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OPTIMAL MOTION CONTROL OF NONHOLONOMIC MECHANICAL SYSTEMS

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In this article has been considered the connection between maximum principles and the integral variations principles of classical mechanics, given in the works [1] and [2] for holonomis systems.

The connection of differential equations of motion and the deviation equations by the function $\varkappa$, which consists of the first variable $\delta \mathcal{H}$ of Hamilton's function H and the first variable δA of the work performed by control forces and reaction of nonholonomic connection, shows that the minimum of Hamilton's action leads to maximum or the supremum of the function $\mathcal{H}$, respectively.

Motion sys.em of N material points in the potential field acted on by control forces are observed. If the motion system is limited by k bilateral nonholonomic connections, it may be expressed in the following way:

$$b_{\mu\alpha}\, \dot{q}^{\alpha} = 0 \tag{1}$$

where: $b_{\mu\alpha} = b_{\mu\alpha}(q^1, \ldots, q^n)$, $(\mu = 1, \ldots k)$, $(\alpha = 1, \ldots n)$ $k < n$

and it can be described by $2n = 6N - 2k$ Hamilton's differential motion equations:

$$\dot{p}_{\alpha} = -\frac{\partial H}{\partial q^{\alpha}} + U_{\alpha} + \sum_{\mu=1}^{k} \Lambda_{\mu}\, b_{\mu\alpha} \tag{2}$$

$$\dot{q}^{\alpha} = \frac{\partial H}{\partial p_{\alpha}}.$$

Their variation equations are:

$$\dot{\eta}^{\alpha} = -\frac{\partial^2 H}{\varrho\, q^{\alpha}\, \partial q^{\beta}}\, \xi^{\beta} - \frac{\partial^2 H}{\partial q^{\alpha}\, \partial p^{\beta}}\, \eta_{\beta} + \frac{\partial u_{\alpha}}{\partial q^{\beta}}\, \xi^{\beta} + \sum_{\mu=1}^{k} \Lambda_{\mu} \frac{\partial b_{\mu\alpha}}{\partial q^{\beta}}\, \xi^{\beta}$$

$$\dot{\xi}^{\alpha} = \frac{\partial^2 H}{\partial p_{\alpha}\, \partial p_{\beta}}\, \eta_{\beta} + \frac{\partial^2 H}{\partial p_{\alpha}\, \partial q^{\beta}}\, \xi^{\beta}. \tag{3}$$

Where: $H = p_\alpha \dot{q}^\alpha - L = H(p, q, t)$ — stands for Hamilton's function,
$q^\alpha \in R_n$ — stands for Langrange's generalized independent, coordinates,
$p_\alpha \in \Phi_{2n} (R_n \subset \Phi_{2n})$ — stands for generalized impulses,
$\eta_\alpha = \delta p_\alpha;\ \xi^\alpha = \delta q^\alpha$ — stands for variables p and q, the repeated indices mean addirion.

Motion is controlled by generalized control forces $U_\alpha(q, u, t)$ depedning on the coordinates $q = q\{q^1, \ldots q^n\}$ and on control $u = u\{u^1, \ldots u^n\}$.

The task is to transform the systems from the beginning motion $A(t_0, q(t_0), p(t_0))$ to the finishing state $B(t_1 > t_0, q(t_1), p(t_1))$ so that in this case the action:

$$S = \int_{t_0}^{t_1} p_\alpha dq^\alpha - (H + U_\alpha + \sum_{\mu=1}^{k} \rho_\mu b_{\mu\alpha} \dot{q}^\alpha) dt, \tag{5}$$

is the smallest at optimal control $u_0 = \text{opt}\, u$. The condition of existing of the function minimum S is that the second variable $\delta^2 S$ of the increment $\Delta S = \delta S + \frac{1}{2} \delta^2 S + \ldots$, is positive. The first variable is:

$$\begin{aligned} \mathcal{J} = \delta S = \int_{t_0}^{t_1} & \delta p_\alpha dq^\alpha - dp_\alpha \delta q^\alpha - \left[\frac{\partial H}{\partial p_\alpha} \delta p_\alpha + \frac{\partial H}{\partial p^\alpha} \delta q^\alpha - U_\alpha \delta p^\alpha + \right. \\ & \left. + \sum_{\mu=1}^{k} \rho_\mu \left(\frac{\partial b_{\mu\alpha}}{\partial q^\alpha} \dot{q}^\alpha - \frac{d}{dt} b_{\mu\alpha} \right) \delta q^\alpha - \sum_{\mu=1}^{k} \dot{\rho}_\mu b_{\mu\alpha} \delta q^\alpha \right] dt. \end{aligned} \tag{6}$$

It is well known that Hamilton's principle formulated in this way would be stationary and does not satisfy the nonholonomic connections, from the first variation must follow the motion equation (2) and the conditions intruded by the nonholonomic connections, what is proved in the work [3], and that are:

$$\sum_{\mu=1}^{k} \Lambda_\mu \left(\frac{\partial b_{\mu\alpha}}{\partial q^\beta} \dot{q}^\beta - \frac{d}{dt} b_{\mu\alpha} \right) = 0. \tag{7}$$

Let us take that Langrage's connestion multipliers are $\dot{\rho}_\mu = \Lambda_\mu$ then the first variation may be written:

$$\begin{aligned} \mathcal{J} = \int_{t_0}^{t_1} & \delta p_\alpha dq^\alpha - dp_\alpha \delta q^\alpha - \left[\frac{\partial H}{\partial p_\alpha} \delta p_\alpha + \frac{\partial H}{\partial q^\alpha} \delta q^\alpha - U_\alpha \delta q^\alpha - \right. \\ & \left. - \sum_{\mu=1}^{k} \Lambda_\mu b_{\mu\alpha} \delta q^\alpha \right] dt. \end{aligned} \tag{8}$$

If the conjugated function in the brackets of the expression (8) is written as $\mathcal{H}$, which connects Hamilton's function and work, it conjugates the differential motion equation and the differential equation of deviation. It is in this case formed as the sum of first variations of Hamilton's functions and the work which is performed by control forces and connection reactions on the possible variations of nonholonomic system.

$$\mathcal{H} = \delta H + \delta A = \frac{\partial H}{\partial p_\alpha} \delta p_\alpha + \frac{\partial H}{\partial q^\alpha} \delta q^\alpha - U_\alpha \delta q^\alpha - \sum_{\mu=1}^{k} \Lambda_\mu b_{\mu\alpha} \delta q^\alpha. \quad (9)$$

In the work [2] this function is taken for the required holonomic system. That is, the function $\mathcal{H} = \mathcal{H}(q, p, \xi, \eta, \Lambda, u, t)$ depend. on phase variables $q = (q^1, \ldots, q^n)$ and $p = (p_1, \ldots, p_n)$ on deviation $\xi = (\xi^1, \ldots, \xi^n)$ and $\eta = (\eta_1, \ldots, \eta_n)$ on control $u = (u^1, \ldots, u^n)$ on Langrage's multipliers of connections $\Lambda = (\Lambda_1, \ldots, \Lambda_n)$ and the time t.

Finally, the first variation according to the signs (4) and (9) can be written:

$$\mathcal{J} = \int_{t_0}^{t_1} (\dot{q}^\alpha \eta_\alpha - \dot{p}_\alpha \xi^\alpha - \mathcal{H})\, dt. \quad (10)$$

The second variation is:

$$\delta \mathcal{J} = \delta^2 S = \int_{t_0}^{t_1} \Big[\Big[\dot{\xi}^\alpha - \frac{\partial \mathcal{H}}{\partial p_\alpha} \delta p_\alpha + \left(-\dot{p}_\alpha - \frac{\partial \mathcal{H}}{\partial \xi^\alpha}\right) \delta \xi^\alpha + \left(\dot{q}^\alpha - \frac{\partial \mathcal{H}}{\partial \eta_\alpha}\right) \delta \eta_\alpha + \\ + \left(-\dot{\eta}_\alpha - \frac{\partial \mathcal{H}}{\partial q^\alpha}\right) \delta q^\alpha - \frac{\partial \mathcal{H}}{\partial u^\alpha} \delta u^\alpha \Big] dt > 0 \quad (11)$$

Since the differential equations (2) and (3) regarding the function are equivalent to the system of differential equation:

$$\dot{q}^\alpha = \frac{\partial \mathcal{H}}{\partial \eta_\alpha} \qquad \dot{\xi}^\alpha = \frac{\partial \mathcal{H}}{\partial p_\alpha}$$
$$\dot{q}_\alpha = \frac{\partial \mathcal{H}}{\partial \xi^\alpha} \qquad \dot{\eta}_\alpha = - \frac{\partial \mathcal{H}}{\partial q^\alpha}, \quad (12)$$

it follows

$$\int_{t_0}^{t_1} \frac{\partial \mathcal{H}}{\partial u^\alpha} \delta u^\alpha\, dt < 0. \quad (13)$$

This represents the condition of maximum function for the control u_0.

$$\mathcal{H}(p, q, \xi, \eta, u_0, \Lambda, t) = \max_{u \in U} (p, q, \xi, \eta, u, \Lambda, t) \quad (14)$$

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ОПТИМАЛЬНОЕ УПРАВЛЕНИЕ ДВИЖЕНИЕМ НЕГОЛОНОМНЫХ МЕХАНИЧЕСКИХ СИСТЕМ

В данной работе доказывается связь между принципом максимума теории оптимального управления и интегральным вариационным принципом Гамилтона классической механики. Сопряжением дифференциальных уравнений движения и уравнения смещения при помощи функции $\mathcal{H}$ показано, что максимум действия Гамилтона для неголономных механических систем сводится к максимум функции.

OPTIMALNO UPRAVLJANJE KRETANJEM NEHOLONOMNIH MEHANIČKIH SISTEMA

Dokazuje se veza između principa maksimuma teorije optimalnog upravljanja sa integralnim varijacionim Hamiltonovim principom u klasičnoj mehanici. Sprezanjem diferencijalnih jednačina kretanja i diferencijalnih jednačina poremećaja pomoću funkcije $\mathcal{H}$ pokazano je da se maksimum dejstva po Hamiltonu svodi na maksimum te funkcije.

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