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TRACTION FREE ELLIPTICAL HOLE UNDER UNIFORM STRESS AT INFINITY

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1. Introduction

In infinitesimal plane strain the problem of an unstressed elliptical hole in an infinite elastic isotropic plate under a uniform tension at infinity has been solved by Muskhelishvili [1] and England [2].

In finite plane strain the solution of an unstressed elliptical hole in an infinite plate of a compressible istropic hyperelastic solid under a uniform stress at infinity has been given in [3].

Here the infinitesimal plane analysis of the displacement and stress field of an infinite elastic isotropic plate with an unstressed elliptical hole under a uniform stress at infinity is approached through the complex stress function. This problem is solved with prescribed values of the complex stress function on the inner and outer edges, through a conformal mapping of the elliptical domain onto a circle with explicit transformations and power series expansions of the complex potential.

2. Mathematical Formulation and Solution

The complex variable formulation of the static two-dimensional infinitesimal strain is governed by the equations [2]:

$$2\,\mu\, u = k\,\Omega\,(Z) - Z\,\overline{\Omega}'\,(\bar{Z}) - \overline{\omega}\,(\bar{Z}), \tag{1}$$

$$\sigma_{11} + \sigma_{12} = 2\,[\Omega'\,(Z) + \overline{\Omega}'\,(\bar{Z})], \tag{2}$$

$$\sigma_{22} - \sigma_{11} - 2\,i\,\sigma_{12} = 2\,[Z\,\overline{\Omega}''\,(\bar{Z}) + \overline{\omega}'\,(\bar{Z})], \tag{3}$$

where $\sigma_{\alpha\beta}$ (α, $\beta = 1, 2$) are the Cauchy stress components, $u = u_1 + iu_2$ the complex displacement, $\Omega\,(Z)$ and $\omega\,(Z)$ analytic functions of $Z = X_1 + iX_2$ to be determined by boundary conditions, $\overline{\Omega}\,(\bar{Z})$ and $\overline{\omega}\,(\bar{Z})$ the conjugate complex of these functions, $k = (\lambda + 3\,\mu)/(\lambda + \mu)$ a constant and λ, μ the Lamé constants.

In the absence of the body forces the equilibrium equations are $\sigma_{\alpha\beta,\beta} = 0$, where a comma followed by a suffix β denotes $\partial/\partial X_\beta$. These equations imply the existence of the complex stress function $\Phi = \Phi_1 + i\,\Phi_2$ such that

$$\sigma_{11} = -\,\Phi_{1,2},\ \sigma_{22} = \Phi_{2,1},\ \sigma_{12} = \sigma_{21} = \Phi_{1,1} = -\,\Phi_{2,2}.$$

It is shown that the complex stress function Φ is given by

$$\Phi = i\,[\Omega(Z) + Z\,\overline{\Omega}'(\bar{Z}) + \bar{\omega}(\bar{Z})]. \tag{4}$$

We introduce the conformal transformation [3]

$$Z = \zeta + m a^2 \zeta^{-1} \tag{5}$$

which transforms the region of the Z-plane outside the ellipse into the region of the ζ-plane outside the circle $\zeta\bar{\zeta} = a^2$, where $0 < m < 1$ and a is a positive constant.

The boundary condition and the condition at infinity of the problem are

$$\Phi = 0 \quad \text{at} \quad \zeta\bar{\zeta} = a^2 \tag{6}$$

$$-i\,\Phi \sim \frac{1}{2}(\hat{\sigma}_1 + \hat{\sigma}_2)\,Z - \frac{1}{2}(\bar{\sigma}_1 - \bar{\sigma}_2)\,e^{2i\hat{\theta}}\,\bar{Z} \quad \text{as} \quad |Z| \to \infty \tag{7}$$

where, $\hat{\sigma}_1$, $\hat{\sigma}_2$ are the principal Cauchy stresses at infinity and $\hat{\theta}$ is the inclination to the X_1 — direction of the principal axis of stress associated with the principal Cauchy stress $\hat{\sigma}_1$.

We define the following transformations [2]

$$\Omega(Z) = \Omega(\zeta + m a^2 \zeta^{-1}) \equiv \Omega(\zeta), \qquad \omega(Z) = \omega(\zeta + m a^2 \zeta^{-1}) \equiv \omega(\zeta)$$

$$\Omega'(Z) = \frac{\zeta^2\,\Omega'(\zeta)}{\zeta^2 - m a^2}, \qquad \omega'(Z) = \frac{\zeta^2\,\omega'(\zeta)}{\zeta^2 - m a^2}. \tag{8}$$

The complex stress function Φ now becomes

$$-i\,\Phi = \Omega(\zeta) + \frac{\bar{\zeta}^2(\zeta^2 + m a^2)}{\zeta(\bar{\zeta}^2 - m a^2)}\,\Omega'(\zeta) + \omega(\zeta). \tag{9}$$

The condition (6) provides the functional relation between the two complex potentials $\omega(\zeta)$ and $\Omega(\zeta)$

$$\bar{\omega}(\bar{\zeta}) = -\,\Omega(a^2\bar{\zeta}^{-1}) - \frac{\bar{\zeta}(m\,\bar{\zeta}^2 + a^2)}{\bar{\zeta}^2 - m a^2}\,\Omega'(\zeta). \tag{10}$$

From equations (9) and (10) we get

$$-i\,\Phi = \Omega(\zeta) - \Omega(a^2\bar{\zeta}^{-1}) + \frac{\bar{\zeta}(\zeta - m\,\bar{\zeta})(\zeta\bar{\zeta} - a^2)}{\zeta(\bar{\zeta}^2 - m a^2)}\,\overline{\Omega}'(\zeta). \tag{11}$$

In view of (5), we conclude from (7) that Φ must be of degree one in ζ and $\bar{\zeta}$ as $|\zeta| \to \infty$. Thus, by inspection of (11), it follows that, if the analytic function $\Omega(\zeta)$ is represented as a Laurent series the powers $\zeta^2, \zeta^3, \ldots$ and $\zeta^{-2}, \zeta^{-3}, \ldots$ must be excluded, leaving

$$\Omega(\zeta) = c\,\zeta + d\,\zeta^{-1}, \tag{12}$$

where c and d are in general complex constant which must be determined. Equations (11) and (12) give

$$-i\,\Phi \sim (c + \bar{c})\,\zeta - (m\bar{c} + a^{-2} d)\,\bar{\zeta} \quad \text{as} \quad |\zeta| \to \infty. \tag{13}$$

The comparison between (7) and (13) yields

$$\left.\begin{aligned} c+\bar{c} &= \frac{1}{2}(\hat{\sigma}_1+\hat{\sigma}_2), \\ m\bar{c}+a^{-2}\,d &= \frac{1}{2}(\hat{\sigma}_1-\hat{\sigma}_2)\,e^{2i\hat{\theta}}, \end{aligned}\right\} \tag{14}$$

using $\zeta = Z(1 - ma^2 Z^{-2} + O(Z^{-4}))$ as $|Z| \to \infty$.
The solution of the system (14) in c and d is

$$\begin{aligned} c &= \frac{1}{4}(\hat{\sigma}_1+\hat{\sigma}_2)+i\,\Gamma, \\ d &= -\frac{1}{4}ma^2(\hat{\sigma}_1+\hat{\sigma}_2)+\frac{1}{2}a^2(\hat{\sigma}_1-\hat{\sigma}_2)\,e^{2i\hat{\theta}}+ima^2\,\Gamma, \end{aligned} \tag{15}$$

where Γ is an arbitrary real constant.

The contribution to $\Omega(\zeta)$ arising from the terms in Γ is $i\,\Gamma Z$. It seems from (4) that $i\,\Gamma\,Z$ has no effect on Φ and it is therefore possible to put $\Gamma = 0$.

The complex potentials are then given by

$$\Omega(\zeta) = \frac{1}{4}(\hat{\sigma}_1+\hat{\sigma}_2)\,\zeta - \frac{1}{4}a^2\,[m(\hat{\sigma}_1+\hat{\sigma}_2)-2(\hat{\sigma}_1-\hat{\sigma}_2)\,e^{2i\hat{\theta}}]\,\zeta^{-1} \tag{16}$$

$$\omega(\zeta) = -\frac{1}{2}\,a^2(1+m^2)(\hat{\sigma}_1+\hat{\sigma}_2)\frac{\zeta}{\zeta^2-ma^2}+\frac{1}{2}(\hat{\sigma}_1-\hat{\sigma}_2).$$

$$\left[\frac{a^2(m\,\zeta^2+a^2)}{\zeta(\zeta^2-ma^2)}\,e^{2i\hat{\theta}}-\zeta\,e^{-2i\hat{\theta}}\right]. \tag{17}$$

In the special case $\hat{\sigma}_2 = 0$, the above expressions agree with those given in [2.]

Having found the complex potentials (16) and (17), it is easy to determine the complex stress function Φ as well as the displacement and stress fields.

REFERENCES

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DIE DEFORMATION EINER ELLIPTISCHEN SPANNUNGSLOSEN OFFNUNG UNTER DER UNIFORMEN SPANNUNG IN UNENDLICHEN

In dieser Arbeit wird die infinitesimale ebene Analysis des Versetzungs- und Spannungsfeldes einer unendlish grossen elastischen isotropen Platte, die eine ellyptische spannungslose Offnung besitzt, durch die komplexe Spannungsfunktion approximiert. Die Platte befindet sich under einer uniformen Spannung im Unendlichen.

Dieses Problem ist mit vorgegebenen Werten der komplexen Spannungsfunktion auf die inneren und ausseren Kanten, durch eine konforme Abbildung des elliptischen Gebietes auf einen Kreis mit expliziten Transformationen und mit Potenzreihen Entwicklung des komplexen Potentials gelost.

DEFORMACIJA JEDNOG ELIPTIČNOG NENAPREGNUTOG OTVORA PRI UNIFORMNOM NAPONU U BESKONAČNOSTI

U ovom radu se infinitezimalna ravanska analiza prenosnog i naponskog polja jedne beskonačno velike elastične izotropne ploče, koja ima eliptični nenapregnuti otvor, aproksimira kompleksnom naponskom funkcijom. Ploča se nalazi pod uniformnim naponom u beskonačnosti.

Ovaj problem se rešava datim vrednostima kompleksne naponske funkcije na unutrašnjim i spoljašnjim ivicama konformnim preslikavanjem eliptične oblasti na krug sa eksplicitnim transformacijama i razvojem kompleksnog potencijala u stepeni red.

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