

OSCILLATION OF MECHANICAL SYSTEM WITH FINITE NUMBER OF DEGREES OF FREEDOM IN INTERACTION WITH THE ELASTIC LAYER

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I. Introduction

Some problems in dynamics of structures [1] — e. g. the oscillations of the fundamentals of machines and other structures being in contact with soil — can be deduced to study of mechanical systems with finite number of degrees of freedom being in dynamic interaction with the elastic layer of V. Z. Vlasov as possible model of deformable soil. The model was formulated by V. Z. Vlasov the first time in his known monography [2] and more detailed elaborated in [9]. This model represents the deformable half-space by finite number of deformable layers assuming the distribution of displacement components through the depth of each layer.

In the case of one-layer model of soil with finite depth H , if the vertical displacement is much greater then horizontal one, the form of the displacement field for the layer of V. Z. Vlasov will be as follows [3]:

$$\begin{aligned} u_1(z_j, t) &= 0 \\ u_2(z_j, t) &= 0 \\ u_3(z_j, t) &= \begin{cases} \tilde{u}_3(z_a, t) \theta(z_3), & z_3 \in (0, H) \\ 0, & (\forall z_3) z_3 \notin (0, H) \end{cases} \end{aligned} \quad (1)$$

where $\tilde{u}_3(Z_a, t)$ is the vertical displacement on the contour of layer $z_3 = 0$ in the direction of the axis z_3 whereas $\theta(Z_3)$ is the known function by which the distribution of displacement $u_3(Z_j, t)$ ($j = 1, 2, 3$) through the depth of layer is assumed. For the mechanical system it will be assumed that it is unconservative and its motion is limited by unstationary holonomic constraints and also the contact with elastic layer is realized in the area \tilde{S}

If $q^\alpha(t)$ ($\alpha = 2, 3, \dots, n$) are the independent generalized coordinates of the mechanical system, the displacement component can be shown as follows:

$$\tilde{u}_3(z_a, t) = \begin{cases} \tilde{U}(z_a, q^\alpha, t), & (z_a) \in \tilde{S} & (a = 1, 2) \\ U(z_a, q^\alpha, t), & (z_a) \in S & (S = R^2/\tilde{S}) \end{cases} \quad (2)$$

The displacement component \tilde{U} in the contact field \tilde{s} is a known function and the displacement component U , out of the contact field, is an unknown function.

In order to determine the motion of the extended mechanical system consisting of initial mechanical system and elastic layer, it is necessary to determine the system of functions $q^\alpha(t)$ ($\alpha = 1, 2, \dots, n$) and function $U(Z_\alpha, q^\alpha, t)$. This last one describes the behaviour of elastic layer, whereas the motion of the mechanical system is described by the system of functions. For the material of the layer it will be assumed that it possesses elastic properties, i. e. that it can be described with the constitutive relation:

$$t^{ij} = C^{ijkl} e_{kl} \quad (3)$$

where C^{ijkl} is in general case an anisotropic tensor of the elastic layer with the properties of algebraic operator.

2. List of symbols

Z_j, Z_α	— Descartes coordinates,	\tilde{S}	— field of contact of the mechanical system and layer,
t	— time,	S	— free contour of the layer,
q^α	— independent generalized coordinates,	t^{ij}	— stress tensor,
$u_i(Z_j, t)$	— displacement components of soil points,	e_{ij}	— strain tensor,
ρ	— soil density,	$\theta(Z_3)$	— function of displacement distribution through the depth of the layer,
G, ν	— shear modulus and Poisson's ratio of the soil,	H	— depth of the layer,
\tilde{U}, U	— vertical displacement of points on the boundary of soil	$A = \int_0^H \theta^2(Z_3) dz_3$	— constant of the layer.

3. Mathematical model of the system

We will investigate the motion of nontatinary holonomic mechanical system with n degrees of freedom subject to nonconservative forces. This system is in interaction with the homogenous elastic layer of V. Z. Vlasov. In the text the layer will be treated as an additional inertial connection.

The equations of motion of the mechanical system will be derived from Hamilton's principle for noncorvesative systems. The form of the kinetic energy of nonstationary holonomic system is the following [4]:

$$T_0^0 = \frac{1}{2} a_{\alpha\beta}^0 \dot{q}^\alpha \dot{q}^\beta + \frac{1}{2} a_\alpha^0 \dot{q}^\alpha + \frac{1}{2} a_0^0 \quad (\alpha, \beta = 1, 2, \dots, n) \quad (4)$$

where $a_{\alpha\beta}^0, a_\alpha^0, a_0^0$ are in general case the functions of generalized coordinates $q^\alpha(t)$ ($\alpha = 1, 2, \dots, n$) and time t .

The potential energy of the mechanical system will be taken in the form:

$$\Pi_0^0 = \frac{1}{2} c_{\alpha\beta}^0 q^\alpha q^\beta + \Pi_*^0(q^\alpha, t) \quad (5)$$

where $c_{\alpha\beta}^0$ are the functions of generalized coordinates and time.

The kinetic and potential energy of the extended system can be shown in the form:

$$T = T_0^0 + \frac{1}{2} \int_{\tilde{s}} \rho A \tilde{U}_t^2 d\tilde{s} + \frac{1}{2} \int_s \rho A U_t^2 ds \quad (6)$$

$$\Pi = \Pi_0^0 + \frac{1}{2} \int_{\tilde{s}} \tilde{\pi} d\tilde{s} + \frac{1}{2} \int_s \pi ds$$

where π is the strain energy density function.

$$\pi = \int_0^H C^{ijkl} e_{ij} e_{kl} dz_3 \quad (i, j, k, l = 1, 2, 3) \quad (7)$$

Since C^{ijkl} is the algebraic operator, the existing dependence $\pi = \pi(\mathbf{t}, U, Ua)$ i. e. $\tilde{\pi} = \tilde{\pi}(\mathbf{t}, \tilde{U}, \tilde{U}a)$ will exist. Since according to the assumption $\tilde{U} = \tilde{U}(Za, q^\alpha, t)$ is a known function, the first integrals in the expression (6) in the area \tilde{s} can be uniquely determined obtaining some new functions of the generalized coordinates and time. If

$$T_0 = T_0^0 + \frac{1}{2} \int_{\tilde{s}} \rho A \tilde{U}_t^2 d\tilde{s} \quad (8)$$

$$\Pi_0 = \Pi_0^0 + \frac{1}{2} \int_{\tilde{s}} \pi d\tilde{s}$$

where T_0 and Π_0 have the same form obviously, as well as T_0^0 and Π_0^0 of the given relations (4) and (5), the expressions (6) change to:

$$T = T_0 + \frac{1}{2} \int_s \rho A U_t^2 ds \quad (9)$$

$$\Pi = \Pi_0 + \frac{1}{2} \int_s \pi ds$$

where — as it was already told — T_0 and Π_0 are given in the expressions:

$$T_0 = \frac{1}{2} a_{\alpha\beta} \dot{q}^\alpha \dot{q}^\beta + \frac{1}{2} a^\alpha \dot{q}^\alpha + \frac{1}{2} a_0 \quad (\alpha, \beta = 1, 2, \dots, n) \quad (10)$$

$$\Pi_0 = \frac{1}{2} c_{\alpha\beta} q^\alpha q^\beta + \Pi^*(q^\alpha, t)$$

In the expressions (10), the coefficients $a_{\alpha\beta}$, a_α , a_0 and $c_{\alpha\beta}$ are also the functions of generalized coordinates and time. The expressions T_0 and Π_0 represent the kinetic and potential energy of the mechanical system together with the part of the layer belonging to the area of cylinder with the base and height H .

Since the kinetic and potential energy of the extendent system were defined, the Lagrangian function can be determined, too:

$$L = T - \Pi \quad (11)$$

taking into account (10):

$$L = L_0 + \int_s \Phi ds \quad (12)$$

where the following symbols were introduced:

$$L_0 = T_0 - \Pi_0 \quad (13)$$

$$\Phi = \frac{1}{2} (\rho A U_t^2 - \pi)$$

The form of Hamilton's principle for nonconservative system is the following:

$$\delta \int_0^t L dt + \int_0^t \delta' A dt = 0 \quad (14)$$

Since according to the assumption the system is affected also by nonconservative forces, the expression for the work on virtual displacements of the system

expressed through the variations of independent generalized coordinates, has the following form:

$$\delta' A = Q_\alpha^* \delta q^\alpha \quad (\alpha = 1, 2, \dots, n) \quad (15)$$

If now, according to the relation (14) the variation of the functional L will be carried out, taking into account that the variation is done through independent generalized coordinates $q^\alpha(t)$ and function $U(Za, t)$, it will be obtained:

$$\begin{aligned} & \int_0^t \left\{ \frac{\partial L}{\partial q^\alpha} - \frac{d}{dt} \frac{\partial L_0}{\partial \dot{q}^\alpha} + Q_\alpha^* + \int_s \left(\frac{\partial \Phi}{\partial q^\alpha} - \frac{d}{dt} \frac{\partial \Phi}{\partial \dot{q}^\alpha} \right) ds \right\} \delta q^\alpha dt + \\ & + \int_0^t \int_s \left\{ \frac{\partial \Phi}{\partial U} - \frac{\partial}{\partial t} \frac{\partial \Phi}{\partial U_t} - \sum_{a=1}^2 \frac{\partial}{\partial z_a} \frac{\partial \Phi}{\partial U_a} \right\} \delta U dt ds + \\ & + \left(\frac{\partial L_0}{\partial \dot{q}^\alpha} \delta q^\alpha \right) \Big|_0^t + \int_s \left(\frac{\partial \Phi}{\partial U_t} \delta U \right) \Big|_0^t ds + \\ & + \int_0^t \int_s \sum_{a=1}^2 \frac{\partial}{\partial z_a} \left(\frac{\partial \Phi}{\partial U_a} \delta U \right) dt ds = 0 \end{aligned} \quad (16)$$

The equalization of the integrand in the first and second integral of (16) with zero will give the equations of motion of the system while the other members are either equal to zero or they give natural boundary conditions for the function $U(Za, t)$. The equations of motion of the system in abbreviated form are as follows:

$$a_{\alpha\beta} \ddot{q}^\beta + [\alpha, \beta\gamma] \dot{q}^\beta \dot{q}^\gamma + c_{\alpha\beta} q^\beta = Q_\alpha^* + \int_s R_\alpha ds + G_\alpha + \psi_\alpha \quad (17)$$

$$\frac{\partial \Phi}{\partial U} - \frac{\partial}{\partial t} \frac{\partial \Phi}{\partial U_t} - \sum_{a=1}^2 \frac{\partial}{\partial z_a} \frac{\partial \Phi}{\partial U_a} = 0$$

$$(\alpha, \beta, \gamma = 1, 2, \dots, n) \quad (a = 1, 2)$$

where:

$$R_\alpha = \frac{\partial \hat{\Phi}}{\partial q^\alpha} - \frac{d}{dt} \frac{\partial \hat{\Phi}}{\partial \dot{q}^\alpha} \quad \hat{\Phi} = \Phi(U(q^\alpha, t), U_t(q^\alpha, t))$$

$$[\alpha, \beta\gamma] = \frac{1}{2} \left(\frac{\partial a_{\alpha\beta}}{\partial q^\gamma} + \frac{\partial a_{\alpha\beta}}{\partial q^\beta} - \frac{\partial a_{\alpha\beta}}{\partial q^\alpha} \right)$$

$$G\alpha = \left(\frac{\partial a_\gamma}{\partial q^\alpha} - \frac{\partial a_\alpha}{\partial q} \right) \dot{q}^\gamma \quad (18)$$

$$\begin{aligned} \psi_\alpha = & \frac{1}{2} \left(\frac{\partial a_0}{\partial q^\alpha} - \frac{\partial c_{\beta\gamma}}{\partial q^\alpha} q^\beta q^\gamma \right) - \frac{\partial a_{\alpha\beta}}{\partial t} \dot{q}^\beta - \\ & - \frac{\partial a_\alpha}{\partial t} + \frac{\partial \Pi_*}{\partial t} - \frac{\partial \Pi_*}{\partial q^\alpha} \end{aligned}$$

The system of integro-differential equations with unknowns $p^\alpha(t)$ and partial equation with unknown $U(Za, t)$ represent with, corresponding initial-boundary condition, a complete system of equations of motions from which the system of functions $q^\alpha(t)$ and $U(Za, t)$ can be determined.

4. Definition of the boundary value problem

The natural contour conditions for the function $U(Za, t)$ obtained from (16) are not always satisfied at the points on the contour $\partial \tilde{s}$ of the area \tilde{s} since at these points the function U and its gradients Ua can have discontinuities. Therefore, it is necessary to define in each specific case the boundary condition for the function $U(Za, t)$. It will be assumed that in the area \tilde{s} a contact over a rigid plate was realised so that the displacement field $U(Za, t)$ is uninterrupted, whereas the gradient Ua has interruption. In that case the complete system of equations of motion together with the initial boundary conditions is the following:

$$a_{\alpha\beta} \ddot{q}^\beta = [\alpha, \beta\gamma] \dot{q}^\beta \dot{q}^\gamma + c_{\alpha\beta} q^\beta = Q_\alpha^* + \int_s R_\alpha ds + G_\alpha + \psi_\alpha$$

$$\frac{\partial \Phi}{\partial U} - \frac{\partial}{\partial t} \frac{\partial \Phi}{\partial U_t} - \sum_{a=1}^2 \frac{\partial}{\partial z_a} \frac{\partial \Phi}{\partial U_a} = 0$$

$$(\alpha, \beta, \gamma = 1, 2, \dots, n) \quad (a = 1, 2) \quad (19)$$

$$U(z_a, 0) = U_0(z_a) \quad q^\alpha(0) = q_0^\alpha$$

$$U_t(z_a, 0) = \dot{U}_0(z_a) \quad \dot{q}^\alpha(0) = \dot{q}_0^\alpha$$

$$U(z_a, t) \Big|_{(z_a) \in \partial \tilde{s}} = \tilde{U}(z_a, t) \quad \lim_{|z_a| \rightarrow \infty} U(z_a, t) = 0$$

5. Conclusion

The aim of the work was to give a mathematical model for description of the oscillations of the mechanical system with finite number of degrees of freedom being in interaction with the elastic layer of V. Z. Vlasov.

It was shown that for the material class of the layer having the properties of elastic continuum a complete equation system (19) was obtained, where besides the integro-differential equations also the partial equations of the layer appeared. Assuming that in (19) $\rho = 0$, i. e. that it is a noninertial layer, the partial equation disappears while the integro-differential equations transform into differential ones by which the motion of the mechanical system with generalized coordinates $q^\alpha(t)$ is described.

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DYNAMISCHE INTERAKTION DES MECHANISCHEN SYSTEMS MIT ENDLICHER ZAHL DES FREIHEITGRADES UND DER ELASTISCHEN SCHICHT

Es wurde die oszillatorische Bewegung des unkonservativen, unstationären holonomen Systems mit n Bewegungsfreiheitsgrad betrachtet, das mit der elastischen Schicht von V. Z. Vlasov in Interaktion ist. Die Schicht wurde als eine ergänzende Trägheitsbindung behandelt.

Es wurde gezeigt, dass für eine bestimmte Klasse des Schichtenmaterials des Problem auf ein System von n Integral-Differenzialgleichungen und auf (ine Partialgleichung der Schicht mit entsprechenden anfänglichen und Umrissbedingungen abgeleitet werden kann.

DINAMIKA INTERAKCIJA MEHANIČKOG SISTEMA SA KONAČNIM BROJEM STEPENI SLOBODE I ELASTIČNOG SLOJA

Posmatrano je oscilatorno kretanje nekonzervativnog, nestacionarnog, holonomnog sistema sa n stepeni slobode kretanja, koji je u interakciji sa deformabilnim slojem V. Z. Vlasova. Sloj je tretiran kao dopunska inerciona veza.

Pokazano je da se za određenu klasu materijala sloja, problem svodi na sistem od n integro-diferencijalnih jednačina i parcijalnu jednačinu sloja, sa odgovarajućim početno-konturnim uslovima.

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