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THE DIRECT METHOD OF DEVELOPMENT OF FINITE ELEMENTS

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1. Introduction

In the finite element method (FEM) the most common way of development of finite elements is the application of some variational principles. The energy in the element and the work of the external load represent a functional. The functional is expressed in terms of some nodal parametars. The variation of the functional on those parametars yields equivalent nodal forces (deformations) and finaly equations of equilibrium (compatibility).

However, in the process of the application of the variational principle it is not allways clear what realy is going on. The essential meaning of the FEM can not be easily seen. For instance, although the equations of equilibrium are completely satisfied, it is not clear why the final stresses along the interelement boundaries on the one side and the other side are not the same. It has been shown that the main reason for that difference is the way of computation of the stresses [1]. The stresses which enter into the equations of equilibrium are computed in one way and the final stresses in another way. The discrepancy between the stresses computed in both ways can be very big. That is partiqulary true in the case of refined elements, with high order deformation shape function. As a result of that discrepancy the final stresses are bad and not allways reliable.

In the case of low order deformation shape function the stresses computed by the application of the variational principle, which enter into the equations of equilibrium, and the stresses computed from the deformation shape function are the same, or the difference is very small. In such case the work of the internal forces (potential energy), can be substituted by the work of the boundary forces. A functional which contains boundary integrals only, for the first time was applied in the development of a mixed rectangular plate bending element [2]. The idea of introduction of boundary integrals only, as a new approach of development of finite elements, was further developed and extended on other problems [3, 4]. The same idea was applied in the development of plane stress elements [5, 6], and recently in the development of shell element.

However, what are the requirements of the deformation shape function (DShF) in order to be possible application of such one concept, in all those papers were not given. Those requirements in short are described in this paper.

In the direct method the problem is not necesserely considered as an energetic one. The boundary stresses (deformations) are computed directly from the deformation shape function and later on trasfered to the nodes. In that way computed nodal forces (deformations) define the element matrix. The application of the method will be ilustrated on the development of one dimensional element, plate bending element, plane stress element an three dimensional element.

2. The deformation shape function

The requirements for the best DShF, which makes possible the application of the direct method, will be analized on the plate bending problem. The diffrential equation of this problem is as follows,

$$\Delta \Delta W = p/D \tag{1}$$

where Δ is the operator $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$, W is normal deflection, p — external load and D — cilindrical stiffness of the plate. The variation of the energy can be defined as follows [7]:

$$\delta U = \int\int (\Delta \Delta W - p)\, \delta W\, dx\, dy + \int M_n \frac{\partial \delta W}{\partial n} ds +$$

$$+ \int M_{ns} \frac{\partial \delta W}{\partial s} da - \int Q^{\text{‘}}\, \delta W\, ds \tag{2}$$

where M_n, M_{ns} and $Q^{\text{‘}}$ are boundary normal moments, twisting moments and shear forces respectively, multiplied by the corresponding deformations. If one wancs this energy variation to give the same nodal forces as the boundary forces give, the first term of the area integral should be equal to zero,

$$\Delta \Delta W = 0 \tag{3}$$

The DShF should satisfy this condition. In other words, the DShF should be the solution of the homogenious differential equation. In that case the energy variation (2) becomes,

$$\delta U = M_n \int \frac{\partial \delta W}{\partial n} ds + \int M_{ns} \frac{\partial \delta W}{\partial s} ds - \int Q_s\, \delta W\, ds - \int\int p\, \delta W\, dx\; dy \tag{4}$$

This equation menas that the work of the internal forces (potential energy) is substituted by the work of the boundary forces. Actually this equation distributes the boundary forces to the corresponding nodes and in that way defines the equivalen nodal forces. The element developed on the base cf the energy variational principle (4) and the element developed by application of the minimum potential energy variational principle should be the same. It means that the boundary forces computed by application of the minimum potential energy variational principle, which are present into the equations of equilibrium, will be the same as the boundary forces computed from the DShF directly (as the final forces are computed). Therefore, the stresses computed from two adjasent elements should be approximately the same.

However, in the case of refined elements, with high order DShF, there will be contribution of the first term of Exp. 2 and the final nodal forces will not correspond to the boundary forces. It means that the forces into the equations will be computed in one way and the final forces in another way. Therefore, although the equations of equilibrium will be completely satisfied, the finaly computed boundary forces will be bad, different from one and the other side of the element boundaries. The reliability of the forces computed in that way is questionable.

As a conclusion, it seems that the best DShF is the one which satisfies the homogenious equation of the problem. Later in the text of this paper will be shown that some modifications in the DShF, which will give improved results, are possible. Those modifications should provide quadratic distribution of the boundary forces.

3. One dimensional element

The solution of the homogenious differential equation of the beam problem is the following third order polinomial,

$$W = a_1 + a_2 x + a_3 x^2 + a^3 x_4^3 \qquad (5)$$

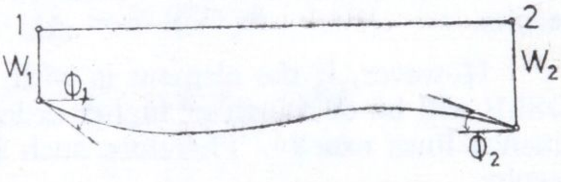

Fig. 1 One Dimensional Beam Element

The 4 coefficients in this expression are easily expressed by the 4 nodal parametars of the element (Fig. 1) and the following expression for the deflections derived,

$$W = W_1 \left(1 - \frac{3x^2}{L^2} + \frac{2x^3}{L^3}\right) + W_2 \left(\frac{3x^2}{L^2} - \frac{2x^3}{L^3}\right) +$$

$$+ \varnothing_1 \left(x - \frac{2x^2}{L} + \frac{x^3}{L^2}\right) + \varnothing_2 \left(- \frac{x^2}{L} + \frac{x^3}{L^2}\right) \qquad (6)$$

By differentiation of this DShF the following boundary forces are derived,

$$M_x = - EI \frac{d^2 W}{dx^2}$$

$$Q_x = - EI \frac{d^3 W}{dx^3}$$

$$M_1 = M_{x=0} = W_1 \frac{6EI}{L^2} - W_2 \frac{6EI}{L^2} + \varnothing_1 \frac{4EI}{L} + \varnothing_2 \frac{2EI}{L}$$

$$M_2 = -M_{x=L} = W_1 \frac{6\,EI}{L^2} - W_2 \frac{6\,EI}{L^2} + Ø_1 \frac{2\,EI}{L} + Ø_2 \frac{4\,EI}{L^2}$$

$$V_1 = -Q_{x=0} = W_1 \frac{12\,EI}{L^3} - W_2 \frac{12\,EI}{L^3} + Ø_1 \frac{6\,EI}{L^2} + Ø_2 \frac{6\,EI}{L^2}$$

$$V_2 = Q_{x=L} = -W_1 \frac{12\,EI}{L^3} + W_2 \frac{12\,EI}{L^3} - Ø_1 \frac{6\,EI}{L^2} - Ø_2 \frac{6\,EI}{L}$$

The coefficients besides the unknown nodal parametars represent the stiffness coefficients of the element matrix. They are the same as in the well known classical slope — deflection method. As it is well known, the classical slope-deflection method gives exact results. Thus, the element developed here, directly from the deformation shape function, will give exact results also, regarless of the number of subdivisions.

The exact results which the element gives are not eccidental, but they are such because the DShF is a third order parabola and represents the solution of the homogenious differential equation. The DShF can be considered as a function which defines the influence lines of the boundary forces: the components associated with W_i define the influence lines for the shear forces Q_i, and the components with $Ø_i$ — the influence lines for the moments M_i. As it is well known, the influence lines for a beam are third order polinomials. Thus, the DShF defines the influence lines exactly. Therefore the element of Fig. 1 allways gives exact results.

However, if the element is with 5 or more degrees of freedom (d.o.f.), the DShF will be of fourth or higher order polinomial and will not represent the influence lines exactly. Therefore such one „refined" element will not give exact results.

The deformation shape function can be considered as a virtual displacement which satisfies the boundary conditions. By application of the principle of virtual work in this case of one dimensional problem, with the DShF (4), the exact results are derived.

The consideration of the DShF as an approximate solution of the problem is wrong and has missled to the development of refined elements. The equations of equilibrium are writen for the nodes and the final results are valid for the nodes only. Inside the element there is no solution. The DShF can represent an approximate solution inside the element, but better not.

4. Plate bending element

The governing differential equation of the plate bending problem was given by Eq. 1. As was mentioned, in order to get same boundary forces as the DShF gives directly, the DShF should satisfy the condition (3). The following DShF satisfies that condition,

$$W = a_1 + a_2 x + \ldots + a_{11} x^3 y + a_{12} x y^3 \tag{7}$$

where the first 10 terms represent a full third order polinomial. With such DShF it is possible to apply the variational principle (4) and the direct method of development of FE.

The application of the direct method of development of FE will be ilustrated on the development of the simple rectangular mixed element of Fig. 2. The unknowns (d.o.f.) of the element are the displacements W_i and the bending moments M_{xi}, M_{yi} (second derivatives) at the nodes. The DShF (7) expressed in terms of those nodal parametars becomes,

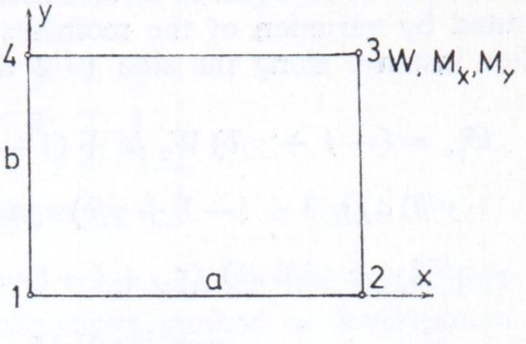

Fig. 2 Plate Bending 12 Degrees of Freedom Element

$$W = W_1\left(1 - \frac{x}{a} - \frac{y}{b} + \frac{xy}{ab}\right) +$$

$$+ W_2\left(\frac{x}{a} - \frac{xy}{ab}\right) + W_3\frac{xy}{ab} + W_4\left(\frac{y}{b} - \frac{xy}{ab}\right) + \bar{M}_{1x} f_1 + \bar{M}_{2x} f_2 +$$

$$+ \bar{M}_{3x} f_3 + \bar{M}_{4x} f_4 + \bar{M}_{1y} f_5 + \bar{M}_{2y} f_6 + \bar{M}_{3y} f_7 + \bar{M}_{4y} f_8 \tag{8}$$

where,

$$\bar{M}_x = \partial^2 W/\partial x^2, \quad \bar{M}_y = \partial^2 W/\partial y^2$$

$$f_1 = -\frac{xa}{3} + \frac{axy}{3b} + \frac{x^2}{2} - \frac{x^3}{6a} - \frac{x^2 y}{2b} + \frac{x^3 y}{6ab}$$

$$f_2 = -\frac{ax}{6} + \frac{axy}{6b} + \frac{x^3}{6a} - \frac{x^3 y}{6ab}$$

$$f_3 = -\frac{axy}{6b} + \frac{x^3 y}{6ab}$$

$$f_4 = -\frac{axy}{3b} + \frac{x^2 y}{2b} - \frac{x^3 y}{6ab}$$

The functions $f_5 \div f_8$ are simmilar to $f_1 \div f_4$ and can be defined by analogy. The element matrix can be defined as follows,

$$K_e = \begin{bmatrix} F & F_k \\ F_k^T & K_k \end{bmatrix} \tag{9}$$

The matrix of the unknowns is as follows,

$$\delta^T = [\bar{M}_{1x} \cdot \cdot \bar{M}_{4x}, \bar{M}_{1y} \cdot \cdot \bar{M}_{4y}, W_1 \cdot \cdot W_4]$$

The first row of the element matrix represents flexibility portion of the element matrix, which defines the compatibility equations. The slopes from one side of the interelement boundary have to be equal to the slopes of the other side and their summ has to be equal to zero. Those slopes, in the standard FEM procedure computed by variation of the moments, can be computed directly from the DShF. For instance along the side 1—4 the following slopes are obtained,

$$\varnothing_{14} = (-1 + y/b)\, W_1/a + (1 - y/b)\, W_2/a + yW_3/ab - yW_4/ab + (-1 + y/b)\, a\bar{M}_1/3 + (-1 + y/b)\, a\bar{M}_{2x}/6 - ya\bar{M}_{3x}/6b - ya\bar{M}_{4x}/3b + (by/3a - y^2/2a + y^3/6ab)\,\bar{M}_{1y} + (-by/3a + y^2/2a - y^3/6ab)\,\bar{M}_{2y} + (-by/6a + y^3/6ab)\,\bar{M}_{3y} + (by/6a - y^3/6ab)\,\bar{M}_{4y}$$

Now these rotations have to be concentrated at the nodes 1 and 4. Their distribution is linear and therefore the equivalent nodal rotation, for instance at node 1, is obtained as follows,

$$\varnothing_1 = \int_0^b \varnothing_{14}(1 - y/b)\, dy = -W_1 b/3a + W_2 b/3a + W_3 b/6a - W_4 b/6a - \bar{M}_{1x}\, ab/9 - \bar{M}_{2x} ab/18 - \bar{M}_{3x}\, ab/36 - \bar{M}_{4x}\, ab/18 + \bar{M}_{1y}\, b^3/45a - \bar{M}_{2y}\, b^3/45a - \bar{M}_{3y}\, 7b^3/360a + \bar{M}_{4y}\, 7b^3/360a \qquad (10)$$

The coefficiets besides W_1 in this equation define the submatrix F_k and those besides $\bar{M}_{ix}$ $\bar{M}_{iy}$ — the submatrix F. Instead of the second derivatives $\bar{M}_x$ and $\bar{M}_y$ can be substituted the real bending moments, according to the following relations,

$$\bar{M}_x = -\frac{1}{D(1-\nu^2)}(M_x - \nu M_y)$$

$$\bar{M}_y = -\frac{1}{D(1-\nu^2)}(-\nu M_x + M_y)$$

The submatrix F can be subdivided as follows,

$$F = F_0 + \Delta F$$

$$F_0 = \begin{bmatrix} F_{ox} & -\nu F_{ox} \\ -\nu F_{oy} & F_{oy} \end{bmatrix}$$

$$\Delta F = \begin{bmatrix} \Delta F_{11} & \Delta F_{12} \\ \Delta F_{21} & \Delta F_{22} \end{bmatrix}$$

The submatrix F_o is the same as in the previously developed elements with independent assumption of the moments and deflections [8, 9]. The expression (10) defines the first row of that submatrix. By analogy can be defined all rows of he matrix. The values of that submatrix are as follows,

$$F_c = \frac{ab}{36\,D\,(1-\nu^2)} \begin{bmatrix} 4 & 2 & 1 & 2 \\ & 4 & 2 & 1 \\ & & 4 & 2 \\ \text{Symm.} & & & 4 \end{bmatrix}$$

The submatrix ΔF appears as additional and it is due to the uncompatibility of the rotations along the boundaries. Thus, the direct method of development of finite elements applied here, authomatically takes into account the uncompatibility. According to Exp. 10, the values of this submatrix are as follows,

$$\Delta F_{11} = \frac{\nu\, b^3/a}{360\,D\,(1-\nu^2)} \begin{bmatrix} 8 & -8 & -7 & 7 \\ & 8 & 7 & -7 \\ & & 8 & 7 \\ \text{Sym.} & & & 8 \end{bmatrix};$$

$$\Delta F_{12} = \frac{b^3/a}{360\,D\,(1-\nu^2)} \begin{bmatrix} -8 & 8 & 7 & -7 \\ & -8 & -7 & 7 \\ & & -8 & -8 \\ \text{Sym} & & & -8 \end{bmatrix};$$

$$\Delta F_{21} = \frac{a^3/b}{360\,D\,(1-\nu^2)} \begin{bmatrix} -8 & -7 & 7 & 8 \\ & -8 & 8 & 7 \\ & & -8 & -7 \\ \text{Sym.} & & & -8 \end{bmatrix};$$

$$\Delta F_{22} = \frac{\nu\, a^3/b}{360\,D\,(1-\nu^2)} \begin{bmatrix} 8 & 7 & -7 & -8 \\ & 8 & -8 & -7 \\ & & 8 & 7 \\ \text{Sym} & & & 8 \end{bmatrix}.$$

It is interesting to note that $\Delta F_{12} \neq \Delta F_{21}^T$. As a result of that $F_{ij} \neq F_{ji}$. It means that the Maxwell's rull in this case doesn't hold completely. A simple explanation of that unusual appearance is that the DShF (8) is not the real solution of the problem. That function should not give uncompatable deormations. Because of practical reasons, in the numerical analysis instead of ΔF_{12} and ΔF_{21} an average submatrix $(\Delta F_{12} + \Delta F_{21})/2$ was used [10]. Such approximation should have some effects when a and b are much different.

The submatrix F_k can be detected from Exp. 10. The values of that submatrix are as follows,

$$F_k = \begin{bmatrix} F_{kx} \\ F_{ky} \end{bmatrix}$$

$$F_{kx} = \frac{F_{kb}}{6\,a} \begin{bmatrix} -2 & 2 & 1 & -1 \\ & -2 & -1 & 1 \\ & & -2 & 2 \\ \text{Sym.} & & & -2 \end{bmatrix};$$

$$F_{ky} = \frac{a}{6\,b} \begin{bmatrix} -2 & -1 & 1 & 2 \\ & -2 & 2 & 1 \\ & & -2 & -1 \\ \text{Sym.} & & & -2 \end{bmatrix}.$$

The submatrix K_k represents stiffness submatrix. The portion of the DShF (8), which is function of the deflections W_i, defines a twisted surface. There are constant twisting moments only, which give constant twisting moments at the boundaries and finaly concentated nodal forces, which are,

$$Q_1 = 2\,M_{xy} = -\,2D\,(1 - \nu)\,\frac{\partial^2 W}{\partial x\,\partial y} = -\,\frac{D\,(1 - \nu)}{ab}\,(W_1 - W_2 + W_3 - W_4)$$

This expression defines the first row of the submatrix K_k. The complete submatrix is as follows,

$$K_k = \frac{D\,(1 - \nu)}{ab} \begin{bmatrix} -1 & 1 & -1 & 1 \\ & -1 & 1 & -1 \\ & & -1 & 1 \\ \text{Sym.} & & & -1 \end{bmatrix}$$

In that way the complete element matrix (9) is defined.

The DShF (8) gives a twisting moment component M_{xy} which is a function of the bending moments M_x, M_y, $-\,M_{xy} = f\,(M_x, M_y)$. The application of an energy approach, as the variational principle (2), results in additional submatrix $\Delta\,F_1$, which is due to that twisting moment component M_{yx}. The element of Ref. 2 is with such submatrix $\Delta\,F_1$. That submatrix is with summ of the coefficients in all rows equal to zero and their values are much smaller of those of F_1. As a result of that, the element of Ref. 2, as well as the previously developed element with ($-\,\Delta\,F_1$) give converging results. On the other hand the direct method applied here gives $\Delta\,F_1 = 0$. This twisting moment components represents a problem in the development of stiffness elements in the standard way, which is solved by application of reduced Gauss integration. This problem needs further investigations.

One disadvantage of the presented element is the linear distribution of the moments. A much better distribution would be a quadratic distribution of the moments along the boundaries. The following deformation shape function provides a higher order moment distribution,

$$W = a_1 + a_2 x + \ldots + a_{13} x^2 y^2 + a_{14} y^3 + a_{15} x^2 y^3 + a_{16} x^3 y^3 \tag{12}$$

where the first 12 terms are the same as (7). For definition of the additional 4 terms in (12) on the element of Fig. 2 as unknowns should be added the normal moments at the midsides.

The DShF (12) doesn't satisfy the condition (3). There is $\partial^2 W / \partial x^2 y^2 \neq 0$. Therefore the variational principle (2) has to be applied. As a result of that, the boundary forces which the variational principle (2) gives, and the boundary forces computed from the DShF directly will not be the same. However, if the twisting moment components $M_{xy} = f(M_x, M_y)$ are excluded, the boundary forces computed in both ways will be approximately the same and the element developed on the base of the DShF (12), by application of the variational principle (4) or the direct method, should be a very good one. Simmilar element in the plane stress problem has given very good results [6].

5. Plane stress element

The governing differential equation of the plane stress problem in terms of the stress function $\varnothing$ is as follows,

$$\Delta \Delta \varnothing = 0 \tag{13}$$

That is the same, biharmonic equation, as the homogenious differential equation of the plate bending problem. The normal stresses N_x, N_y of this problem are equivalent to the bending moments M_x, M_y of the plate bending problem. Therefore, the condition (13) requires linear variation of the normal stresses, as the condition (3) requires linear variation of the bending moments. The following DShF provides such linear variation of the normal stresses,

$$U = a_1 + a_2 x + a_3 y + a_4 x^2 + + a_5 xy + a_6 x^2 y$$

$$\tag{14}$$

$$V = b_1 + b_2 x + b_3 y + b_4 y^2 + + b_5 xy + b_6 xy^2$$

These DShFs can be defined by the normal stresses N_x, N_y at the corner nodes (4 × 2) and the midside node displacements (2 × 2), Fig. 3.

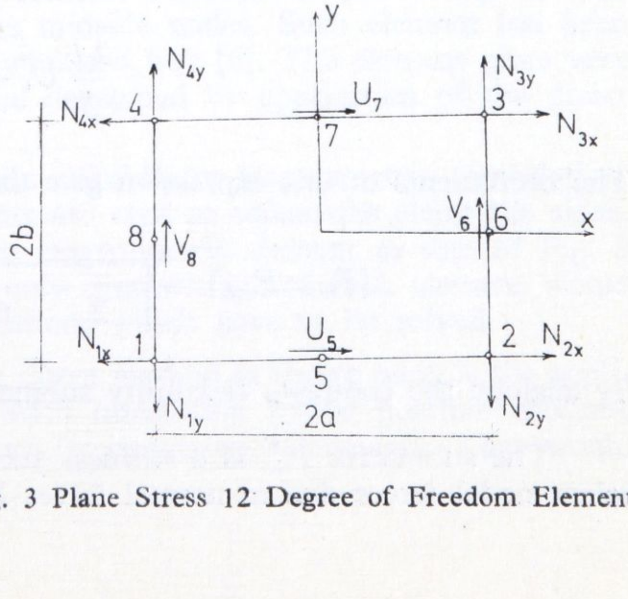

Fig. 3 Plane Stress 12 Degree of Freedom Element

The DShF (14) for the u component of the element of Fig. 3 becomes,

$$U = \frac{1}{4}\left\{\left[2\,U_5 + \bar{N}_{1x}\left(x - \frac{x^2}{2\,a}\right) + \bar{N}_{2x}\left(x + \frac{x^2}{2\,a}\right)\right]\left(1 - \frac{y}{b}\right) + \right.$$

$$\left. + \left[2\,U_7 + \bar{N}_{3x}\left(x + \frac{x^2}{2\,a}\right) + \bar{N}_{4x}\left(x - \frac{x^2}{2\,a}\right)\right]\left(1 + \frac{y}{b}\right)\right\} \qquad (15)$$

where $\bar{N}_x = \partial\,U/\partial\,x$ is the strain. Simmilar is the DShF for the v component of the deformations. The element matrix and the matrix of the unknowns can be defined as follows,

$$K_e = \begin{bmatrix} F_n & F_{uv} \\ F_{uv}^T & K_{uv} \end{bmatrix}$$

$$\delta^T = [\,\bar{N}_{1x} \cdot\cdot\, \bar{N}_{4x}, \bar{N}_{1y} \cdot\cdot\, \bar{N}_{4y}, U_5, U_7, V_6, V_8] \qquad (16)$$

If in Eq. 15 is substituted, for instance $\bar{N}_{1x} = 1$ and all other parametars equal to zero, for $x = -\,a$ the following deformations along the side 1—4 are obtained,

$$U_{14} = \frac{1}{4}\left\{\left[2\,U_5 \frac{3\,a}{2}\,\bar{N}_{1x} - \frac{a}{2}\,\bar{N}_{2x}\right]\left(1 - \frac{y}{b}\right) + \left[2\,U_7 - \frac{a}{2}\,N_{3x} - \right.\right.$$

$$\left.\left. - \frac{3\,a}{2}\,\bar{N}_{4x}\right]\left(1 - \frac{y}{b}\right)\right\}$$

Now these deformations have to be concentrated at the nodes 1 and 4 and in that way the equivalent nodal deformations found. The distribution of the deformations is linear and therefore the equivalent nodal deformation, for instance at the node 1 is as follows,

$$U_1 = \int \frac{1}{2}\,U_{14}\left(1 - \frac{y}{b}\right) \cdot dy = \frac{ab}{12}\,(-\,6\,\bar{N}_{1x} - 2\,\bar{N}_{2x} - \bar{N}_{3x} - 3\,\bar{N}_{4x}) +$$

$$+ \frac{2\,b}{3}\,U_5 + \frac{b}{3}\,U_7$$

The coefficients of this expression give the first row of the element matrix (16),

$$[F_{nli}, F_{uvli}] = \left[\frac{-\,ab}{2}, \frac{-\,ab}{6}, \frac{-\,ab}{12}, \frac{-\,ab}{4}, \frac{2\,b}{3}, \frac{b}{3}\right] \cdot \qquad (17)$$

By analogy the complete flexibility submatrices can be developed.

The submatrix F_{uv}^T is a stiffness submatrix. Its coefficients represent equivalent nodal forces due to normal forces N_x, N_y. That matrix developed in the energetic way contains shear force components which are function of the normal stresses, $-N_{xy} = f(N_x, N_y)$. On the other hand this submatrix should be transposed submatrix F_{uv}. In order that condition to be satisfied, those shear force components should be,

$$N_{xy} = f(N_x, N_y) = 0$$

The element developed with this condition very well simulates the bending stresses and gives very good results.

The stiffness submatrix K_{uv} can be developed by application of unit displacements, Fig. 4. The application of, for instance $U_5 = 1$, cuases shear stresses, which according to (15) are equal to,

$$\tau_{xy} = G\frac{\partial u}{\partial y} = \frac{E}{2(1+\nu)b}$$

These shear stresses give forces which have to be transfered to the corresponding nodes 5 ÷ 8. Those forces define the coefficients of the first row of the submatrix K_{uv}, as follows,

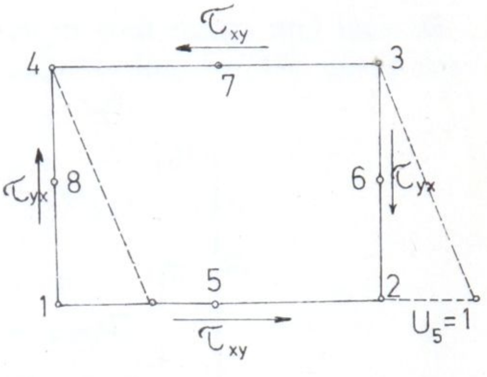

Fig. 4 Development of the Stiffness Submatrix by Application of Unit Displacement

$$K_{\mathrm{uvli.}} = \frac{E}{2(1+\nu)}\left[\frac{a}{b}, \frac{-a}{b}, 1, -1\right]$$

The complete submatrix K_{uv} can be developed by analogy. So, without computation of any energy and variation of it, the element matrix is developed directly from the DShF. The element developed in that way gives very good results [5, 6].

The accuracy which an element can give primarely depends on the offdiagonal terms of the element matrix, or the order of boundary stress distribution [1]. A quadratic distribution of the boundary stresses would be much better than linear distribution as is the distribution of the stresses of the element of Fig. 3. With such quadratic distribution can be the element simmilar to that of Fig. 3, with the two deformation components at the midside nodes. Such element has been developed as isoparametric one, in the standard way [6]. The element gives very good results. The same element can be developed by application of the direct method.

The element of Fig. 3 is not quite suitable for isoparametric formulation. However, if the deformation components are cept as unknowns along the sides, not in x or y directions, the equivalent isoparametric element to that of Fig. 3 should be a very good one also. The only disadvantage of such element would be the unsymmetry of the system equations which have to be solved.

One of the main advantages of the direct method as shown here, is the application of boundary integration only. Such integration makes possible explicite definition of curved boundary elements (isoparametric elements). That would be the final aim.

6. Three dimensional element

The three dimensional elements can be developed in the same way as the plane stress elements: from the DShF directly can be computed the boundary stresses and deformations, which afterwards have to be transfered to the nodes. The equivalent nodal forces and deformations define the element matrix.

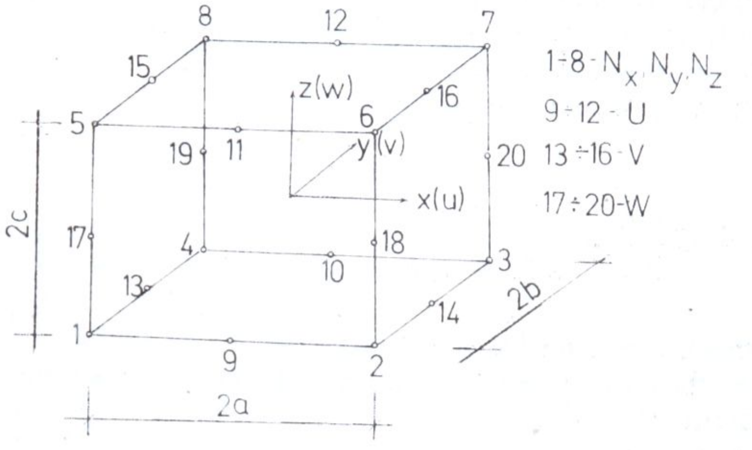

Fig. 5 Three Dimensional 36 Degrees of Freedom Element

The application of the direct method in the three dimensional problem wil be ilustrated on the development of the simple element of Fig. 5. That elemen corresponds to the plane stress element of Fig. 3. At the corner nodes the unknown nodal parametars are the normal stress components and at the midside nodes — the deformation components along the sides. In that way assumed unknown nodal parametars represent independent degrees of freedom. The DShF which is defined by those degrees of freedom will satisfy simmilar condition to that of (13) for the three dimensional problem. According to expression (15) the following DShF for the U component of the three dimensional element is derived,

$$U = \frac{1}{8}\left\{\left[2\,U_9 + \bar{N}_{1x}\left(x - \frac{x^2}{2\,a}\right) + \bar{N}_{2x}\left(x + \frac{x^2}{2\,a}\right)\right](1 - y/b) + \right.$$

$$\left. + \left[2\,U_{10} + \bar{N}_{3x}\left(x + \frac{x^2}{2\,a}\right) + \bar{N}_{4x}\left(x - \frac{x_2}{2\,a}\right)\right](1 + y/b)\right\}(1 - z/c) +$$

$$+ \frac{1}{8}\left\{\left[2\,U_{11} + \bar{N}_{5x}\left(x - \frac{x^2}{2\,a}\right)\right] + \bar{N}_{6x}\left(x + \frac{x^2}{2\,a}\right)\right](1 - y/b) +$$

$$\left. + \left[2\,U_{12} + \bar{N}_{7x}\left(x + \frac{x^2}{2\,a}\right) + \bar{N}_{8x}\left(x - \frac{x^2}{2\,a}\right)\right](1 + y/b)\right\}(1 + z/c) \qquad (19)$$

The element matrix and the matrix of the unknowns can be defined as follows,

$$K_e = \begin{bmatrix} F_n & F_{uvw} \\ F^T_{uvw} & K_{uvw} \end{bmatrix} \tag{20}$$

$$\delta^T = [\bar{N}_{1x} \cdots \bar{N}_{8x}, \bar{N}_{1y} \cdots \bar{N}_{8y}, \bar{N}_{1z} \cdots \bar{N}_{8z}\, U_9 \ldots U_{12},$$
$$V_{13} \ldots V_{16}, W_{17} \ldots W_{20}]$$

The element matrix can be developed by application of unit forces and unit deformations. For instance the application of $\bar{N}_{1x} = 1$, according to (19), gives the following deformed surfaces:

$$\text{— for } x = -a, \quad U = -\frac{3\,a}{16}(1 - y/b)(1 - z/c)$$

$$x = a, \quad U = \frac{a}{16}(1 - y/b)(1 - z/c)$$

The volumes of these deformations $\int U \cdot dy \cdot dz$ are equal to — $3\,abc/4$, — on the aurface 1, 5, 8, 4, and $abc/4$, — on the surface 2, 3, 7, 6 (Fig. 5). These volumes define the summ of the first 8 terms of the submatrix F_n of the element matrix (20). The distribution of these volume deformations to the corresponding nodes gives the coeeficients of the first row of the submatrix F_n. For instance for the node 1 the following equivalent nodal deformation is derived.

$$U_1 = \int_{-b}^{b} \int_{-c}^{c} U(1 - y/b)(1 - z/c) \cdot dy \cdot dz = -\,abc/3$$

In that way computed first colomn, i.e. first row of the submatrix F_n defined as,

$$F_n = [F_{nx}\, F_{ny}\, F_{nz}]$$

is as follows,

$$F_{nxli} = \frac{-\,abc}{36}[12, 4, 2, 6, 6, 2, 1, 3]$$

The other components of the first row of F_n are $F_{nyli} = 0$ and $F_{nzli} = 0$. But if as unknowns are substituted the stresses N_x, N_y, and N_z instead of the strains $\bar{N}_x = \partial u/\partial x$, $\bar{N}_y = \partial v/\partial y)$ and $\bar{N}_z = \partial w/\partial z$, those submatrices will get some values, which will be a function of the Poison's coefficient ν.

The submatrix F_{uvw} can be developed in a simmilar way. For instance the parametar U_9 gives the following surface deformations,

$$\mathrm{U} = \frac{1}{4} U_9 (1 - y/b)(1 - z/c)$$

The volume of these deformations on one surface of the element is as follows,

$$\int U \cdot dy \cdot dz = bc\, U_9,$$

The distribution of this deformation to the corresponding nodes is the same as the distribution of the deformations due to $\bar{N}_{1x}$ previously defined. In that way computed equivalent nodal deformations give the first colomn, i.e. the first row of the submatrix F_{uvw}, which is,

$$F_{uvw1i} = \frac{bc}{9} [4, 2, 2, 1, 0 \ldots 0]$$

The complete submatrices F_n and F_{uvw} can be derived by analogy.

The stiffness submatrix K_{uvw} can be computed in a simmilar way as the submatrix K_{uv} of the plane stress problem, by application of unit nodal deformations and computation of the shear forces along the boundaries. Those shear forces distributed to the corresponding nodes will give the coefficients of the submatrix K_{uvw}. The derivation of that submatrix here is not given.

The three dimensional element developed in that way is not tested yet. The expected accuracy of the element should be very good, as is the accuracy of the corresponding plane stress element of Fig. 3. The element has to be developed with curved boundaries, explicitely defined. The sugestion given for the plane sress element holds here also. The deformations at the midside nodes have to be hold in direction of the sides.

7. Conclusions

The boundary forces and deformations computed directly from the deformation shape function and then distributed to the nodes define the element matrix. That approach is called direct method of development of finite elements.

For the application of that method to be possible, the deformation shape function has to satisfy the homogenious differential equation of the problem with some small modifications. In such case the boundary forces computed by application of a variational principle and directly from the deformation shape function will be approximately the same. Otherwise, as is the case with the refined elements, there is descripancy in the forces computed in the both ways. That discripancy is the main problem in the current practice of the FEM.

The application of the direct method is ilustrated on the development of simple one dimensional, plate bending, plane stress and three dimensional elements. The mixed two dimensional elements developed in that way give very good and allways reliable results.

The method is and simple in the proces of development it is easy to follow what realy is going on. The method involves boundary integration only. Such integration can lead to explicite definition of the isoparametric elements. That would be the final aim.

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DIREKTNI METOD ZA RAZVIJANJE KONAČNIH ELEMENATA

Funkcija deformacija u elementu trebalo bi da zadovoljava rešenje homogene differencijalne jednačine problema, sa mogućim manjim odstupanjima. U takvom slučaju sile (deformacije) po granicama elemenata proračunte direktno iz funkcije deformacija, zatim prenete na čvorove, definiraju matricu elementa. Takav postupak nazvan je direktni metod za razvijanje konačnih elemenata.

Primena direktnog metoda ilustrirana je na razvijanju grednog elementa, elementa za savijanje ploča, elementa za ravninsko stanje naprezanja i trodimenzionalnog elementa.

Elementi razvijeni ovim metodom su vrlo dobri i uvek pouzdani. Ovaj metod može dovesti do razvijanja izoparametrijskih elemenata definiranih explicitno.

ДИРЕКТНИЙ МЕТОД ДЛЯ РАЗВИТИЯ КОНЕЧЫХ ЕЛЕМЕНТОВ

Функция деформаций в елементе должна удоволетворят решению хомогенного диференциаллного уравнения проблемы, с можностю меньшых модификаций. В таком случае силы (деформации) по краям елемента полученные прямо из функции деформации и затем перенесенных в узлах дают матрицу елемента. Такой подход назван директний метод развития конечных елементов.

Применение директного метода показано на развитию балочного елемента, елемента плиты, елемента плоскостного состояния напряжений и обьемного елемента. Елементы развитие етим методом дают очен хорошие и надежные результаты. Еты метод может привести к експлицитному дефинировению изопараметрических елементов.

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