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ON STABILITY OF APPROXIMATE SOLUTIONS FOR AN EQUATION WITH A SMALL PARAMETER

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1. Introduction

There is a considerable research about methods of solution of nonlinear equations with a small parameter due to their importance for theoretical physics and mechanics (especially for vibration theory). Nevertheless, some important questinos in the application of these methods still have not been answered and one of them is their applicability on infinite time interval. However, for an examination of stability of motion of a system it is of fundamental importance to see is it possible to extend time interval of a solution into infinity and to estimate discrepancy between a solution and the desired motion of the system.

More precisely, the task is in the folowing. A seeking to find a solution by the „small parameter method" consists in an improvement of the fundamental solution which is obtained from the original when the small parameter vanishes. The procedure is correct only if in the vicinity of the fundamental solution there exsists exact solution of the system. If exact solution is continuously dependent on inital conditons then applicabliity conditions of the small parameter method are known [5] but obtained solution holds on a finite time interval. An extension of the upper bound of this interval to the infinity may be considered as a kind of stability problem. However, this stability is not covered by Liapounov definition since undisturbed motion (basic solution) and disturbed motion are not solutions of same equation. Still, there exists a special kind of stability defined by M. Bertolino [2] called „almost-stability of a function" which is not necessarily a solution of the equation. As far as we know, a criterion for this stability is not known until now.

In this paper we propose a procedure for examanition of Bertolino stability for the case when the disturbed motion is an exact solution of the equation with a small parameter. For this purpose the averaging method theoretically founded by Krilov, Bogoljubov and Mitropolski [5, 6] has been used without additional explanations. Details and full explanalation of these calculations are explicitly given in [5, 6].

2. Application of Bertolino stability on systems with a small parameter

Consider a system of differential equations

$$\dot{x} = A(t)\,x + f(t) + \varepsilon\,\Phi(t, x) \tag{1}$$

where are

$(R \equiv (-\infty, \infty),\ R^+ \equiv [0, \infty))$:

$x \in R^n$ — n-dimensional state vector,

$A(t)$ — a square matrix of order n with elements a_{ij}: $R_+ \to R$,

$t \in R^+ = a$ scalar variable (usually chosen as time),

$f : R^+ \to R^n$ — a vector function of the scalar variable,

$\Phi : R^+ \times R^n \to R^n$ = another vector function of the scalar variable and the state vector,

ε — a small scalar parameter

$\dot{x}$ — the time derivative of the state vector.

In the sequel we assume that the vectorial equation (1) fulfils existence as well as uniqueness conditions.

The systems (1) may represent, for instance, differential equations of motion of a mechanical system in the vicinity of a stationary motion or about an equilibrium position. Therefore, its analysis should be interesting for an application in mechanics or elsewhere. Of course, there exists a whole sequence of other procsses which can be adequately described by the vectorial equation (1).

In the linear approximation (1) simplifies into

$$\dot{x} = A(t)\,x + f(t) \tag{2}$$

whose Cauchy's solution satisfying the initial condition $t_0 = 0$, $x_0(0) = x_0$ amounts to

$$x(t(= X(t)\,x_0 + \int_0^t X(t)\,X^{-1}(\tau)\,f(\tau)\,d\tau \tag{3}$$

where $X(t)$ is the normalized fundamental matrix of the homogeneous equation

$$\dot{x} = A(t)\,x \tag{4}$$

related to (2).

Let

$$x = x(t, x_0) \tag{5}$$

be the solution of the differential equation (1) which satisfies the inital condition $t_0 = 0$, $x(0) = x_0$. Taking into account smallness of ε, it is reasonable to state the question how to estimate properly the discrepancy between the exact solution (5) and the function (3) on the real positive axis R^+. This question becomes of essential value if the method of small parameter should be applied in order to obtain the approximate solution of (1). Namely, by this method an improvement of the solution (3) is performed and this procedure is correct only if in the proximaty of (3) there exists the solution (4) of (1). Thus, the distance betwen (3) and (5) must be estimated. The task can be extended in the following way. Let

$$\hat{x} = x(t, \hat{x}_0) = \dot{x}(t) \tag{6}$$

be an arbitrary motion of the system (1) correspoding to a new initial condition $\hat{x}(0) = \hat{x}_0$ where $||\hat{x}_0 - x_0||$ is small (in other words $\hat{x}_0$ belongs to the vicinity of x_0). Then the extended task is to establish an estimation of discrepancy between (6) and (3) on R^+. As a special case, if $\hat{x}_0 = x_0$ the former problem of discrepancy between (3) and (5) on R^+ is obtained. Since the difference betwen x_0 and $\hat{x}$ may be understood as a disturbance of initial conditions a comparison of (3) and (6) on R^+ is a stability problem. However, this is not a Liapounov stability problem because the function (3) is not a solution of (1). Nevertheless, a proper kind of stability correspoding to the stated extended problem can be definied and a suitable definition is found in [2]. Definition (M. Bertolino). A function $\psi(t) = O, \forall\, t \in R^+$

is an almost stable approximate solution of the equation

$$\dot{x} = F(t, x), \tag{a}$$

$x \in R^n$, $F : R^+ X\, R^n \to R^n$ if to any arbitrary $\varepsilon \geqslant l > 0$

(l — fixed number) there corresponds $\delta(\varepsilon, t_0) > 0$ such that for any solution $x(t)$ of (a) for which

$$||x(0) - \psi(0)|| < \delta \tag{b}$$

holds, the inequality

$$||x(t) - \psi(t)|| < \varepsilon \tag{c}$$

is satisfied for all $t \geqslant 0$.

According to this definitios the stated problem maj be reformulated as: to examine conditions under which the function (3) is an almost stable approiximate solution of the differential equation (1).

If as a motion (3) is taken and the disturbance is denoted by ξ ($\xi \in R^n$) then the disturbed motion (denoted by $\hat{x}$) is given by

$$\hat{x}(t) = X(t)\, x_0 + \int_0^t X(t)\, X^{-1}(\tau)\, f(\tau)\, d\tau + \xi(t). \tag{7}$$

Since the perturbed motion has to fulfil (1) replacement of (7) into (1) gives the differential equation for disturbance

$$\dot{\xi} = A(t)\,\xi + \varepsilon\, S(t, \xi) \tag{8}$$

where

$$S(t, \xi) \equiv \Phi\left[t, X(t)\, x_0 + \int_0^t X(t)\, X^{-1}(\tau)\, f(\tau)\, d\tau + \xi(t)\right].$$

It is worth noting that in general a vanishing of disturbance, $\xi = 0$, is not a solution of (8). In this way the term $S(t, \xi)$ in (8) acts as a function of permanent perturbations.

However, the stability in the sense of Bertolino's definition is not the same as the stability in the presence of permanent perturbations (cf [7]. First, an almost stable approximate solution allows that there exists its vicinity $||\xi|| < l$ without disturbed motions (ie. that it is not possible to approach an undisturbed motion by means of some disturbed motion in arbitrary way) and this is not the case in the definition of a stable solution in the presence of permanent perturbations. On the other hand, in order to have a motion stable in the presence of permanent perturbations it is assumed that for $||\xi|| < \delta$ the absolute value of exication $||S(t, \xi)||$ is small enough for all t and this is not fulfilled in our case.

Now, in order to solve the above stated problem, let us find an abitrary solution of the equation (8). For this purpose we introduce a new variable $s \in R^n$ by the substitution

$$\xi = X(t)\, s \tag{9}$$

Differentiating this equality with respect to time and taking into account (cf. (3), (4), (8) and (9) for $f = 0$) the relation

$$\dot{X}(t) = A(t)\, X(t) \tag{10}$$

(8) is tranformed into the differential equation

$$\dot{s} = \varepsilon\, X^{-1}(t)\, S(t, X(t)\, s) \tag{11}$$

for $s(t)$. It is a straightforward matter to see that for $\varepsilon = 0$ the solution $s = c =$const follows so that in this case $X(t)\, c$ represents the solution of the homogeneous part of the equation (2). According to the averaging method (cf. [5, 6]) a solution of (11) could be seeked in the form

$$s = \sigma + \varepsilon\, u(\sigma, t) \tag{12}$$

where the variable σ obeys the following differential equation

$$\dot{\sigma} = \varepsilon\, S_1(\sigma) + \varepsilon^2\, S_2(\sigma) + \dots \tag{13}$$

while functions u, S_1, $S_2, \ldots$ should be additionally determined. If we consider only the first approximation of (13), then (11) and (12) will give

$$\dot{\sigma} = \varepsilon\, S_1(\sigma) \tag{14}$$

where

$$S_1(\sigma) = \lim_{T \to \infty} \frac{1}{T} \int_0^T X^{-1}(t)\, \Phi\,[t, X(t)\, \sigma]\, d\, t$$

and integration is performed only on the explicit t. A solution

$$\hat{\sigma} = \hat{\sigma}\,(t, \varepsilon) \tag{15}$$

of (14) represents an approximation of the correspoding exact solution of (11) being correct up to the order of ε on the interval $[0, 1/\varepsilon]$ (cf. [5, 6]).

The following question is now in order: is the approximation (15) valid also on the whole interval R^+. Paying attention to the fact that (11) may be approximately written in the form

$$\dot{\sigma} = \varepsilon\, S_1(\sigma) + \varepsilon\, 0\,(\varepsilon)$$

it is clear that by rejecting the term $0\,(\varepsilon)$ a solution of the reduced equation is obtained with an error which can infinitely increase with time (when $t \to +\infty$). Therefore, it is necesary to examine conditions on the equation (11) in order to keep the error in the prescribed boundaries.

Suppose that:

1. the equation (14) has an asymptotically stable solution $\sigma = 0$,

2. the function

$$u\,(s, t) = \int_0^t \{X^{-1}(\tau)\, S\,[\tau, X(\tau)\, s] - S_1\, s)\}\, d\,\tau \tag{16}$$

is continuous and uniformly bounded on R^+

as well as its derivatives with respect to σ and t.

Now, let $s\,(t, \varepsilon)$ be a solution of (11). Then, replacing it into (12) and differentiating so obtained equality in the sense of the equation (11) we obtain

$$\varepsilon\, X^{-1}(t)\, S\,[t, X(t)\, s] = \dot{\sigma} + \varepsilon \frac{\partial u}{\partial \sigma}\, \dot{\sigma} + \varepsilon \frac{\partial u}{\partial t}. \tag{17}$$

If in the above equation u (which was undetermined until now) is replaced by (16), we get

$$\dot{\sigma} = \varepsilon\, S_1(\sigma) - \varepsilon \frac{\partial u}{\partial \sigma} |\varepsilon\, S_1(\sigma) + 0\,(\sigma)| \tag{18}$$

Due to the assumptions included in the theorem it is easy to show that for arbitrary $\Delta > 0$ there exists always $\delta > 0$ such that

$$\left\| \frac{\partial u}{\partial \sigma} [\varepsilon S_1 (\sigma) + 0 (\sigma)] \right\| < \Delta$$

if $||\sigma|| < \delta$. Hence, (18) is a differential equation of a perturbed motion (whith respect to $\sigma = 0$) in the presence of permanent small perturbation

$$\lambda (\sigma, t, \varepsilon) = \frac{\partial u}{\partial \varphi} [\varepsilon S_1 (\sigma) + 0 (\varepsilon)]. \tag{19}$$

It is known (cf. [7]) that the asymptotically stable solution $\sigma = 0$ of the autonomous system (14) is stable also in the presence of permanent perturbation (19). Consequently, if assumptions 1. and 2. are valid, the solution of (14) will aproximate the correspoding solution of (11) with accuracy to the order ε on the whole R^+.

Furthermore, it is easy to see that from the stability of the solution $s = 0$ of the equation (11) follows the stability of the function (3) in the sens of Bertonilo's definition under the condition that $X(t)$ is a uniformly bounded matrix on R^+.

3. Conclusion

By means of the procedure presented in the paper conditions under which the function (3) can be almost stable approximate solution of the system (1) with a small parameter are determined. If these conditions are fulfilled then an approximate solution on R^+ may be determined by the small parameter method. Stability problem of instationary nonlinear systems is very interesting for theory as well as for practice. Commonly used method for stability examination by means of linearized system in the case of instationary systems demands a special caution. Namely, for nonlinear systems as a peculiarity appear nonlinear resonance modes. By linearization trace of these resonances disappears and cannot be detected. It is known that for nonlinear systems also a phenomenon known as ''spatial instability'' could appear [8, 9] and cannot by analized by means of the linearized model.

Results of this paper have been used for conditions of spatial stability of a car as a multidegree-of-freedom system. These conditions are used for an optimization of some geometric and dynamic characteristics of the considered car.

Naturally, it is clear that the proposed method may be used for estimation of stability of other mechanical and nonmechanical systems described by ifferential equations (1).

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SUR LA STABILITE DES SOLUTIONS APPROXIMATIVES D'UNE EQUATION A PETIT PARAMETRE

On considère un système dynamique général non autonome non linéaire à petit paramètre. Ce système peut, par exemple, représenter les équations différentalles du mouvement d'un systéme dynamique au voisinage du mouvement stationnaire ou de l'équilibre.

Outre la solution exacte on introduit les solutions du système linéarisé et quasilinéarisé, dans ce dernier cas elles ont la forme de séries potetielles du petit paramétre. Dans la définition clasique de la stabilité de Liapounov la perturbation représente des solutions pour différentes conditions initiales. Cette définition n'est pas appropriée pour l'analyse de la différence entre les solutions exactes et approchées lors de la résolution approximative du problème du mouvement des systèmes dynamipues réels. C'est pourquoi on applique ici la généralisation 2 de Bertolino de la définition de Liapounov aux différentes solutions conditions initiales. Les auteurs ne possident pas d'informations quant à l'application de la définition de Bertolino ou problème de la stabilité des systèmes dynamiques.

Les conditions suffisantes pour la stabilité d'un système au sens de la définition de Bertolino. ont été formalées.

O STABILNOSTI PRIBLIŽNIH REŠENJA JEDNAČINE SA MALIM PARAMETROM

U radu se posmatra opšti dinamički neautonomni nelinearni sistem sa malim parametrom. Ovakav sistem može, na primer, da reprezentuje diferencijalne jednačine kretanja nekog mehaničkog sistema u okolini stacionarnog kretanja ili ravnoteže.

Pored tačnog rešenja uvode se i rešenje linearizovanog kao i kvazilinearizovanog sistema pri čemu se rešenje kvazilinearizovanog sistema daje u obliku stepenog reda po malom parometru. U klasičnoj Ljapunovljevoj definiciji stabilnosti poremećaj predstavlja razliku istog rešenja pri različitim početnim uslovima. Ova definicija je nepodesna za ocenu različitosti tačnog i približnog rešenja pri približnom rešavanju problema kretanja realnih dinamičkih sistema. Zbog toga se u radu koristi Bertolinova [2] generalizacija Ljapunovljeve definicije na različita rešenja i različite početne uslove. Koliko je autorima poznato Bertolinova definicija nije dosad primenjivana na problem ispitivanja stabilnosti dinamičkih sistema.

U radu su definisani dovolni uslovi za stabilnost sistema u smislu Bertolinove definicije.

Razvijena metoda se veoma uspešno primenjuje na ispitivanje stabilnosti stacionarnog kretanja realnih mehaničkih sistema kada ovo stacionarno kretanje nije tačno rešenje diferencijalnih jednačina kretanja [9].

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