) along the integrating line of differential equations (16) is the eq. (9). It satisfies the conditions (13). So, it can be concluded that the roating motion of the rotor with zero deflection of the mass center is unstable.

4. Rotor with damping force

The damping force is proportional to velocity of mass center

$$Z_D = -D\,\frac{dz}{dT} \tag{18}$$

where

D is coefficient of damping

Substituting the eq. (18) into eq. (2) and using the eq. (4) it is

$$m(\tau)\,\frac{d^2 z}{dT^2} + z + D\,\frac{dz}{dT} = -\mu K\left(1 - \frac{\delta}{|z|}\right)z \tag{19}$$

or

$$\begin{aligned} m(\tau)\,\frac{d^2 x}{dT^2} + x + D\,\frac{dx}{dT} &= -\mu K\left(1 - \frac{\delta}{(x^2+y^2)^{1/2}}\right)x \\ m(\tau)\,\frac{d^2 y}{dT^2} + y + D\,\frac{dy}{dT} &= -\mu K\left(1 - \frac{\delta}{(x^2+y^2)^{1/2}}\right)y \end{aligned} \tag{20}$$

Let us suppose the Lyapunov function in the form

$$V = m(\tau)\left[\left(\frac{dx}{dT}\right)^2 + \left(\frac{dy}{dT}\right)^2\right] + 2\left(\sqrt{x^2+y^2}\sqrt{\frac{\mu K+1}{2}} - \frac{\delta\mu K\sqrt{2}}{2\sqrt{\mu K+1}}\right)^2 \tag{21}$$

The function (21) is a positive-definite function which satisfies the relations:

$$\begin{aligned} &V \geqslant W_1 > 0 \qquad \text{for} \quad x, y, \frac{dx}{dT}, \frac{dy}{dT} \neq 0, \quad m(\tau) \geqslant 1 \\ &V = W_1 = 0 \qquad \text{for} \quad x, y, \frac{dx}{dT}, \frac{dy}{dT} = 0 \end{aligned} \tag{22}$$

where

$$W_1 = \left(\frac{dx}{dT}\right)^2 + \left(\frac{dy}{dT}\right)^2 + 2\left(\sqrt{x^2+y^2}\sqrt{\frac{\mu K+1}{2}} - \frac{\mu\delta K\sqrt{2}}{2\sqrt{\mu K+1}}\right)^2 \tag{23}$$

is a positive-definite function.

The first derivative of the function (21) along the integrating line of differential equations (20) is

$$\frac{dV}{dT} = \left(\mu \frac{dm}{d\tau} - 2D\right)\left[\left(\frac{dx}{dT}\right)^2 + \left(\frac{dy}{dT}\right)^2\right] \tag{24}$$

The function (24) is a negative function for

$$0 < \mu \frac{dm}{d\tau} < 2D \tag{25}$$

Using the first Lyapunov theorem for stability [2] it can be concluded that the rotating motion with zero deflection of mass center is stable for $0 < \mu \frac{dm}{d\tau} < 2D$.

4. Rotor with internal and structural damping

The internal and structural damping is after the paper [5]

$$Zs = -k\left(\frac{dz}{dT} - i\,\Omega\, z\right) \tag{26}$$

where

k — the coefficient of structural and internal damping

Ω — the angular velocity of the rotor

The mathematical model of the rotor is

$$m(\tau)\frac{d^2 z}{dT^2} + z + k\left(\frac{dz}{dT} - i\,\Omega\, z\right) = -\mu K\left(1 - \frac{\delta}{|z|}\right) z \tag{27}$$

or

$$\begin{aligned} m(\tau)\frac{d^2 x}{dT^2} + x + k\left(\frac{dx}{dT} + y\,\Omega\right) &= -\mu K x + \mu K \frac{x\,\delta}{(x^2+y^2)^{1/2}} \\ m(\tau)\frac{d^2 y}{dT^2} + y + k\left(\frac{dy}{dT} - x\,\Omega\right) &= -\mu K y + \mu K \frac{y\,\delta}{(x^2+y^2)^{1/2}} \end{aligned} \tag{28}$$

Let us suppose the Lyapunov function in the form

$$\begin{aligned} V = m(\tau)\left[\left(\frac{dx}{dT}\right)^2 + \left(\frac{dy}{dT}\right)^2\right] + 2\left(\sqrt{x^2+y^2}\,\sqrt{\frac{\mu K+1}{2}} - \frac{\delta\,\mu K\sqrt{2}}{2\sqrt{\mu K+1}}\right)^2 + \\ + 3\,m(\tau)\left(x\frac{dx}{dT} + y\frac{dy}{dT}\right) + 3\,m(\tau)\left(1 + \frac{1}{\mu} + K\right)(x^2+y^2) \end{aligned} \tag{29}$$

The function (29) satisfies the relations

$$\begin{aligned} V \geqslant W_2 > 0 \quad &\text{for} \quad x,\ y,\ \frac{dx}{dT},\ \frac{dy}{dT} \neq 0, \quad m(\tau) \geqslant 1 \\ V = W_2 = 0 \quad &\text{for} \quad x,\ y,\ \frac{dx}{dT},\ \frac{dy}{dT} = 0 \end{aligned} \tag{30}$$

and is a positive-definite function when

$$W_2 = \left(\frac{dx}{dT}\right)^2 + \left(\frac{dy}{dT}\right)^2 + 2\left(\sqrt{x^2+y^2}\sqrt{\frac{\mu K+1}{2}} - \frac{K\sqrt{2}}{2\sqrt{\mu K+1}}\right)^2 + $$
$$+ 3\left(x\frac{dx}{dT} + y\frac{dy}{dT}\right) + 3\left(1+\frac{1}{\mu}+K\right)(x^2+y^2) \tag{31}$$

is a positive definite function.

The first derivative of the function (29) along the differential equations (28) is

$$\frac{dV}{dT} = \left[\left(\frac{dx}{dT}\right)^2 + \left(\frac{dy}{dT}\right)^2\right]\left(\mu\frac{dm}{d\tau} - 2k + 3m\right) - 2Kk\Omega\left(y\frac{dx}{dT} - x\frac{dy}{dT}\right) +$$
$$+ \left(x\frac{dx}{dT} + y\frac{dy}{dT}\right)\left[3\mu\frac{dm}{d\tau} - 3k + 6m\left(1+\frac{1}{\mu}+K\right)\right] +$$
$$+ (x^2+y^2)\left[3\mu\frac{dm}{d\tau}\left(1+\frac{1}{\mu}+K\right) - 3 - 3\mu K\right] + 3\mu K\delta(x^2+y^2)^{1/2} \tag{32}$$

Let us assume a function W_3 in the form

$$W_3 = \left[\left(\frac{dx}{dT}\right)^2 + \left(\frac{dy}{dT}\right)^2\right](\mu - 2k + 3) - 2k\Omega\left(\frac{dx}{dT}y - x\frac{dy}{dT}\right) + \left(\mathrm{x}\frac{dx}{dT} + y\frac{dy}{dT}\right)$$
$$\left[3\mu - 3K + 6\left(1+\frac{1}{\mu}+K\right)\right] + 3(x^2+y^2)\mu + 3\mu K(x^2+y^2)^{1/2} \tag{33}$$

In the eq. (33) it is possible to separate two systems of square forms

$$\begin{aligned}&\left(\frac{dx}{dT}\right)^2\left(\mu - k + 2 + \frac{2}{\mu} + 2K\right) + 3x\frac{dx}{dT}\left(\mu - k + 2 + \frac{2}{\mu} + 2K\right) + \mu x^2\\ &\left(\frac{dy}{dT}\right)^2\left(\mu - k + 2 + \frac{2}{\mu} + 2K\right) + 3y\frac{dy}{dT}\left(\mu - k + 2 + \frac{2}{\mu} + 2K\right) + \mu y^2\end{aligned} \tag{34}$$

and

$$\begin{aligned}&\left(\frac{dx}{dT}\right)^2\left(1 - k - \frac{2}{\mu} - 2K\right) - 2k\Omega y\frac{dx}{dT} + 2\mu y^2\\ &\left(\frac{dy}{dT}\right)^2\left(1 - k - \frac{2}{\mu} - 2K\right) + 2k\Omega x\frac{dy}{dT} + 2\mu x^2\end{aligned} \tag{35}$$

The forms are positive definite for

$$\begin{aligned}&0 < \mu - k + 2 + \frac{2}{\mu} + 2K < \frac{4}{9}\mu\\ &1 - k - 2K - \frac{2}{\mu} < 0\\ &0 < \Omega < \frac{1}{k}[2(1 - k - {}^2 - 2K)]^{1/2}\end{aligned} \tag{36}$$

So, the function (33) is positive definite.

Hence,

$$\frac{dV}{dT} \leqslant - W_3 < 0 \quad \text{for } x,\ y,\ \frac{dx}{dT},\ \frac{dy}{dT} \neq 0 \text{ and } \frac{dm}{d\tau} \geqslant 1$$

$$\frac{dV}{dT} = W_3 = 0 \quad \text{for } x,\ y,\ \frac{dx}{dT},\ \frac{dy}{dT} = 0 \tag{37}$$

Therefore dV/dT is negative definite function.

Applying the second Lyapunov theorem [2] it can be concluded that the rotating, with zero deflection of mass center, is asymptotically stable for the conditions (36).

5. Conclusion

It can be concluded:

1. The rotating motion of the rotor with zero deflection of mass center, when the radial clearance in the bearing exsists, is unstable.

2. When beside the clearance in the bearing there exsists the nonlinear elastic force in the shaft, the rotating motion of the rotor with zero deflection of mass center is unstable, too.

3. The external damping and the internal and structural damping stabilisy the rotating motion for this kind of rotors. For the case when the structural and internal damping force acts the rotating is asimptotically stable.

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DIE ROTATIONSTABILITÄT DES ROTORS MIT VARIABLER MASSE

In dieser Arbeit wird die Rotationstabilität des Rotors mit variabler Masse analysiert. Das Model des Rotors besteht aus einer masseloen Welle und einer Scheibe mit veränderlicher Masse. Der Rotor ist in zwei Lager gelagert. Dort gibt es Lagerluft in Äussere Lagerring und Lagerhäuse. Die Stabilität ist mit der Lyapunov Methoden bestimmt. Die Schlussfolgerung ist das die Rotation nichtstabil ist wenn die an der Welle eingreifende Kraft linear oder nichtlinear ist. Die Dämpfungen stabilisieren die Rotation.

STABILNOST OBRTNOG KRETANJA ROTORA SA PROMENLJIVOM MASOM

U ovom radu analizirana je stabilnost obrtanja rotora s promenljivom masom pri nultom pomeranju centra mase. Rotor je razmatran kao vratilo-disk sistem. Rotor je postavljen u ležajeve koji imaju zazor između spoljašnjeg prstena ležaja i kućišta. Za ispitivanje stabilnosti obrtanja primenjene su metode Ljapunova. Zaključeno je da je za slučaj da je sila elastičnosti u vratilu rotora bilo linearna ili nelinearna obrtno kretanje nestabilno. Ako na rotor deluje spoljašnja sila viskoznog prigušenja (proporcionalna brzini obrtanja centra rotora) kretanje je stabilno. Ako na rotor deluje sila strukturnog i unutrašnjeg prigušenja obrtno kretanje je asimptotski stabilno.

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