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VARIATIONAL FORMULATION IN THE ANALYSIS OF FINITE ELASTIC—PLASTIC DEFORMATION

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1. Introduction

In the previous papers [1, 2, 3] we have analysied the structure of the constitutive laws for the materials which are in the conditions of finite elastic-plastic deformation. Various aspects of the analysis were considered, such as isothermal [1] and non-isothermal behaviour [2], or the inclusion of plastic anisotropy, i. e. the Bauschinger effect and anisotropic h^rdening in the analysis [3]. This work i oriented toward the formulation of the variational principle for velocity fields which is valid for finite deformation, and to incorporation of the established constitutive laws into its structure. At present we shall consider the isotropic, non-isothermal case and derive the corresponding exspression for the rate-potential function [7] which can be used in the finite element formulations of elastic-plastic boundary-value problems [5, 6].

2. Constitutive law

We have shown in [1] in a kinematically rigorous manner that materials which deform in the elastic-plastic regime obey the following constitutive laws:

$$D = \Lambda\,[\mathring{\tau}] \tag{2.1}$$

$$\mathring{\tau} = \mathfrak{L}\,[D] \tag{2.2}$$

where τ is the Kirchhoff stress tensor, D is the velocity strain, $(\mathring{\ })$ is the Jaumann derivative with respect to total spin W, and Λ and $\mathfrak{L}$ are the forth order operators which depend on the current state of material, i. e. stress and other quantities which define the state. The symbol [ ] is used for the inner product or trace, such that, for example (2.2), means

$$\mathring{\tau}_{ij} = \mathfrak{L}_{ijkl}\, D_{kl}. \tag{2.3}$$

It is shown in [4] that tensors Λ and $\mathfrak{L}$ possess the symmetry properties of the type:

$$\begin{gathered} \mathfrak{L}_{ijkl} = \mathfrak{L}_{jikl} = \mathfrak{L}_{ijlk} \\ \mathfrak{L}_{ijkl} = \mathfrak{L}_{klij} \end{gathered} \tag{2.4}$$

which are important in the subsequent derivation of rate potential function and which lead to symmetry of the elastic-plastic stiffness matrix in the finite element formulations of the appropriate boundary-value problems [5, 6].

The special attention of the general theory deserves the case of finite elastic-plastic deformation with small elastic component of strain, because it includes almost all common technological processes involving plastic deformation. In this case it is easy to show (for details, see [4] that tensor $\wedge$ has the representation

$$\wedge_{ijkl} = \frac{1}{2\mu}\left(\delta_{ik}\,\delta_{jl} - \frac{\lambda}{3\lambda + 2\mu}\,\delta_{ij}\,\delta_{kl}\right) + \frac{1}{h}\,\frac{\partial f}{\partial \tau_{ij}}\,\frac{\partial f}{\partial \tau_{kl}}, \tag{2.5}$$

where λ and μ are the Lamé lasticity constants, $f = g(\tau) - c = 0$ is the yield function, g being an isotropic scalar function of stress τ (in the case of isotropic hardening), while c and h are the scalar functions of history of plastic deformation. The constitutive law (2.1) therefore reads

$$D_{ij} = \left[\frac{1}{2\mu}\left(\delta_{ik}\,\delta_{jl} - \frac{\lambda}{3\lambda + 2\mu}\,\delta_{ij}\,\delta_{kl}\right) + \frac{1}{h}\,\frac{\partial f}{\partial \tau_{ij}}\,\frac{\partial f}{\partial \tau_{kl}}\right]\overset{\circ}{\tau}_{kl} \tag{2.6}$$

To get the explicit representation of the $\mathfrak{L}$ tensor in (2.2), we need to invert (2.6). The inversion is rather instructive. First we rewrite (2.6) in the form

$$D_{ij} = \frac{1}{2\mu}\,\overset{\circ}{\tau}_{ij} - \frac{\lambda}{2\mu\,(3\lambda+2\mu)}\,\overset{\circ}{\tau}_{nn}\,\delta_{ij} + \frac{1}{h}\,\frac{\partial f}{\partial \tau_{ij}}\,\frac{\partial f}{\partial \tau_{kl}}\,\overset{\circ}{\tau}_{kl}. \tag{2.7}$$

Contraction $i = j$ gives

$$D_{nn} = \frac{1}{3\lambda + 2\mu}\,\overset{\circ}{\tau}_{nn} + \frac{1}{h}\,\frac{\partial f}{\partial \tau_{nn}}\,\frac{\partial f}{\partial \tau_{kl}}\,\overset{\circ}{\tau}_{kl}. \tag{2.8}$$

But plastic yielding is not influenced by a hydrostatic component of stress [8], i. e. $\partial f/\partial \tau_{nn} = 0$, and (2.8) gives

$$\overset{\circ}{\tau} = (3\lambda + 2\mu)\,D_{nn}\,, \tag{2.9}$$

so that (2.7) becomes

$$D_{ij} = \frac{1}{2\mu}\,\overset{\circ}{\tau}_{ij} - \frac{\lambda}{2\mu}\,D_{nn} + \delta_{ij} + \frac{1}{h}\,\frac{\partial f}{\partial \tau_{ij}}\,\frac{\partial f}{\partial \tau_{kl}}\,\overset{\circ}{\tau}_{kc}. \tag{2.10}$$

Multiplying both sides of (2.10) with $\partial f/\partial \tau_{ij}$, we find that

$$\frac{\partial f}{\partial \tau_{ij}}\,\overset{\circ}{\tau}_{ij} = \frac{1}{\dfrac{1}{2\mu} + \dfrac{1}{h}\,\dfrac{\partial f}{\partial \tau_{kl}}\,\dfrac{\partial f}{\partial \tau_{kl}}}\,\frac{\partial f}{\partial \tau_{ij}}\,D_{ij}. \tag{2.11}$$

which substituted into (2.10) leads, after solving for $\overset{\circ}{\tau}_{ij}$, to

$$\overset{\circ}{\tau}_{ij} = 2\mu \left[D_{ij} + \frac{\lambda}{2\mu} D_{nn}\,\delta_{ij} - \frac{1}{\dfrac{h}{2\mu}\dfrac{\partial}{\partial \tau_{kl}}\dfrac{\partial}{\partial \tau_{kl}}} \frac{\partial}{\partial \tau_{ij}} \frac{\partial f}{\partial \tau_{\alpha\beta}} D_{\alpha\beta} \right], \tag{2.12}$$

or, equivalently

$$\overset{\circ}{\tau}_{ij} = 2\mu \left[\delta_{i\alpha}\,\delta_{j\alpha} + \frac{\lambda}{2\mu} \delta_{\alpha\beta}\,\delta_{ij} - \frac{1}{\dfrac{h}{2\mu} + \dfrac{\partial f}{\partial \tau_{kl}}\dfrac{\partial f}{\partial \tau_{kl}}} \frac{\partial}{\partial \tau_{ij}} \frac{\partial f}{\partial \tau_{\alpha\beta}} \right] D_{\alpha\beta} \tag{2.13}$$

Relation (2.13) is the explicit representation of the constitutive law (2.2) for the case of isotropic, isothermal finite elastic-plastic deformation with small elastic component of strain.

3. Variational Principle

Consider the virgin configuration of the body $\mathcal{B}_o$, the configuration $\mathcal{B}_t$ at time t, and the configuration $\mathcal{B}_z$ at time $z \geqslant t$ (Fig 3.1).

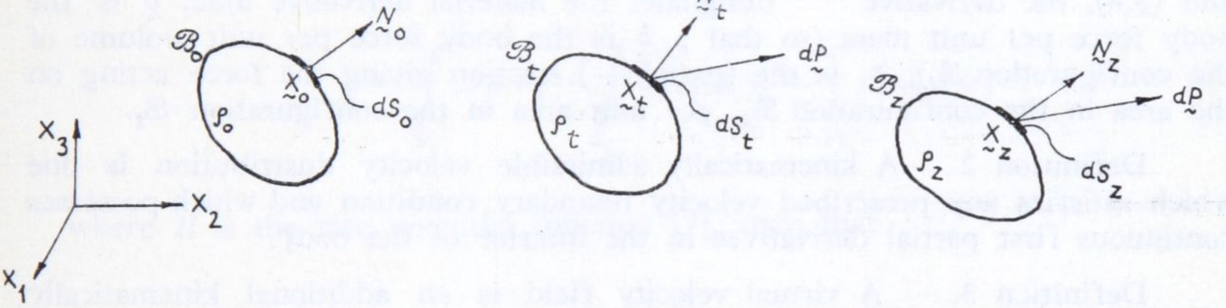

Fig. 3.1

Introduce the deformation gradients corresponding to the motions from $\mathcal{B}_o$ to $\mathcal{B}_t$ and from $\mathcal{B}_o$ to $\mathcal{B}_z$ as F (t) and $F\,(z)$, respectively. Then the relative deformation gradient $F_t\ (z)$ corresponding to the motion from $\mathcal{B}_t$ to $\mathcal{B}_z$ is defined by

$$F\,(z) = F_t\,(z) \cdot F\,(t). \tag{3.1}$$

Let the Cauchy stress in the configuration $\mathcal{B}_z$ be T. Introduce the first Piola--Kirchhoff stress tensor T_1 betwen configurations $\beta_t - \beta_z$ by

$$d\underset{\sim}{P} = (\underset{\sim}{N}_t\, T_1)\, d\, S_t, \tag{3.2}$$

where $d\underset{\sim}{P}$ is the force action on element $d\, S_z$ and $\underset{\sim}{N}_t$ is normal to $d\, S_t$. Then due to Nanson's relation [9]

$$\underset{\sim}{N}_z\, d\, S_z = \frac{\rho_t}{\rho_z} \underset{\sim}{N}_t\, F_t^{-1}\,(z)\, d\, S_t, \tag{3.3}$$

we have

$$T_1 = \frac{\rho_t}{\rho_z} F_t^{-1}(z)\, T. \tag{3.4}$$

We now introduce several definitions and theorems which correspond to similar and well known ones, such as are given, for example in [9].

Definition 1. — When the rates of external (body and surface) forces are prescribed as acting on a deformable body, a statically admissible stress rate field is defined a one satisfying the rate equilibrium equations

$$\nabla_{\underset{\sim}{x}_t} \dot{T}_1 + \rho_t \dot{\underset{\sim}{b}} = 0 \tag{3.5}$$

in the interior of the body, and the boundary conditions

$$\underset{\sim}{N}_t \dot{T}_1 = \dot{\underset{\sim}{t}} \tag{3.6}$$

at the boundary where the rate of boundary tractions are prescribed. In (3.5) and (3.6), the derivative" · " designates the material derivative d/dz, $\underset{\sim}{b}$ is the body force per unit mass (so that $\rho_t \underset{\sim}{b}$ is the body force per unit volume of the configuration $\mathcal{B}_t$), $\underset{\sim}{t}_1$ is the (pseudo—) traction giving the force acting on the area in the configuration $\mathcal{B}_z$, per unit area in the configuration $\mathcal{B}_t$.

Definition 2. — A kinematically admissible velocity dustribution is one which satisfies any prescribed velocity boundary condition and which possesses continuous first partial derivatives in the interior of the body.

Definition 3. — A virtual velocity field is an additional kinematically admissible velocity distribution applied on an equilibrium configuration. As such, it vanishes where the velocities are prescribed and it has continuous first partial derivatives in the interior of the body.

Definition 4. — The virtual power δP of the rate of the external surface tractions $\dot{\underset{\sim}{t}}_1$ and body forces $\dot{\underset{\sim}{f}}_1 = \rho_t \dot{\underset{\sim}{b}}$, if these rates are assumed to remain unchanged during the application of the virtual velocities $\delta \underset{\sim}{v}$, is

$$\delta \mathcal{P} = \int_{S_t} \dot{\underset{\sim}{t}}_1 \delta \underset{\sim}{v}\, d S_t + \int_{V_t} \dot{\underset{\sim}{f}}\, \delta \underset{\sim}{v}\, d V_t \tag{3.7}$$

where S_t and V_t are the surface and volume in the configuration $\mathcal{B}_t$.

Theorem 1. — If the stress rate field is statically admissible the virtual power on any virtual velocity field is

$$\delta P = \int_{V_t} \dot{T}^1_{Ji} \frac{\partial (\delta v_i)}{\partial X_J}\, d V_t, \tag{3.8}$$

where X_J are the coordinates in the configuration $\mathcal{B}_t$, T^1_{Ji} are the components of T_1, and δv_i are the components of the virtual velocity field $\delta \underset{\sim}{v}$.

Theorem 2. (The Key theorem). — If

$$\int_{V_t} \dot{T}^1_{Ji} \frac{\partial (\delta v_i)}{\partial X_J} d V_t = \int_{S_t} \dot{t}^1_i \, \delta v_i \, d S_t + \int_{V_t} \dot{f}^1_i \, \delta v_i \, d V_t \tag{3.9}$$

for a certain assumed stress rate field $\dot{T}_1$ and for every virtual velocity field $\delta \underset{\sim}{v}$, then the stress rate field is statically admisible, i. e. it satisfies the equilibrium equations (3.5) and the boundary conditions (3.6).

We named Theorem 2 the Key theorem because it derectly leads to the variational principle

$$\delta I = 0 \tag{3.10}$$

with

$$I = \int_{V_t} E \, d V_t - \int_{S_t} \dot{t}^1_i \, v_i \, d S_t - \int_{Vt} \dot{f}^1_i \, v_i \, d V_t \tag{3.11}$$

where E is the rate potential function [7], such that

$$\dot{T}^1_{Ji} = \frac{\partial E}{\partial \left(\dfrac{\partial v_i}{\partial X_J} \right)}, \tag{3.12}$$

as is seen by comparing (3.9) with (3.10) and (3.11). We see, therefore, that (3.12) necessarilly follows from the Theorem 2 and need not to be assumed, as is done in [7].

The variational integral (3.11) will directly serve our purposes. We need only consider the configuration $\mathcal{B}_t$ to be the current configuration, so that integrals in (3.11) are written with respect to the known geometry, and to take $z = t$. Then, the variational principle (3.10) with variational integral (3.11), gives the rate equilibrium equations and boundary conditions at the current time.

In what follows we shall derive the explicit expression for the rate potential E. In particular, we shall show that, for time independent plasticity, E is a homogeneous function of degree two in the velocity gradients.

To derive the form of E and variational integral (3.11), we start from (3.9) by substituting into it the established constitutive law (2.13), i. e. by experssing $\dot{T}_1$ in terms of the Kirchhoff stress τ. In that connection, we introduce the Kirchhoff stress corresponding to $\mathcal{B}_t - \mathcal{B}_z$ as

$$\tilde{\tau} = \frac{\rho_t}{\rho_z} T, \tag{3.13}$$

so that for $z = t$ it coincides with the Cauchy stress T. The first Piola--Kirchhoff stress (3.4) now can be written as

$$T_1 = F_t^{-1}(z)\,\tilde{\tau}. \tag{3.14}$$

Differentiating with respect to z and evaluating at $z = t$, gives

$$\dot{T}_1 = -L\,T + \dot{\tilde{\tau}} \tag{3.15}$$

Further, we recall the Kirchoff stress coresponding to $\mathcal{B}_o - \mathcal{B}_z$

$$\tau = \frac{\rho_o}{\rho_z} T, \tag{3.16}$$

which in view of (3.13) can be written as

$$\tau = \frac{\rho_o}{\rho_t} \tilde{\tau}. \tag{3.17}$$

Differentiation with respect to z gives

$$\dot{\tilde{\tau}} = \frac{\rho_t}{\rho_o} \dot{\tau}. \tag{3.18}$$

This is valid for any $z \geqslant t$ and, in particular, it is valid at $z = t$. Hence, substitution in (3.15) gives

$$\dot{T}_1 = -L\,T + \frac{\rho_t}{\rho_o} \dot{\tau}. \tag{3.19}$$

or, due to (3.16),

$$\dot{T}_1 = \frac{\rho_t}{\rho_o} (-L\,\tau + \dot{\tau}). \tag{3.20}$$

But, from the constitutive law (2.2), we have

$$\dot{\tau} = W\tau - \tau W + \mathfrak{L}[D], \tag{3.21}$$

and (3.20) becomes

$$\dot{T} = \frac{\rho_t}{\rho_o} (- D\,\tau - \tau\,W + \mathfrak{L}\,[D]). \tag{3.22}$$

Since $W = L - D$, (3.22) can be rewritten as

$$\dot{T}_1 = \frac{\rho_t}{\rho_o} (\tau\,D - D\,\tau - \tau\,L + \mathfrak{L}\,[D]). \tag{3.23}$$

The relation (3.23) is the desired expression for $\dot{T}_1$. Its substitution into the integrand of the left integral in (3.9) gives after some manipulation

$$\dot{T}_{Ji} \frac{\partial\,(\delta\,v_i)}{\partial\,X_J} = \frac{\rho_t}{\rho_o} \mathfrak{L}_{ijmn}\,D_{mn}\,\delta D_{ij} + \frac{\rho_t}{\rho_o} \tau_{ij}\,(-\,2D_{ij}\,\delta\,D_{ik} + L_{kj}\,\delta\,L_{ki}), \tag{3.24}$$

i. e.

$$\dot{T}_{Ji}\,\delta \left(\frac{\partial\,v_i}{\partial\,X_J}\right) = \delta \left[\frac{1}{2}\,\frac{\rho_t}{\rho_o} \mathfrak{L}_{ijmn}\,D_{mn} D_{ij} + \frac{1}{2}\,\frac{\rho_t}{\rho_o} \tau_{ij}\,(-\,2\,D_{ik}\,D_{kj} + L_{ki}\,L_{kj})\right]. \tag{3.25}$$

But the geometry V_t remains fixed during the application of variation δ and the integral on the left hand side of (3.9) becomes

$$\int\limits_{V_t} \dot{T}_{Ji}\,\delta \left(\frac{\partial\,v_i}{\partial\,X_J}\right) d\,V_t = \delta \int\limits_{V_t} \left\{\frac{1}{2}\,\frac{\rho_t}{\rho_o}\,[\mathfrak{L}_{ijmn}\,D_{mn}\,D_{ij} + \tau_{ij}\,(-\,2D_{ik}\,D_{kj} + L_{ki}\,L_{kj})]\right\} d\,V_t. \tag{3.26}$$

Therefore, the rate potential function E in (3.11) has the representation

$$E = \frac{1}{2}\,\frac{\rho_t}{\rho_o}\,[\mathfrak{L}_{ijmn}\,D_{mn}\,D_{ij} + \tau_{ij}\,(-\,2D_{ik}\,D_{kj} + L_{ki}\,L_{kj})] \tag{3.27}$$

which is seen to be a quadratic in velocity gradients L.

Whit E given by (3.27), variational integral (3.11) becomes precisely defined and it is ready for the use in numerical treatments of boundary value problems [6].

The explicit representation of $\mathfrak{L}_{ijmn}$ in the case of finite elastic-plastic deformation with small elastic component of strain is given in (2.13), from which

$$\mathfrak{L}_{ijmn} = 2\mu \left[\delta_{im}\,\delta_{jn} + \frac{\lambda}{2\,\mu}\,\delta_{mn}\,\delta_{ij} - \frac{1}{\dfrac{h}{2\,\mu} + \dfrac{\partial f}{\partial\,\tau_{kl}}\,\dfrac{\partial f}{\partial\,\tau_{kl}}}\,\frac{\partial f}{\partial\,\tau_{ij}}\,\frac{\partial f}{\partial\,\tau_{mn}}\right]. \tag{3.28}$$

This, together with (3.27) and (3.11), is the basis for the accurate finite element analysis of metal forming processes, such a extrusion or drawing, as discussed in [5].

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FORMULATION VARIATIONELLE EN ANALYSE DE DEFORMATIOS FINIES ELASTIQUES — PLASTYQUES

Cette communibation est concernée par l'établissement d'une fonction de changement de potentiel et de l'intégrale variationnlle correspondante pour la plattité isothermale, isotropique et indépendante du temps.

VARIJACIONA FORMULACIJA U ANALIZI KONAČNE ELASTO-PLASTIČNE DEFORMACIJE

U cilju određivanja istorije raspodjele naponskog i deformacionog stanja u procesu elasto-plastične deformacije, neophodno je izvršiti integraciju po vremenu tokom čitavog procesa deformacije. Ovo se može ostvariti nakon formulisanja pogodnog varijacionog principa koji važi u uslovima konačnih deformacija, a u koji je ugrađen konstitutivni zakon inkrementalnog tipa za elasto-plastičnu deformaciju. Metod konačnih elemenata je onda najpogodniji za određivanje numeričkog rješenja konkretnih problema.

Imajući navedeno u vidu, u ovom radu je izvedena potencijalna funkcija sa korespondentnim varijacionim integralom za slučaj izotermičke i izotropne, vremenski nezavisne elasto-plastične deformacije.

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