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VARIATIONAL EQUATIONS OF MOTION OF THE MECHANICAL SYSTEM OF VARIABLE MASS AND THEIR INTEGRATION

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The mechanical system of N particles M_i, $(i = 1, 2, \dots, N)$ is observed in the present paper, with masses m_i, whose position in the undisturbed motion in determined by vectors of positions $\vec{r}_i$. If the masses of all system particles constant, then the motion of the sys em is de ermined by N vector differential equations

$$m_i \ddot{\vec{r}}_i = \vec{F}_i (t, \vec{r}, \vec{v}), \quad (i = 1,2, \dots N) \tag{1}$$

The differential equation of the undisturbed motion in the phase space (q^α, p_α), $(\alpha = 1,2, \dots, n > N)$ are according to [1]

$$\dot{q}^\alpha = a^{\alpha\beta} p_\beta, \quad D p_\alpha / dt = Q_\alpha (t, q, \dot{q}) \tag{2}$$

where p_α is the generalised pulse, Q_α the generalised force and D/dt the operatos of the absolute differentiation.

If the disturbance of vectors of the position $\overleftarrow{r}_i$ and speeds $\vec{v}_i = \dot{\vec{r}}_i$, occurs in the mechanical system (1), the new disturbed values of thesə quantities are:

$$\begin{gathered} \vec{r}_i^{\,*} = \vec{r}_i + \rho_i = \vec{r}_i + \xi^\alpha \partial_\alpha \vec{r}_i. \quad \overset{*}{q}^\alpha = q^\alpha + \xi^\alpha \\ \vec{v}_i^{\,*} = \vec{v}_i + \partial_{\alpha\beta} \vec{r}_i \xi^\alpha \dot{q}^\beta + \partial_\alpha \vec{r}_i \dot{\xi}^\alpha \end{gathered} \tag{3}$$

Apart from these quantities $\vec{F}_i$ are changed. too. as well as the generalized pulses $p_\alpha^* = p_\alpha + \eta_\alpha$. We do not go into the fakt eow we ariive at the differential equations of the disturbed motion, but their final form given in [1, 2] is only used.

$$\frac{D \xi^\alpha}{d t} = a^{\alpha\gamma} \eta_\gamma, \quad \frac{D \eta_\alpha}{d t} = \Psi_\alpha \tag{3}$$

These equations (4) are differential equations of the disturbed motion of the mechanical system in the phase space (ξ^α, η_α). In these equations:

$$\eta_\gamma = p^*_\gamma = p_\gamma = a_{\alpha\gamma}\,(\dot{\xi}^\alpha + \Gamma^\alpha_{\delta\beta}\,\xi^\delta\,\dot{q}^\beta) \tag{5}$$

$$\psi_\alpha = \sum_{i=1}^{N} (\vec{F}_i^{\,*} - \vec{F}_i)\cdot\partial_\alpha\,\vec{r}_i \tag{6}$$

In a more general case, when the mechanical system consisting of particles of variable mass $m_i\,(t)$ is considered, in Maščerski's sense, the differential equations of the disturbed motion according to [2] have the follo-ving form:

$$\ddot{\xi}^\gamma + 2\,\Gamma^\gamma_{\alpha\beta}\,\dot{q}^\alpha\,\dot{\xi}^\beta + \dot{q}^\alpha\,\dot{q}^\beta\,\xi^\delta\,\partial_\delta\,\Gamma^\gamma_{\alpha\sigma} =$$
$$\frac{\partial\,Q_\gamma}{\partial\,q^\delta}\,\xi^\delta + \frac{\partial\,Q^\gamma}{\partial\,\dot{q}^\delta}\,\dot{\xi}^\delta + \frac{\partial\,\psi^\varsigma}{\partial\,q^\delta}\,\xi^\delta + \frac{\partial\,\psi^\gamma}{\partial\,\dot{q}^\delta}\,\dot{\xi}^\delta \tag{7}$$

Equations (7) are variational equations of of motion of the zystem of changeable mass in a variable configurational space with:

$$(d\,s)^2 = a_{\alpha\beta}\,dq^\alpha\,dq^\beta, \quad (\alpha,\ \beta = 1, \ldots, n) \tag{8}$$

In the case that equations of motion of the undisturbed motion $q^\alpha\,(t)$ finite, the equation (7) reduces to:

$$\ddot{\xi}^\gamma = A^\gamma_\delta\,(t)\,\xi^\delta + B^\gamma_\delta\,(t)\,\dot{\xi}^\delta, \quad (\gamma,\ \beta = 1, \ldots, n) \tag{9}$$

where:

$$A^\gamma_\delta\,(t) = \frac{\partial\,Q^\gamma}{\partial\,q^\delta} + \frac{\partial\,\psi^\gamma}{\partial\,q^\delta} - \dot{q}^\alpha\,\dot{q}^\beta\,\partial_\delta\,\Gamma^\gamma_{\alpha\beta} \tag{10}$$

$$B^\gamma_\delta\,(t) = \frac{\partial\,Q^\gamma}{\partial\,\dot{q}^\delta} + \frac{\partial\,\psi^\gamma}{\partial\,\dot{q}^\delta} - 2\,\Gamma^\gamma_{\delta\delta}\,\dot{q}^\alpha \tag{11}$$

Variational equations (9) are linear differenti^l equations of the second order according to disturbances ξ^α. These equations can be solved analytically in the ȷinite form only cfor some special coefficient values $A^\gamma_\theta\,(t)$ and $B^\gamma_\theta\,(t)$. Then the solutions are presented by Legendre's and Bessel's polvnomials-or similary by some otheretypes of polynomials. In the gener l case some approximative analytical or numerical methods are used. The present paper will describe an analytical and a numerical methods for resolving system (9) in the general case, using the dizcrete model of the linear nonstationary dynamical system. That is why differential equationѕ (9) must be transformee into a normal Caushy's form using the matrix motation. To that end, the following substitutions are used: $\xi^1 = x_1$, $\xi^2 = x_3$, $\xi^3 = x_5, \ldots, \xi^n = x_{2n-1}$: $\dot{\xi}' = x_2$,

$\dot{\xi}^2 = x_4, \ldots, \dot{\xi}^n = x_{2n}$. In such a way the phase vector of the system state $x = \mathrm{col}\,(x_1, x_2, x_3\ x_4 \ldots, x_{2n-1}, x_{2n})$, is obtained, whose od coordiuates are the disturbauces ξ^α, aud eveu coordiuates their derivative $\dot{\xi}^\alpha$. With these substitutions system (9) cau be preseu ed by a matrix differential equation

$$\dot{x}(t) = A(t)\, x(t),\ x(t_0) = x_0 \tag{12}$$

where the time variable of the spuare matrix $A(t)$ is of the type $2n \times 2n$

$$A(t) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ A_1^1 & B_1^1 & A_2^1 & B_2^1 & A_3^1 & B_3^1 & \cdot & \cdot & \cdot & A_n^1 & B_n^1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ A_1^2 & B_1^2 & A_2^2 & B_2^2 & A_3^2 & B_3^2 & \cdot & \cdot & \cdot & A_n^2 & B_n^2 \\ 0 & & & & & & & & & & \\ A_1^3 & & & & & & & & & & \\ \cdot & & & & & & & & & & \\ \cdot & & & & & & & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ A_1^n & B_1^n & A_2^n & B_2^n & A_3^n & B_3^n & \cdot & \cdot & \cdot & A_n^n & B_n^n \end{bmatrix} \tag{13}$$

Elements $A_\delta^\gamma(t)$ and $B_\delta^\gamma(t)$ of matrix $A(t)$ are time variable functions which means that system (12) is noustationary.

The problem of determing the variation $\xi^\alpha(t)$ reduces to the problem of determining the phase vector $x(t)$ of system (12), where the initial condition $x(t_0) = x_0$, is indispensable, which was defined by the starting values of disturbances and thein derivatives $\xi(0)$ and $\dot{\xi}(0)$. According to Cauchy's formula, system (12) has a solution:

$$x(t) = \Phi(t, t_0)\, x(t_0) \tag{14}$$

where $\Phi(t, t_0)$ is the fundamental matrix of sxstem (12). It should be kept in mind that the determination of matrix $\Phi(t, t_0)$ in a general case is a complex and frequently unsonvable problem in the finite form. This problem is more largely explained in papers [3, 4, 5] and here only some of the results from these papers will be employed. These results reduce to the following: Phase vector $x(t)$ of a conptinual dynamical system (12) cannot be determined in the general case in the finite form, but is can be determined in discrete time moments t_{k+1} by means of the discrete model

$$x(t)_{k+1}) = E(t_k)\; x(t_k), \quad x(t_0) = x_0 \tag{15}$$

where the discretisatinn of the time axis is made in the following way $t_{k+1} - t_k = T$, $(k = 0,1,2, \ldots, s)$, $sT = t - t_0$, and matrix $E(t_k)$ is defined by the folloving recurrence formula:

$$E(t_k) = \sum_{n=0}^{m} \frac{A_n(t_k)}{n!} T^n, \; A_0(t) = I, \quad \overset{(0)}{A} = A \tag{16}$$

$$A_n(t) = \sum_{p=0}^{n-1} \binom{n-1}{p} \overset{(p)}{A}(t)\, A_{n-1-p}(t), \quad \overset{(1)}{A} = \dot{A} \tag{17}$$

For the practical application of expression (16), $m = 1$ provides satisfactory results, which is usually used for numerical computations. Formula (16) enables to improve the numerical calculation with $m = 2$, or more, independently from the size of the step of the discretization T. So now the precision of the numerical c lculation can be increased in two ways. The furst way is the change of the discretization step (decrase of T), and the second is the increase of number m in formula (16). In a favourable combination these two ways may bring an approximate solution very close to the accurate solution, which is shown by examples made in papers [3] and [4].

The discretisation step T can be adjusted even duting the investigation of the work of the discrete model (15) on the computer, when it may be decreas if the first results show great deviations from the expected results. However,i n the cases when the computer has a limited capacity with respect to the number of decimal places, positive value $T < 1$ cannot unlimitedly be decreased, because the accumulated error can affect the final result in such a way that it may be practically unisable. In this situation, when due to the quoted reasons the discretisation step T cannot be decreased further, we can increase the approximation precision (16) by increasing the number m, which is clearly seen in paper [3]. In the ideal case (when $m \to \infty$) the discrete model (15) represents the accurate solution of system (12), and the coordinate of phase vector x are integrals of variational equations (9). This last example is possible to be carried out in such cases, when it is possible to determine the boundary functions of all elements of matrix $E(t)$ form (16). An example of this hind is represented in paper [4].

REFERENCE

[1] Vujičić V. A., *Kovarijantna dinamika,* Matemat. instit. Beograd 1981.

[2] Vujičić V. A., *Covariant epuations of disturbed motion of mechcanical systems.* Tenzor, N. S. Vol. 22 (1981).

[3] Mikičić ó. D., *O linearnom nustacionarnom sistemu sa brzo promenljivim parametrima,* Automatika, 22 (5—6), Zagreb 1981.

[4] Mikičić, ú. D., *Prilog odredivanju konačnih jednačina kretania nestacioniranih mehnničkih sistema,* Zbornik radova Matematičkog instituta, Nova serija, knj. 4 (12), Beograd 1984.

[5] Mikičić, J. D., *On a diskrete model of an unstationary sistem,* Theoretical and Appliend Mechanics, No 1, Beograd 1976.

ВАРИАЦИОННЫЕ УРАВНЕНИЯ ДВИЖЕНИЯ МЕХАНИЧЕСКОГО СИСТЕМЫ ПЕРЕМЕННОЙ МАССЫ И ВОЗМОЖНОТЬ ИНТЕГРАЦИЙ

В єтой статье рассмхтриваются вариационыя уравнений механической системы переменной массы $\ddot{\xi}^{\varsigma} = A^{\gamma}_{\delta}(t)\,\xi^{\delta} + B^{\gamma}_{\delta}(t)\,\dot{\xi}^{\delta}$, $(\xi,\ \delta = 1, \ldots, n)$. Здес показамо, как можно дискретныи модел $x(t_{n+1}) = E(t_n)\,x(t_n) + F(t_n)\,U$ изпользовать для нахождения вариаций $\xi^{\gamma}(t)$.

VARIJACIJE JEDNAČINE KRETANJA MEHANIČKOG SISTEMA PROMENLJIVE MASE I NJIHOVA INTEGRACIJA

U radu se posmatraju varijacione jednačine kretanja mehaničkog sistema promenljive mase oblika $\ddot{\xi}^{\gamma} = A^{\gamma}_{\delta}(t)\,\xi^{\delta} + B^{\gamma}_{\delta}(t)\,\dot{\xi}^{\delta}$ Ovde je pokazano koko se može diskretan model linearnog sistema $x(t_{k+1}) = E(t_k)\,x(t_k) + F(t_k)\,u(t_k)$ iskoristiti za rešavanje varijacija $\xi^{\gamma}(t)$.

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