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LAMINAR VISCOUS NONCOMPRESSIBLE FLUID FLOWS IN INTERNAL CHANNELS WITH SLOPES, STEPS AND OBSTACLES

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1. Introduction

In this paper the singular boundary integral technique is applied to solve some recirculation phenomena in laminar motion of isochoric viscous fluids.

The vorticity-velocity and vorticity-velocity-pressure formulations are employed to solve steady fluid motion, governed by Navier-Stokes equations. With the vorticity variable the computation procedure is partitioned into its kinetic and kinematic parts. The first one is described by transport equation for vorticity and the second by an integral equation which can be recognised as Biot--Savart law for a bounded region.

Linear and quadratic continuous boundary elements and internal cells are used to discretise the integral equations. All boundary and domain integrations are performed analitically to increase the accuracy and speedeness of the computation. Some typical examples of recirculation flows are evaluated and compared against other published results.

2. Governing Equation

Three dimensional steady motion of an isochoric viscous fluid is governed by the Navier-Stokes equation and by mass conservation law

$$\nu \, \Delta \, \underline{v} = \underline{\nabla} \, p/\rho + (\underline{v} \, \underline{\nabla}) \, \underline{v} \tag{1}$$

$$\underline{\nabla} \, \underline{v} = 0 \tag{2}$$

which are expressed with the primitive variables velocity v and pressure p. The material properties such as viscosity ν and density ρ are assumed to be constant.

Introducing the vorticity vector

$$\underline{\omega} = \underline{\nabla} \times \underline{v} \tag{3}$$

the fluid flow problem is divided into its kinematic and kinetic parts. The kinematic relation between $\underline{v}$ and $\underline{\omega}$ is described by equations (2) and (3). The kinetic aspect of the flow is given by the momentum transport equation (1).

Using some vector identities, the equation (1) can be rewritten as

$$\nu \Delta \underline{v} = \nabla h - \underline{v} \times \underline{\omega} = - \nu \nabla \times \underline{\omega} \tag{4}$$

h being the total head defined by

$$h = p/\rho + v^2/2 \tag{5}$$

and $v^2 = \underline{v}\,\underline{v}$, or in the form without pressure

$$\nu \Delta \underline{\omega} = - (\underline{\omega} \nabla)\, \underline{v} + (\underline{v} \nabla)\, \underline{\omega} \tag{6}$$

The equation (4) defines velocity-vorticity-pressure ($v - \omega - p$) formulation, and equation (6) defines velocity-vorticity ($v - \omega$) formulation respectively. Both equations are nonlinear due to the kinematic relation between v and ω.

For the plane problems the vorticity vector has only one component perpendicular to the plane of the flow and it can be treated as a scalar

$$\omega = \partial v_y/\partial x - \partial v_x/\partial y \tag{7}$$

Due to the orthogonality of $\underline{v}$ and $\underline{\omega}$, term $(\underline{\omega} \nabla)\, \underline{v}$ vanishes and the vorticity transport equation (6) reduces to the transport equation for the scalar quantity ω

$$\nu \Delta \omega = (\underline{v} \nabla)\, \omega \tag{8}$$

3. Integral representation for kinetic part (v—ω) formulation

Inhomogeneous elliptic equation for the scalar function $\omega(s)$

$$\Delta \omega(s) + b(s) = 0 \quad \text{in} \quad \Omega \tag{9}$$

with Dirichlet's and Neumann's boundary conditions

$$\omega(S) = \overline{\omega}(S) \quad \text{on} \quad \Gamma_1, \qquad \partial \omega(S)/\partial n(S) = \overline{q}(S) \quad \text{on} \quad \Gamma_2 \tag{10}$$

can be transformed by Green's theorems or by weighted residual statement (1) into the following integral form

$$c(\xi)\, \omega(\xi) + \int_\Gamma \omega(S)\, q^*(\xi, S)\, d\Gamma(S) = \int_\Gamma q(S)\, u^*(\xi, S)\, d\Gamma(S) + \\ + \int_\Omega b(s)\, u^*(\xi, s)\, d\Omega(s) \tag{11}$$

where $u^*(\xi, s)$ is fundamental solution and its normal derivative $q^*(\xi, s) = \partial u^*(\xi, s)/\partial n(s)$ with ξ is source point and S or s are field points on the boundary or in the domain, respectively. For the two dimensional problems, which are subject of the paper, the fundamental solution is given by

$$u^*(\xi, S) = \frac{1}{2\pi} \ln \frac{1}{r(\xi, S)} \tag{12}$$

$$q^*(\xi, S) = \frac{1}{2\pi} d(\xi, S)/r^2(\xi, S) \tag{13}$$

with $d(\xi, s) = (x_i(\xi) - x_i(s))\, n_i(s)$, $i = 1, 2$ and $n_i(s)$ are direction cosines of the normal at the field point S.

The coefficient $c(\xi)$ has values 0.0 (if ξ is outside Ω), 1.0 (if ξ is inside Ω), 0.5 (if ξ lies on a smooth Γ) or $\beta/2\pi$ (when ξ lies on a non-smooth Γ, where β is internal angle of the boundary at ξ point).

Now, one can easily write the boundary integral equation for the steady diffusion — convective equation for vorticity [7] by taking the body force term $b(s)$ in (9) equivalent to the convective term in (8), ie.

$$c(\xi)\,\omega(\xi) + \int_\Gamma \omega\, q^*\, d\Gamma = \int_\Gamma q\, u^*\, d\Gamma - \frac{1}{\nu}\int_\Omega (\underline{v}\,\underline{\nabla})\,\omega\, u^*\, d\Omega \tag{14}$$

The domain integral contains derivatives of vorticity. It can be transformed by Gaussian divergence theorem, and the integral formulation (14) becomes for a solenoidal velocity field

$$c(\xi)\,\omega(\xi) + \int_\Gamma \omega\, q^*\, d\Gamma = \int_\Gamma q\, u^*\, d\Gamma - \frac{1}{\nu}\int_\Gamma \omega\, v_n\, u^*\, d\Gamma + \frac{1}{\nu}\int_\Omega \omega\, \underline{v}\, \underline{\nabla}\, u^*\, d\Omega \tag{15}$$

where the fluxes $\underline{\nabla}\, u^*(\xi, s) = \partial u^*(\xi, s)/\partial x_i(s) = \overset{*}{q_i}(\xi, s)$ and the normal velocity component $v_n(s) = v_i(s)\, n_i(s)$, $i = 1, 2$.

Notice that the formulation (15) describes the transport process for vorticity in an integral form. The first two boundary integrals represent the diffusion from the boundary, and the third integral describes convection from the boundary. The domain integral is due to the convection effect.

4. Integral representation for kinetic part (v—ω—p) formulation

One can derive the boundary integral statement for the momentum equation (4) by using Green's theorem for vectors [13] yielding

$$\underline{\omega}(\xi) + \int_\Gamma (\omega\, \underline{n})\, \underline{\nabla}\, u^*\, d\Gamma = \int_\Gamma (\underline{\omega} \times \underline{n}) \times \underline{\nabla}\, u^*\, d\Gamma - \frac{1}{\nu}\int_\Omega \underline{\nabla} h \times \underline{\nabla}\, u^*\, d\Omega + \frac{1}{\nu}\int_\Omega (\underline{v} \times \underline{\omega}) \times \underline{\nabla}\, u^*\, d\Omega \tag{16}$$

Accounting fov vectors identities

$$\underline{\nabla} \times (h \underline{\nabla} u^*) = h \underline{\nabla} \times \underline{\nabla} u^* - \underline{\nabla} u^* \times \underline{\nabla} h, \qquad \underline{\nabla} \times \underline{\nabla} u^* = 0 \tag{17}$$

and applying Gauss theorem to the domain integral in (16) containing the total head

$$\int_\Omega \underline{\nabla} h \times \underline{\nabla} u^* \, d\Omega = \int_\Omega \underline{\nabla} \times (h \underline{\nabla} u^*) \, d\Omega = - \int_\Gamma h \underline{\nabla} u^* \times \underline{n} \, d\Gamma \tag{18}$$

the following boundary integral equation can be formulated [14]

$$\underline{\omega}(\xi) + \int_\Gamma (\underline{\omega}\, \underline{n}) \underline{\nabla} u^* \, d\Gamma = \int_\Gamma (\underline{\omega} \times \underline{n}) \times \underline{\nabla} u^* \, d\Gamma + \frac{1}{\nu} \int_\Gamma h \underline{\nabla} u^* \times \underline{n} \, d\Gamma +$$

$$+ \frac{1}{\nu} \int_\Omega (\underline{v} \times \underline{\omega}) \times \nabla u^* \, d\Omega \tag{19}$$

Integral equation (19) can be also given as

$$c(\xi)\, \omega(\xi) + \int_\Gamma (\nabla \underline{u^*}\, \underline{n})\, \underline{\omega} \, d\Gamma = \int_\Gamma (\nabla u^* \times \underline{n}) \times \underline{\omega} \, d\Gamma +$$

$$+ \frac{1}{\nu} \int_\Gamma h \nabla u^* \times \underline{n} \, d\Gamma + \frac{1}{\nu} \int_\Omega (\underline{v} \times \underline{\omega}) \times \underline{\nabla} u^* \, d\Omega \tag{20}$$

written now generally for boundary and domain points. Mathematical description of the plane flows is much simpler, since $\underline{\omega} = \omega$ and $(\nabla u^* \times \underline{n}) \times \underline{\omega} = 0$. Thus, the vector equation (20) reduces to scalar equation

$$c(\xi)\, \omega(\xi) + \int_\Gamma \omega q^* \, d\Gamma = - \frac{1}{\nu} \int_\Gamma h q_t^* \, d\Gamma + \frac{1}{\nu} \int_\Omega \omega \underline{v} \underline{\nabla} u^* \, d\Omega \tag{21}$$

5. Integral representation for the kinematic part

Integral equation for kinematic part can be derived by introducing the vector potential $\underline{\Psi}$ of the solenoidol velocity field

$$\underline{v} = \underline{\nabla} \times \underline{\Psi}, \qquad \underline{\nabla}\, \underline{\Psi} = 0 \tag{22}$$

or in the form of vector Poisson's equation for $\underline{\Psi}$

$$\Delta \underline{\Psi} = - \underline{\omega} \tag{23}$$

With the Green's theorem for vectors [11]

$$\int_\Omega (\underline{E} \Delta \underline{F} - \underline{F} \Delta \underline{E}) \, d\Omega = \int_\Gamma (\underline{E} \times (\underline{\nabla} \times \underline{F}) + \underline{E} (\underline{\nabla} \underline{F}) -$$

$$- \underline{F} \times (\underline{\nabla} \times \underline{E}) - \underline{F} (\underline{\nabla} \underline{E}))\, \underline{n} \, d\Gamma \tag{24}$$

where $\underline{E} = \Psi$ and $\underline{F}$ is the fundamental solution to vector Laplace equation $\Delta \underline{F} = 0$ [14], given by

$$\underline{v}^* (\xi, s) = \underline{\nabla} (u^* (\xi, s)) \times \underline{e} = \underline{\nabla} \times (u^* (\xi, s) \, \underline{e}) \tag{25}$$

with the properties

$$\underline{\nabla} \, \underline{v}^* = 0, \qquad \underline{\nabla} \times \underline{v}^* = \underline{\nabla} (\underline{e} \, \underline{\nabla} \, u^*), \qquad \xi \neq s \tag{26}$$

where $\underline{e}$ is constant unit vector, the following integral equation can be formulated [7]

$$c(\xi) \, \underline{v}(\xi) + \int_\Gamma (\underline{v} \, \underline{n}) \, \underline{\nabla} \, u^* \, d\Gamma = \int_\Gamma (\underline{v} \times \underline{n}) \times \underline{\nabla} \, u^* \, d\Gamma + \int_\Omega \underline{\omega} \times \underline{\nabla} \, u^* \, d\Omega \tag{27}$$

Relation (27) expresses the kinematics of the flow in the integral form. The domain integral presents the contribution of the vorticity field to the velocity field, and the boundary integrals express the effect of potential flow in the domain.

The boundary integral statement (27) can be formulated using some vector identities to

$$c(\xi) \, \underline{v}(\xi) + \int_\Gamma (\underline{\nabla} \, u^* \, \underline{n}) \, \underline{v} \, d\Gamma = \int_\Gamma (\underline{\nabla} \, u^* \times \underline{n}) \times \underline{v} \, d\Gamma + \int_\Omega \underline{\omega} \times \underline{\nabla} \, u^* \, d\Omega \tag{28}$$

It is easy to include the free-stream velocity $\underline{v}_\infty$ on Γ_∞ into formulation in the case of external flow, yielding the following equation

$$c(\xi) \, \underline{v}(\xi) + \int_\Gamma (\underline{\nabla} \, u^* \, \underline{n}) \, \underline{v} \, d\Gamma = \int_\Gamma (\underline{\nabla} \, u^* \times \underline{n}) \times \underline{v} \, d\Gamma + \int_\Omega \underline{\omega} \times \underline{\nabla} \, u^* \, d\Omega + \underline{v}_\infty \tag{29}$$

By imposing a no slip and no-permeability conditions on a non-rotating surface Γ the equation (29) for external flow simplifies to

$$c(\xi) \, \underline{v}(\xi) = \int_\Omega \underline{\omega} \times \underline{\nabla} \, u^* \, d\Omega + \underline{v}_\infty \tag{30}$$

The equation (30) is the Biot-Savart law of induced velocities. Thus, the equation (28) can be recognised as an extension of the Biot-Savart law for internal flows bounded by Γ.

The derived integral equations are completely equivalent to the continuity equation along with the vorticity expression. The boundary integrals contain the velocity boundary conditions. Equations permit the explicit computation of velocity $\underline{v}$ in domain using domain values for vorticity $\underline{\omega}$ and the known velocity boundary conditions.

For the plane problems the vector equation simplifies to two scalar equations, i.e. for internal flows

$$c(\xi)\, v_x(\xi) + \int_\Gamma v_x\, q^*\, d\,\Gamma = \int_\Gamma v_y\, q_t^*\, d\,\Gamma - \int_\Omega \omega\, q_y^*\, d\,\Omega \tag{31}$$

$$c(\xi)\, v_y(\xi) + \int_\Gamma v_y\, q^*\, d\,\Gamma = \int_\Omega \omega\, q_x^*\, d\,\Omega - \int_\Gamma v_x\, q_t^*\, d\,\Gamma \tag{32}$$

6. Boundary element discretisation

Let us divide the boundary Γ into E boundary elements with N_e boundary nodes, and the domain Ω into C internal cells with N_c internal points. The functions ω, q, h, v_x, v_y and the products ωv_x and ωv_y are approximated within each boundary element or internal cell according to the space shape functions, $\varnothing(\eta)$, and $\varphi(\eta_1, \eta_2)$, and the nodal vectors, where one can write for the boundary

$$\begin{aligned} \omega(\eta) &= \underline{\varnothing}^T \underline{W}^n & \omega\, v_x(\eta) &= \underline{\varnothing}^T \underline{W}\, \underline{V}_x^n \\ q(\eta) &= \underline{\varnothing}^T\, \underline{Q}^n & \omega\, v_y(\eta) &= \underline{\varnothing}^T \underline{W}\, \underline{V}_y^n \\ h(\eta) &= \underline{\varnothing}^T\, \underline{h}^n & v_x(\eta) &= \underline{\varnothing}^T\, \underline{V}_x^n \\ & & v_y(\eta) &= \underline{\varnothing}^T\, \underline{V}_y^n \end{aligned} \tag{33}$$

and in the domain

$$\begin{aligned} \omega(\eta_1, \eta_2) &= \underline{\varphi}^T \underline{W}^n & \omega\, v_x(\eta_1, \eta_2) &= \underline{\varphi}^T \underline{W}\, \underline{V}_x^n \\ & & \omega\, v_y(\eta_1, \eta_2) &= \underline{\varphi}^T \underline{W}\, \underline{V}_y^n \end{aligned} \tag{34}$$

One can now write the discretised form of (15), (21) and (31), (32) as

$$\begin{aligned} c(\xi)\, \omega(\xi) &+ \sum\left(\int \underline{\varnothing}^T q^*\, d\,\Gamma_e\right)\underline{W}^n = \sum\left(\int \underline{\varnothing}^T u^*\, d\,\Gamma_e\right)\underline{Q}^n - \\ &- \frac{1}{\nu}\left(\sum\left(\int \underline{\varnothing}^T n_x\, u^*\, d\,\Gamma_e\right)\underline{W}\,\underline{V}_x^n + \sum\left(\int \underline{\varnothing}^T n_y\, u^*\, d\,\Gamma_e\right)\underline{W}\,\underline{V}_y^n - \right. \\ &\left. - \sum\left(\int \underline{\varphi}^T q_x^*\, d\,\Omega_c\right)\underline{W}\,\underline{V}_x^n - \sum\left(\int \underline{\varphi}^T q_y^*\, d\,\Omega_c\right)\underline{W}\,\underline{V}_y^n\right) \end{aligned} \tag{35}$$

$$c(\xi)\,\omega(\xi) + \sum\left(\int \underline{\O}^T q^* \, d\,\Gamma_e\right)\underline{W}^n = -\frac{1}{\nu}\sum\left(\int \underline{\O}^T q_t^* \, d\,\Gamma_e\right)\underline{h}^n +$$

$$+\frac{1}{\nu}\left(\sum\left(\int \underline{\varphi}^T q_x^* \, d\,\Omega_c\right)\underline{W}\,\underline{V}_x^n + \sum\left(\int \underline{\varphi}^T q_y^* \, d\,\Omega_c\right)\underline{W}\,\underline{V}_y^n\right) \quad (36)$$

$$c(\xi)\,v_x(\xi) + \sum\left(\int \underline{\O}^T q^* \, d\,\Gamma_e\right)\underline{V}_x^n = \sum\left(\int \underline{\O}^T q_t^* \, d\,\Gamma_e\right)\underline{V}_y^n -$$

$$-\sum\left(\int \underline{\O}^T q_y^* \, d\,\Omega_c\right)\underline{\omega}^n \quad (37)$$

$$c(\xi)\,v_y(\xi) + \sum\left(\int \underline{\O}^T q^* \, d\,\Gamma_e\right)\underline{V}_y^n = \sum\left(\int \underline{\varphi}^T q_x^* \, d\,\Omega_c\right)\underline{W}^n -$$

$$-\sum\left(\int \underline{\O}^T q_t^* \, d\,\Gamma_e\right)\underline{V}_x^n \quad (38)$$

With the integrals, which are functions of the geometry only,

$$h_e^n = \int \O^n q^* \, d\,\Gamma_e \qquad c_{xe}^n = \int \O^n n_x u^* \, d\,\Gamma_e \qquad d_{xc}^n = \int \varphi^n q_x^* \, d\,\Omega_c$$

$$g_e^n = \int \O^n u^* \, d\,\Gamma_e \qquad c_{ye}^n = \int \O^n n_y u^* \, d\,\Gamma_e \qquad d_{yc}^n = \int \varphi^n q_y^* \, d\,\Omega_c \quad (39)$$

$$h_{et}^n = \int \O^n q_t^* \, d\,\Gamma_e$$

the discretised formulation (35) can be replaced by the following expression

$$c(\xi)\,\omega(\xi) + \sum \underline{h}^T \underline{W}^n = \sum \underline{g}^T \underline{Q}^n - \frac{1}{\nu}\left(\sum \underline{c}_x^T \underline{W}\,\underline{V}_x^n +\right.$$

$$\left.+\sum \underline{c}_y^T \underline{W}\,\underline{V}_y^n - \sum \underline{d}_x^T \underline{W}\,\underline{V}_x^n - \sum \underline{d}_y^T \underline{W}\,\underline{V}_y^n\right) \quad (40)$$

Applying equation (40) to all boundary nodes, the following matrix equation can be obtained

$$\underline{c}(\xi)\underline{W}(\xi) + \underline{\hat{H}}\underline{W} = \underline{G}\,\underline{Q} - \frac{1}{\nu}(\underline{C}_x\underline{W}\,\underline{V}_x + \underline{C}_y\underline{W}\,\underline{V}_y -$$

$$-\underline{D}_x\underline{W}\,\underline{V}_x - \underline{D}_y\underline{W}\,\underline{V}_y) \quad (41)$$

or for unknown vorticity fluxes on the boundary

$$\underline{G}\,\underline{Q} = \underline{H}\,\underline{W} + \frac{1}{\nu}(\underline{C}_x\underline{W}\,\underline{V}_x + \underline{C}_y\underline{W}\,\underline{V}_y - \underline{D}_x\underline{W}\,\underline{V}_x - \underline{D}_y\underline{W}\,\underline{V}_y) \quad (42)$$

Vorticity values in the domain can be computed in an explicit point-by-point manner from (40) for $c(\xi) = 1$

$$\underline{W}(\xi) = -\underline{\hat{H}}\underline{W} + \underline{G}\,\underline{Q} - \frac{1}{\nu}\left(\underline{C}_x\underline{W}\,\underline{V}_x + \underline{C}_y\underline{W}\,\underline{V}_y - \underline{D}_x\underline{W}\,\underline{V}_x - \underline{D}_y\underline{W}\,\underline{V}_y\right) \tag{43}$$

Following the same procedure one can write for the equation (36)

$$c(\xi)\,W(\xi) + \sum \underline{h}^T\underline{W}^n = -\frac{1}{\nu}\sum \underline{h}_t^T\,\underline{h}^n + \frac{1}{\nu}\left(\sum \underline{d}_x^T\,\underline{W}\,\underline{V}_x^n + \sum \underline{d}_y^T\,\underline{W}\,\underline{V}_y^n\right) \tag{44}$$

and formulated for all boundary nodes

$$\underline{c}(\xi)\underline{W}(\xi) + \underline{\hat{H}}\underline{W} = -\frac{1}{\nu}\underline{H}_t\,\underline{h} + \frac{1}{\nu}\left(\underline{D}_x\underline{W}\,\underline{V}_x + \underline{D}_y\underline{W}\,\underline{V}_y\right) \tag{45}$$

or in the form

$$\underline{H}\underline{W} = -\frac{1}{\nu}\underline{H}_t\,\underline{h} + \frac{1}{\nu}\left(\underline{D}_x\underline{W}\,\underline{V}_x + \underline{D}_y\underline{W}\,\underline{V}_y\right) \tag{46}$$

which can be reordered for unknown total head on the boundary

$$\underline{H}_t\,\underline{h} = \underline{D}_x\underline{W}\,\underline{V}_x + \underline{D}_y\underline{W}\,\underline{V}_y - \nu\,\underline{H}\underline{W} \tag{47}$$

One can easy prove using Stokes theorem that the boundary integral containing total head in (21) vanishes for its constant value

$$\int_\Gamma \underline{\nabla}\,u^*\,\underline{t}\,d\,\Gamma = \int_\Omega \underline{\nabla}\times\underline{\nabla}\,u^*\,d\,\Omega = 0 \tag{48}$$

This means that the homogeneous set of equations, corresponding to the (47) has a non-trivial solution. As a consequence at most $N_e - 1$ equations are independent of each other. In the computation procedure one can assigne an arbitrary value of total head at any boundary node [12].

At the end one can express the kinematic equations (37) and (38) as

$$c(\xi)\,v_x(\xi) + \sum \underline{h}^T\,\underline{V}_x^n = \sum \underline{h}_t^T\,\underline{V}_y^n - \sum \underline{d}_y^T\,\underline{W}^n \tag{49}$$

$$c(\xi)\,v_y(\xi) + \sum \underline{h}^T\,\underline{V}_y^n = \sum \underline{d}_x^T\,\underline{W}^n - \sum \underline{h}_t^T\,\underline{V}_x^n \tag{50}$$

Applying equations (49) and (50) to all boundary nodes, the following $(2\,N_e)$ matrix equation system can be obtained

$$\underline{c}(\xi)\,\underline{V}_x(\xi) + \underline{\hat{H}}\,\underline{V}_x = \underline{H}_t\,\underline{V}_y - \underline{D}_y\underline{W} \tag{51}$$

$$\underline{c}(\xi)\,\underline{V}_y(\xi) + \underline{\hat{H}}\,\underline{V}_y = \underline{D}_x\underline{W} - \underline{H}_t\,\underline{V}_x \tag{52}$$

Using the boundary velocity conditions, which are prescribed by tangential or normal components

$$v_t(\xi) = v_x(\xi)\, t_x(\xi) + v_y(\xi)\, t_y(\xi) \tag{53}$$

$$v_n(\xi) = v_x(\xi)\, n_x(\xi) + v_y(\xi)\, n_y(\xi) \tag{54}$$

for all boundary nodes ($\xi = 1, N_e$) in equations (51) and (52) the system of equations for boundary vorticity can be derived [8]. Velocity components in the domain are given explicitely for $c(\xi) = 1$ as

$$\underline{V}_x(\xi) = -\underline{\hat{H}}\,\underline{V}_x + \underline{H}_t\,\underline{V}_y - \underline{D}_y \underline{W} \tag{55}$$

$$\underline{V}_y(\xi) = -\underline{\hat{H}}\,\underline{V}_y - \underline{H}_t\,\underline{V}_x + \underline{D}_x \underline{W} \tag{56}$$

7. Solution procedure

With the velocity boundary conditions (53) or (54) the equations (51) and (52) can be reformulated as a set of algebraic equations for unknown boundary vorticity values. One can follow with explicit computation of the velocites in the domain using (55) and (56). Evaluation of the unknown vorticity boundary fluxes follows from (42). As a final step in iterative procedure we compute the vorticity values in the domain using (43). When the computation of vorticity and velocity fields have ended the code computes boundary pressure values from (47).

The computation scheme consists from the following steps:

— kinematic part

a) starting with initial vorticity values ($\omega^\circ = 0$) in N_c internal points $\underline{\omega}^i$

b) computing of boundary vorticity values from (51), (52) and (53) $\underline{\omega}^{i+1}$

c) computing of velocities in N_c internal points $\underline{v}^{i+1}$ from (55) and (56) in an explicit manner

— kinetic part

d) with known values of $\underline{\omega}^{i+1}$, $\underline{\omega}^i$, $\underline{v}^{i+1}$ evaluation of vorticity fluxes q^{i+1} on the boundary from (42)

e) explicit computation of internal vorticity $\underline{\omega}^{i+1}$ from (43).

Due to the existing nonlinearity, a point under-relaxation iterative procedure has to be employed for the purpose of obtaining a convergent solution [12]

$$\omega^{i+1} = k\, \omega^{i+1} + (1 - k)\, \omega^i \tag{57}$$

f) computation of the boundary pressure values from (47) after the iterative procedure ended.

8. Examples

Flow over a backward facing step

This problem presents the flow within a channel which contains a down stream facing step expansion. Due to this abrupt change in domain geometry the recirculation zone exists even at very low Reynolds numbers. Some experimental data and numerical computed results exist [4], [9].

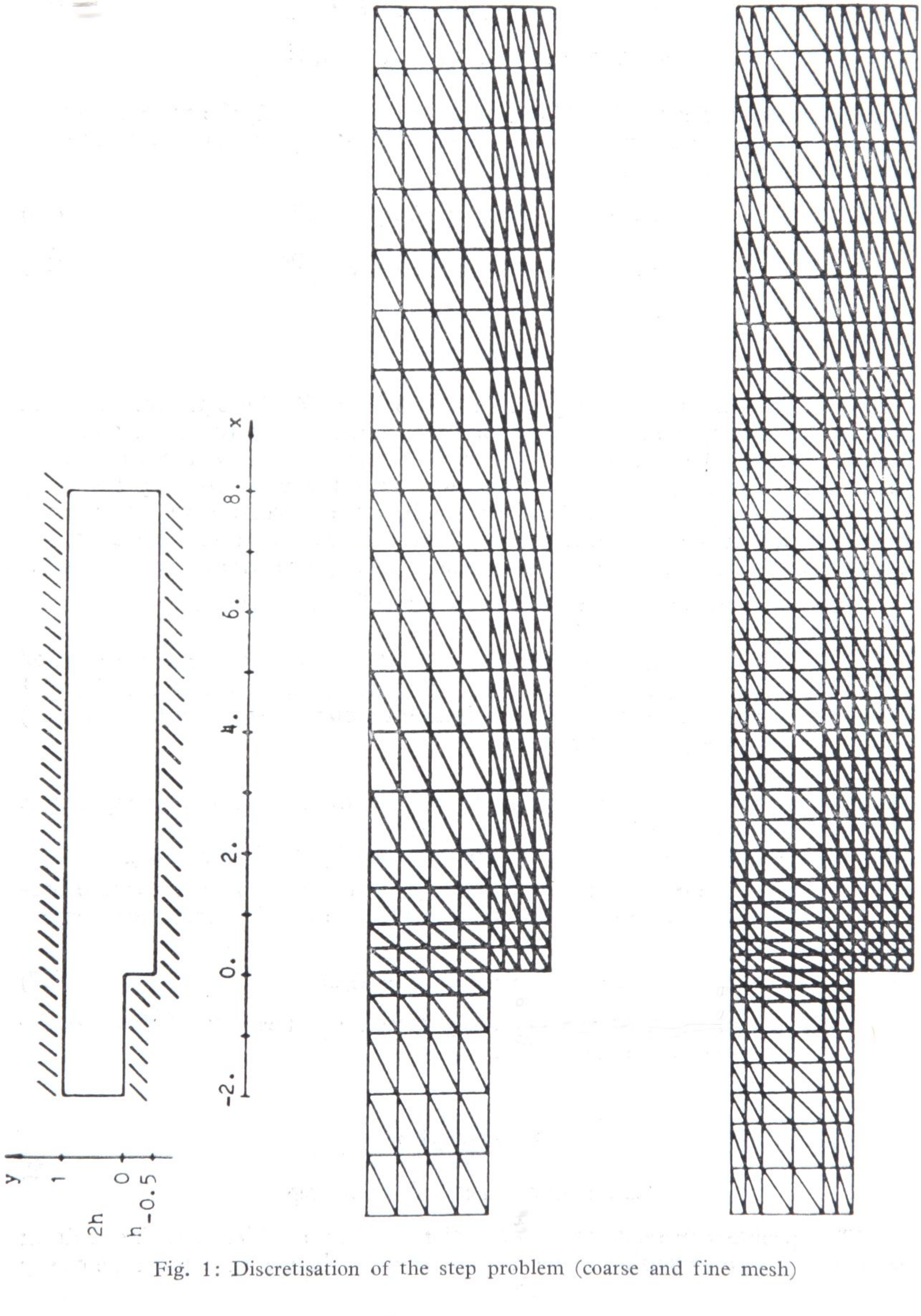

Fig. 1: Discretisation of the step problem (coarse and fine mesh)

The channel contains an inlet and outlet boundary with two no-slip boundaries between them, Figure 1. Inlet and outlet boundary conditions are prescribed according to exact parabolic variation for fully developed laminar flow in a channel. Inlet boundary velocities are defined by $v_x = 4y(1-y)$ with the mean velocity $\bar{v}_x = 2/3$.

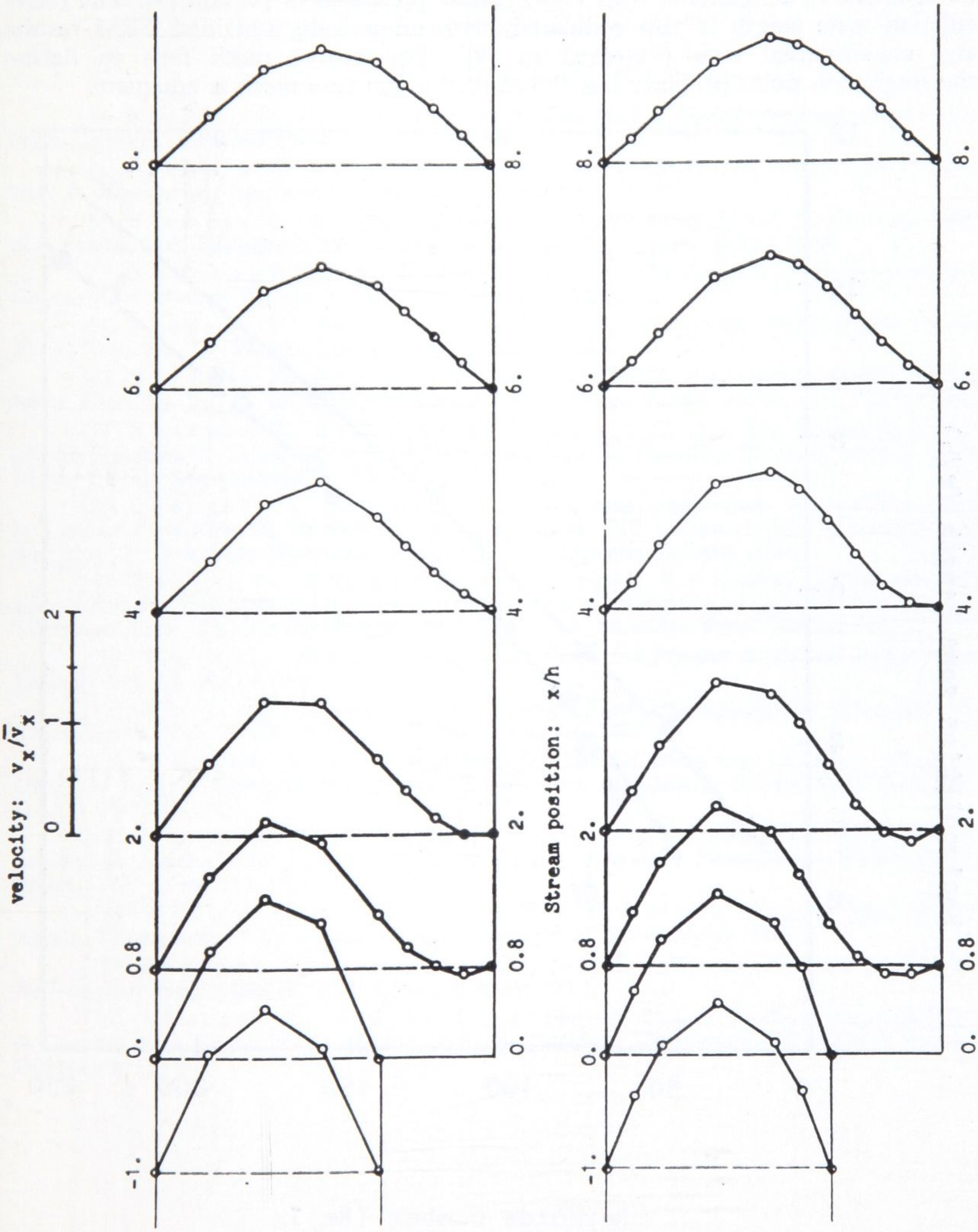

Fig. 2. Velocity profiles ($v_x/\bar{v}_x$) at Re_h = 73. for coarse and fine mesh.

Two meshes have been used to solve this problem at $Re_h = \bar{v}h/\nu = 73$ defined for mean velocity and step height. Figure 1 presents two boundary elements meshes, the coarse and the fine one respectively which are both coarser comparing to the FEM meshes presented in [4].

Figure 2 presents the velocity profiles computed by BEM which seem to be adequate in comparison with FEM results presented in [4] and [9]. The recirculacion zone length is also compared, with numerically obtained FEM results and experimental data presented in [9]. The coarse mesh fails to define the stagnation point properly but the result for the fine mesh is adequate.

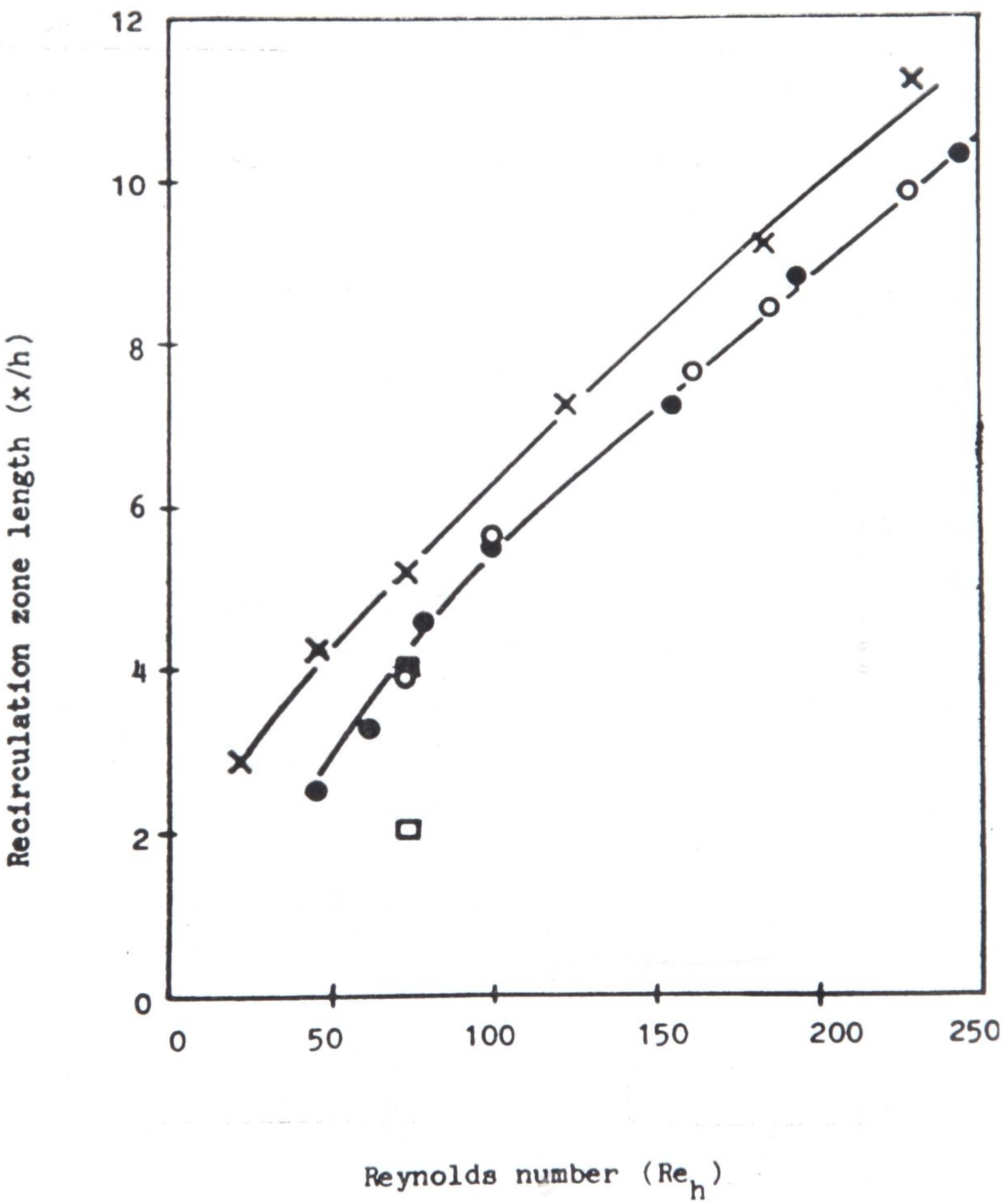

Fig. 3: Recirculation zone length-comparison with finite elements and experimental results

Other cases

Flows of laminar viscous isochoric fluids in a converging-diverging (ramping) channel, and in a channel with a cross-cylindrical obstacle, has been also analysed recently by the present Authors, but these results are presented elsewhere [15] and [16].

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ПРОТЕКАНИЕ ЛАМИНАРНОЙ ВИСКОЗНОЙ НЕСЖИМАЕМОЙ ЖИДКОСТИ В ВНУТРЕННЉХ КАНАЛАХ С ПЕРЕМЕННЉМХ ПЕРЕСЕЧЕНИЕМ (РАМПЉ, ЛЕСТНИЦЉ и ПЕРЕПЯТСТВИЯ)

В статъе использован сингулярный краевой метод для разрешения некоторых возвраженых проточных прсблем движения ламинарной изохраной вискозной жидкости. Использованы выхрево-скоростная и вихрево-скоростно-авленческая формулировки для разрешения стабилиризованного движения, для которого действительно уравнение Навиер-Стокеса. Введением вихревости вњчислительный процес разделяется на кинетическую и кинематическую части. Первая часть изображается транстпортным уравнением для вихревости, а вторая интегральным уравнением, которые отличается как Биот-Саваратов закон для ограниченной области.

Оцениваются и некоторње примеры возвратных проточнњх течений, которые сравниваљтся с, в других местах, опубликованными результатами.

PRETOK LAMINARNE VISKOZNE NESTISLJIVE TEKOČINE V NOTRANJIH KANALIH S SPREMENLJIVIM PREREZOM (RAMPE, STOPNICE IN OVIRE)

V članku je uporabljena singularna robna integralska metoda za reševanje nekaterih povratnih pretočnih problemov gibanja laminarne izohorne viskozne tekočine.

Uporabljeni sta vrtinčno-hitrostna in vrtinčno-hitrostno-tlačna formulacija za razrešitev ustaljenega gibanja tekočine, za katerega velja Navier-Stokes enačba. Z vpeljavo vrtinčnosti se računski proces razdeli na kinetski in kinematski del. Prvi je opisan s transportno enačbo za vrtinčnost, drugi pa z integralsko enačbo, ki jo razpoznamo kot Biot-Savartov zakon za ograjeno področje.

Uporabljeni so linearni in kvadratni konformni robni elementi in notranje celice za diskretizacijo integralskih enačb. Vse robne in področne integracije so izvedene analitično, da se poveča natančnost in hitrost računalniškega izračuna. Izvrednoteni so nekateri primeri povratnih pretočnih tokov, ki jih primerjamo z drugje objavljenimi rezultati.

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