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THE FISSION OF CAPILLARY-GRAVITY SOLITARY WAVES ON THE SHELF

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Two recently appeared notes have to some extent refreshed the interest in studying capillary-gravity solitary waves. In the first of them [1], recalling the „celebrated — but perhaps not widely enough read!'' original paper by Korteweg and de Vries [2], Benjamin points out some incorrectnesses in a paper by Shinbrot [3], that are referred to the case when the Weber number $W = g\rho h_o^2/T < 3$ (g — the acceleration due to gravity, ρ — the density of the liquid, h_o — the depth of the liquid at rest, T — the surface tension coefficient), in which solitary waves are waves of depression rather than waves of elevation as they are for $W_e > 3$. In the second note [4], an insight into the exceptional case when $W_e = 3$, in which waves on hollow water suffer no dispersion within the order of approximations used in the derivation of the Korteweg-de Vries ($K - dV$) equation, is given.

Although it is will known that capillarity affects solitary waves only when the liquid is relatively small and that consequently the effect of an uneven bottom on the propagation of solitary waves does not have such a practical importance as it has for purely gravity waves, we in this note, for completeness of the theory, state the $K - dV$ equation with varying coefficients describing the propagation of capillary-gravity solitary waves over an uneven bottom and study by means of it the fissions of solitary waves on the shelf. The first $K - dV$ equation with varying coefficients for capillary—gravity solitary waves moving over an uneven bottom was derived by Kakutani [5]. Including capillarity and applying the same procedure we obtain the following evolution equation (for convenience we do not use the same notations as in [1] and [4]!):

$$\frac{h'}{2h^{1/2}}\eta + 2h^{1/2}\eta_\xi + \frac{3}{h}\eta\eta_\tau + \frac{h}{3}\left(1 - \frac{3}{We \cdot h^2}\right)\eta_{\tau\tau\tau} = 0, \tag{1}$$

in which τ and ξ are the following scaled coordinates:

$$\tau = \varepsilon^{1/2}\left(\int^x \frac{dx}{h^{1/2}} - t\right), \xi = \varepsilon^{3/2}x. \tag{2}$$

In (1) and (2) $h = h(\xi)$ is the varying depth of the liquid, $\eta = \eta(\tau,\xi)$ is the perturbation on the free surface, x is the horizontal coordinate increasing in the direction of the wave propagation, t is the time and $\varepsilon > 0$ is a small parameter representing the ratio of the wave amplitude to the depth. All lengths and the time are normalized by a characteristic depth h_0 and $h_0/(gh_0)^{1/2}$, respectively. It is noticed that the product $We \cdot h^2$ represents actually a local Weber number.

The equation (1) belongs to the class of so-called perturbed $K - dV$ equations with varying coefficients. The theory of this equation for purely gravity waves ($We \rightarrow \infty$) was given by Ono [6] and by Johnson [7]. The main and the most interesting result of this theory is referred to the evolution of a solitary wave moving over a region of varying depth sandwitched between two regions of constant but different depth — over a shelf. It is shown that under certain conditions a solitary wave may desintegrate by passing over the shelf into an exact number of new solitary waves — the phenomenon known in the literature as the fission of solitary waves. It occurs if:

$$\frac{h_1}{h_2} = \left[\frac{n(n+1}{2}\right]^{4/9} \tag{3}$$

where h_1 and h_2 ar the depths in front and behind the shelf respectively and n is the number of solitary waves emerging on the shelf. The expression (3) is called the eigendepth relation, because it follows directly from the solution of an eigenvalue problem (s. [7]).

The results by Ono and by Johnson were in a simple way generalized to include the propagation of solitary waves along a channel of varying width [8] and the propagation of wave packets over an uneven bottom — the problem described by a nonlinear Schrödinger equation with varying coefficients [9]. For a channel with a contraction, the eigenwidth relation in which b_1 and b_2 are respectively the widths in front and behind the contraction is:

$$\frac{b_1}{b_2} = \left[\frac{n(n+1)}{2}\right]^{2} \tag{4}$$

while the eigendepth relation describing the fission of an envelope-kole solition is:

$$\frac{h_1}{h_2}\left[\frac{n(n+1)}{2}\right]^{8/27} \tag{5}$$

The fission of solitary waves in all these cases obviously occurs exclusively by decreasing the depth of the liquid or by decreasing the width of the channel.

The results by Ono and by Johnson can be also generalized to include the fission of capillary-gravity solitary waves, described by equation (1)· From their theories the following eigendepth relation is staightforwardly obtained:

$$\left(\frac{h_1}{h_2}\right)^{9/4} \frac{1 - 3/(We \cdot h_1^2)}{1 - 3/(We \cdot h^2)} = \frac{n\,(n+1)}{2}. \tag{6}$$

It obviously reduces for $We \to \infty$ to (3). If the following notations $We_1 = We \cdot h_1^2$ and $We_2 = We \cdot h_2^2$ for local Weber number in front and behind the shelf respectively are introduced, it will be $We_2 > 3$ or $We_2 < 3$ depending on whether $We_1 > 3$ or $We_1 < 3$, i.e. an elevation-like (depression-like) solitary wave can desintegrate only into a sequence of elevation-like (depressios-like) solitary waves — the transition from an elevation-like solitary wave to a depression-like and vice versa is not possible by the mechanism of fission on shelf. The relation (6) is graphically presented in Fig. 1 in the form of the dependence of h_1/h_2 on We_1 for $n = 1, 2, 3$. The results for $n = 2$ and $n = 3$ will be analyzed first. For $We_1 \to \infty$, h_1/h_2 tends to the values determined by (3).

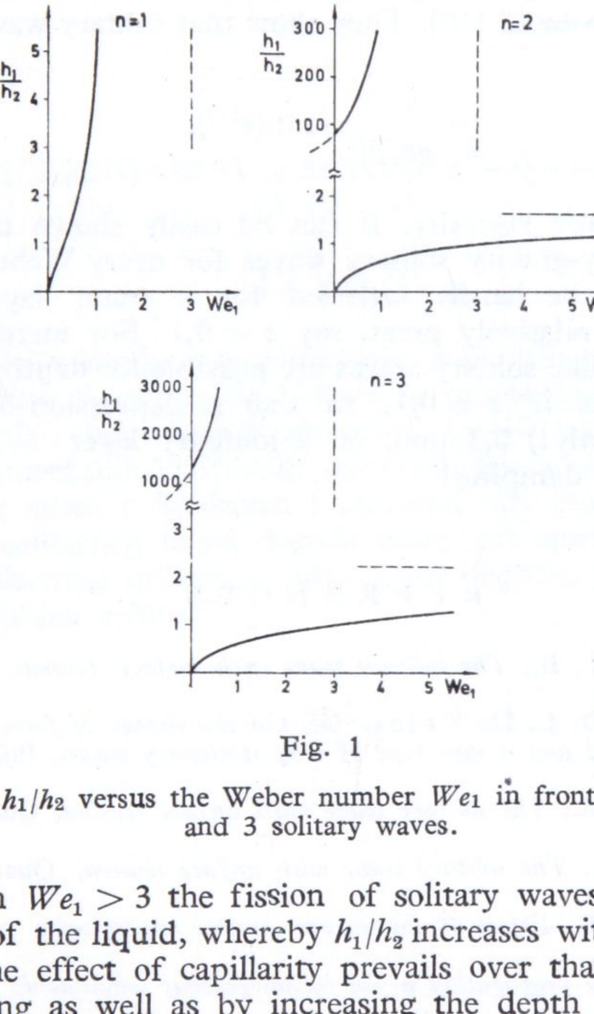

Fig. 1

The eigendepth ratio h_1/h_2 versus the Weber number We_1 in front of the shelf for $n = 1, 2$ and 3 solitary waves.

In the region $We_1 > 3$ the fission of solitary waves occurs only by decreasing the depth of the liquid, whereby h_1/h_2 increases with We_1. In the region $We_1 < 3$, where the effect of capillarity prevails over that of gravity, the fission occurs by decreasing as well as by increasing the depth of the liquid! What is more, since the ratios $h_1/h_2 > 1$ are very great and almost impracticable (s. Fig. 1), the fission in this region practically occurs by increasing the depth of liquid. In this region too, h_1/h_2 increases with We_1. The diagram for $n = 1$ Fig. 1 reveals the kind of fission in only one solitary wave emerges on the shelf. Only depression-like ($We_1 < 3$) solitary waves can desintegrate in this way. This fission occurs also by decreasing (for $We_1 > 1/3$) as well as by increasing (for $We_1 < 1/3$) the depth of the liquid and can be physically explained in the following manner. By passing over the shelf the solitary wave is transformed to some degree. For $n = 2, 3, \ldots$ such a transformed solitary wave represents for the $K - dV$ equation valid on the shelf a perturbation which eventually evolves into 2, 3, ... new solitary waves, while for $n = 1$ it fits exactly into the one-solitary wave solution of this equation. This kind of fission can be more precisely elucidated using the procedure employed by Ono [6].

Finally, a comment concerning the viscous damping of capillary-gravity olitary waves will be given. In [1] some difficulties in producing the depression-like capillary-gravity solitary waves which are possible on water only for sdepths less than 4,8mm are emphasized, since the waves are „strongly affected by viscosity, so that experimental checks on perfect-fluid predictions are generally difficult" but the wave amplitude with which the experimental checks were carried out is not quoted. The effect of viscosity, however, depends on the wave amplitude. For purely gravity solitary waves, it was analyzed in detail) by Kakutani and Matsuuchi [10]. They show that solitary waves are not affected· by viscosity if:

$$\frac{\nu}{h_0 (gh_0)^{1/2}} < 0\,(\varepsilon^{5/2}), \qquad (7$$

where ν is the kinematic viscosity. It can be easily shown that this condition also holds for capillary-gravity solitary waves for every Weber number. Really, the condition (7) can be hardly satisfied for a water layer shallower than 4,8 mm, even if ε is relatively great, say $\varepsilon = 0{,}1$. For mercury, however, for which the depression-like solitary waves are possible for depths less than 3,3 mm, the condition (7) holds if $\varepsilon = 0{,}1$, so that a depression-like solitary wave, which amplitude is (only!) 0,3 mm, on a mercury layer 3 mm deep should not suffer the viscous damping!

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DIE FISSION DER SOLITAEREN KAPILLAR-SCHWEREWELLEN AN DER SCHWELLE

ZUSAMMENFASSUNG

In der vorliegenden Arbeit wird die Fission der solitären Kapillar-Schwerewellen an der Schwelle mit Hilfe der Korteweg-de Vries-Gleichung mit den variablen Koeffitienten behandelt. Es zeigt sich, dass die konvexe/konkave solitäre Welle in eine disintegrieren kann, d. h. die Umwandlung der konvexen solitären Welle in die konkave Welle und umgekehrt ist an der Schwelle nicht möglich. Während die Fission der konvexen solitären Welle nur bei der abnehmeden Flüssigkeitstiefe vorkommt, kann die Fission der konkaven solitären Welle sowohl bei der abnehmenden als auch bei der zunehmenden Flüssigkeitstiefe auftreten.

FISIJA KAPILARNO-GRAVITACIONIH SOLITARNIH TALASA NA PRAGU

IZVOD

U radu se fisija kapilarno-gravitacionih solitarnih talasa na pragu razmatra pomoću jednačine Korteweg-de Vries-a sa promenljivim koeficijentima. Pokazuje se da se jedan konveksan/konkavan solitarni talas može dezintegrisati isključivo u niz konveksnih/konkavnih solitarnih talasa, tj. da preobražaj konveksnog solitarnog talasa u konkavan i obrnuto, nije moguć na pragu. Dok se fisija konveksnog solitarnog talasa događa samo pri smanjenju dubine tečnosti, dotle se fisija konkavnog solitarnog talasa može dogoditi pri smanjenju, a takođe i pri povećanju dubine tečnosti.

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