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DYNAMIC STABILITY OF ELASTIC CURVED BEAMS

V. B. Lazić

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A curved beam of variable cross sections whose radius of the curvature is positive lies in a plane containing a cross sectional principal axis of inertia. The boundary conditions are homogeneous. A linear elastic material of the beam is supposed.

The usual assumptions of the second order theory as well as the technical theory of bending are accepted. The influence of the axial forces on the beam deformation is neglected.

The curved beam is subjected to a distributed loading lying in the plane of the beam and varying alog the beam axis. The resultant of the external forces acting on any given element of the beam does not change in magnitude with the deformation of this element. We consider that kind of the external load which turns with the beam element during the deformation (the hydrostatical kind of the external load).

The external load consists of distributed forces in the normal ($\lambda = u$) and the tangential ($\lambda = v$) directions with respect to the beam axis; the first is positive toward the center of the curvature and the second in the direction of the increasing arc coordinate s. Each of them has two components: time-independent $p_{\lambda c}(s)$ and time-dependent $\pi(t)\, p_{\lambda t}(s)$ the magnitudes of the last being periodic functions of time:

$$\pi(t + T) = \pi(t). \tag{1}$$

Introdcing two load parameters α and β we get:

$$q_\lambda(s, t) = \alpha\, p_{\lambda c}(s) + \beta\, \pi(t)\, p_{\lambda t}(s), \quad \lambda = u, v. \tag{2}$$

Since the beam is considered as a „flexibile system“ (the equilibrium conditions are applied to the deformated shape) the influence of the deformation on equilibrium conditions is introduced as an additional load [2]:

$$\begin{aligned} \Delta q_u(s, t) &= [\alpha N_{oc}(s) + \beta \pi(t) N_{ot}(s)]\, \varkappa(s, t), \\ \Delta q_v(s, t) &= [\alpha T_{oc}(s) + \beta \pi(s) T_{ot}(s)]\, \varkappa(s, t), \end{aligned} \tag{3}$$

where $N_{oc}(T_{oc})$ and $N_{ot}(T_{ot})$ are the axial (shear) forces due to the external load $p_{\lambda c}$ and $p_{\lambda t}$ $(\lambda = u, v)$, respectively, referring to the curved beam as a „rigid system'' (the equilibrium conditions are applied to the non-deformated shape); $\varkappa(s, t)$ is the unknown function (see Appendix). The inertia forces are:

$$q_{ui}(s, t) = -m(s)\frac{\partial^2 (u, s)}{\partial t^2} = -m(s)\,\ddot{u}(s, t),$$

$$q_{vi}(s, t) = -m(s)\frac{\partial^2 v(s, t)}{\partial t^2} = -m(s)\,\ddot{v}(s, t); \tag{4}$$

$u(s, t)$ and $v(s, t)$ being the unknown functions, too.

Using the well-known procedure [1] we establish two integral-differential equations where appears the functions $u(s, t)$ and $v(s, t)$:

$$u(s, \mathrm{t}) = \int_L K_{uu}(s, z)\,q_u(z, t)\,dz + \int_L K_{uv}(s, z)\,q_v(z, \mathrm{t})\,dz +$$

$$+ \int_L K_{uu}(s, z)\,\Delta q_u(z, t)\,dz + \int_L K_{uv}(s, z)\,\Delta q_v(z, t)\,dz -$$

$$- \int_L K_{uu}(s, z)\,m(z)\,\ddot{u}(z, t)\,dz - \int_L K_{uv}(s, z)\,m(z)\,\ddot{v}(z, t)\,dz, \tag{5}$$

$$v(s, t) = \int_L K_{vu}(s, z)\,q_u(z, t)\,dz + \int_L K_{vv}(s, z), q_v(z, t)\,dz +$$

$$+ \int_L K_{vu}(s, z)\,\Delta q_u(z, t)\,dz + \int_L K_{vv}(s, z)\,\Delta q_v(z, t)\,dz -$$

$$- \int_L K_{vu}(s, z)\,m(z)\,\ddot{u}(z, t)\,dz - \int_L K_{vv}(s, z)\,m(z)\,\ddot{v}(z, t)\,dz, \tag{6}$$

where it is necessary to introduce the expressions for the external load Eq. (2) and additional load Eq. (3); $K_{\lambda\mu}(s, z)$ $(\lambda, \mu = u, v)$ being the corresponding influence functions (see Appendix).

Eqs. (5) and (6) are inter-dependent because the known relation (A. 1) exists so that in further text we can dwell only on one of them. We will etain Eq. (5) .

Applying the well-known method for solving Eq. (5) we introduce the individual functions $U_j(s)$ and $V_j(s)$ $(j = 1, 2, \ldots)$ of the matrix kernel (A. 3) corresponding to the equations of vibrations of the curved beam [1]. These functions are orthonormal:

$$\int\limits_L m(s)\,[U_j(s)\,U_k(s) + V_j(s)\,V_k(s)]\,ds = \delta_{jk}, \quad \delta_{jk} = \begin{cases} 1 & \text{for } j = k \\ 0 & \text{for } j \neq k \end{cases} \tag{7}$$

and the influence functions $K_{uu}(s, z)$ and $K_{uv}(s, z)$ in Eq. (5) can be expanded in a series absolutely and uniformly convergent with respect to the arguments s and z:

$$K_{uu}(s, z) = \sum_j \frac{1}{\omega_j^2}\,U_j(s)\,U_j(z), \quad K_{uv}(s, z) = \sum_j \frac{1}{\omega_j^2}\,U_j(s)\,V_j(z), \tag{8}$$

ω_j^2 $(j = 1,2,\ldots)$ being the individual values of the matrix kernel (A. 3), i.e. angular natural frequencies of the curved beam. The unknown functions $u(s, t)$ and $v(s, t)$ can be also expanded in a series uniformly converging with the time-dependent coefficients:

$$u(s, t) = \sum_j f_j(t)\,U_j(s), \quad v(s, t) = \sum_j f_j(t)\,V_j(s). \tag{9}$$

On the basis of Eqs. (A. 1) and (A. 2) the function $\varkappa(s, t)$ can be represented in the shape of the series:

$$\varkappa(s, t) = \sum_j f_j(t)\,\varkappa_j(s), \tag{10}$$

where $\varkappa_j(s)$ are expressed through functions $U_j(s)$ and $V_j(s)$ in the following way:

$$\varkappa_j(s) = -\frac{d}{ds}\left[\frac{d\,U_j(s)}{ds} + \frac{V_j(s)}{r(s)}\right], \quad j = 1, 2, \ldots, . \tag{11}$$

Introduction Eqs. (9), (8), (2), (3) and (11) in (5) and then Eq. (7), we obtain the following expression for the j^{th} $(j = 1, 2, \ldots)$ differential equation with respect to the unknown function $f_j(t)$:

$$\frac{1}{\omega_j^2}\,\ddot{f}_j(t) + f_j(t) - \alpha \sum_k a_{jk}\,f_k(t) - \beta\,\pi(t) \sum_k b_{jk}\,f_k(t) =$$

$$= \alpha\,g_j + \beta\,\pi(t)\,h_j, \quad j = 1, 2, \ldots, \tag{12}$$

where the coefficients are as follows:

$$a_{jk} = \frac{1}{\omega_j^2} \int_L [U_j(s) N_{oc}(s) + V_j(s) T_{oc}(s)] \varkappa_k(s)\, ds, \quad j, k = 1, 2, \ldots, \tag{13}$$

$$b_{jk} = \frac{1}{\omega_j^2} \int_L [U_j(s) N_{ot}(s) + V_j(s) T_{ot}(s)] \varkappa_k(s)\, ds \quad j, k = 1, 2, \ldots, \tag{14}$$

$$g_j = \frac{1}{\omega_j^2} \int_L [U_j(s) p_{uc}(s) + V_j(s) P_{vc}(s)]\, ds, \quad j = 1, 2, \ldots, \tag{15}$$

$$h_j = \frac{1}{\omega_j^2} \int_L [U_j(s) p_{ut}(s) + V_j(s) p_{vt}(s)]\, ds, \quad j = 1, 2, \ldots . \tag{16}$$

When we keep the finite number n of Eq. (12) we may represent them in the matrix form:

$$C\ddot{f} + [E - A - \beta\pi(t) B] f = \alpha g + \beta \pi(t) h, \tag{17}$$

where the following matrices:

$$A = ||a_{jk}||_{n,n}, \qquad B = ||b_{jk}||_{n,n}, \qquad C = \left\| \frac{\delta_{jk}}{\omega_j^2} \right\|_{n,n} \tag{18}$$

and the column vectors:

$$f = |f_1, f_2, \ldots, f_n|' \quad g = |g_1, g_2, \ldots, g_n|' \quad h = |h_1, h_2, \ldots, g_n|', \tag{19}$$

are introduced.

It is known that the type of the matrix equation ot the dynamic stability depends on the properties of the material. For the linear elastic material the quation has the form given by Eq. (17). The shape of the beam, the change of the mass law, as well as the cross section geometrical properties along the beam axis, the boundary conditions as well as the kind of the external load (gravitational or hydrostatical) influence the elements of the matrices A, B, C and the column vectors g and h only. In Ref [1] different expressions for elements a_{jk} and b_{jk} of the matrices A and B were obtained for the elastic curved beam subjected to the hidrostatical kind of the external load (for $p_{\lambda c}(s) = p_{\lambda t}(s) = p_\lambda(s)$, $\lambda = u, v$).

When we introduce:

$$\beta = 0, \ p_{\lambda c}(s) = p_\lambda(s) \quad \lambda = u, v,$$
$$u = u(s), \quad v = v(s), \quad N = N_o(s), \quad T = T_o(s), \tag{20}$$

we arrive at the problem of the static stability of the elastic curved beam, $N_o(s)$ and $T_o(s)$ being the axial and the shear forces due to the external load $p_\lambda(s)$ $(\lambda = u, v)$ for the beam as a „rigid system''.

After some transformations of Eqs. (5) and (6) according to (A. 1), keeping in mind Eq. (20), we get:

$$\varphi(s) - \alpha \int_L [K_{\varphi u}(s, z) N_o(z) + K_{\varphi v}(s, z) T_o(z)] \varkappa(z) \, dz = \varphi_o(s), \tag{21}$$

where the influence functions $K_{\varphi\lambda}(s, z)$ $(\lambda = u, v)$ are determined by Eq. (A. 4) and:

$$\varphi_o(s) = \alpha \int_L [K_{\varphi u}(s, z) p_u(z) + K_{\varphi v}(s, z) p_v(z)] \, dz, \tag{22}$$

representing the rotation of the cross sectional plane of the beam as a „rigid system" due to the external load.

When we differentiate Eq. (21) with respect to the argument s and use (A. 6) and (A. 5) we arrive at the well-known integral equation of the static stability of the elastic curved beam subjected to the hidrostatical kind of the external load [2]:

$$M(s) - \alpha \int_L \left[M_u(s, z) \frac{N_o(z)}{EJ(z)} + M_v(s, z) \frac{T_o(z)}{EJ(z)} \right] M(z) \, dz = M_o(s), \tag{23}$$

where $M_\lambda(s, z)$ $(\lambda = u, v)$ are the influence functions defined by Eq. (A. 6); $M_o(s)$ is the bending moment of the curved beam as a „rigid system'' due to the external load $\alpha p_\lambda(s)$ $(\lambda = u, v)$.

The equation of the static stability of the elastic curved beam (23) referring to the hydrostatical kind of external load cannot be derived as a special case from the corresponding equations of dynamic stability, formulated in the mentioned Ref. [1].

APPENDIX

The following symbols are used in this paper:

s, z	coordinate along the beam axis
t	time
$u(s, t)$	deflection directed toward the normal of the beam axis, positive toward the center of the curvature

$v(s, t)$	deflection directed toward the tangent of the beam axis, positive toward the increasing arc coordinate s
$\varphi(s, t)$	rotation of the cross sectional plane
$\varkappa(s, t)$	change in the curvature of the beam axis
N	axial force, positive when exerting a pressure on the element
T	shear force, positive when turning the element clockwise
M	bending moment, positive when the lower fibres of the element are stretched
E	Young's modulus
$J(s)$	centroidal moment of inertia of the cross section
r (s)	mass
$u(s)$	radius of the curvature
$K_{\lambda\mu}(s, z)$	influence function for the generalized displacement ($\lambda = u$ deflection in the normal direction; $\lambda = v$ deflection in the tangential direction: $\lambda = \varphi$ rotation of the cross sectional plane) of point s due to the unit force ($\lambda = u$ directed toward the normal of the beam axis; $\lambda = v$ directed toward the tangent of the beam axis) at point z
$M_\lambda(s, z)$	influence function for the bending moment at point s, due to the unit force ($\lambda = u$ directed toward the normal of the beam axis; $\lambda = v$ directed toward the tangent of the beam axis) at point z

The following well-known expressions are used in this paper:

$$\varphi(s, t) = \frac{\partial u(s, t)}{\partial s} + \frac{v(s, t)}{r(s)}; \tag{A. 1}$$

$$\varkappa(s, t) = -\frac{\partial \varphi(s, t)}{\partial s}; \tag{A. 2}$$

$$\left\| \sqrt{m(s)\, m(z)}\, K_{\lambda\mu}(s, z) \right\|_{2, 2}, \qquad \lambda, \mu = u, v; \tag{A. 3}$$

$$K_{\varphi\lambda}(s, z) = \frac{\partial K_{u\lambda}(s, z)}{\partial s} + \frac{K_{v\lambda}(s, z)}{r(s)}, \quad \lambda = u, v; \tag{A. 4}$$

$$M(s) = EJ(s)\, \varkappa(s); \tag{A. 5}$$

$$M_\lambda(s, z) = -EJ(s) \frac{\partial K_{\varphi\lambda}(s, z)}{\partial s}, \qquad \lambda = u, v. \tag{A. 6}$$

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ДИНАМИЧЕСКАЯ УСТОЙЧИВОСТЬ УПРУГИХ КРИВОЛИНЕЙНЫХ СТЕРЖНЕЙ

Для упругих криволинейных стержней выведено матричное уравнение динамической устойчивости. Внешная нагрузка которая лежит в плоскости к которой принадлежить ось криволинейного стержня поворочивается, составляя с изогнутой осью стержня первоначальный угол. Полученые коэффициенты матричного уравнения различные от тех которые известны в литературе. Как частный случай интегро — дифференциальных уравнений динамической устойчивости получено интеграьное уравнение статической устойчивости криволинейных стержней которое известно в литературе.

DINAMIČKA STABILNOST KRIVIH ŠTAPOVA OD ELASTIČNOG MATERIJALA

Za kriv štap od elastičnog materijala izvedena je matrična jednačina dinamičke stabilnosti. Spoljno opterećenje, koje leži u ravni ose štapa, obrće se prilikom deformacije nosača zadržavajući prema njemu prvobitni ugao.

Dobijeni su izrazi za koeficijente matrične jednačine koji se razlikuju od izraza poznatih u literaturi. Kao specijalan slučaj integro-diferencijalnih jednačina dinamičke stabilnosti dobijena je, poznata u literaturi, integralna jednačina statičke stabilnosti krivog štapa koji je izložen delovanju hidrostatičke vrste spoljnog opterećenja.

Vera B. Lazić
Višegradska 16/II
11000 Beograd, Jugoslavija