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UNIVERSAL SOLUTION OF THE INCOMPRESSIBLE LAMINAR BOUNDARY LAYER FLOW ON A SPINNING BODY OF REVOLUTION OF ARBITRARY SHAPE

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1. Introduction

Because of its practical importance, the flow on a body of revolution of arbitrary shape, spinning about its axsis, which is parallel to stream, was intensively studied in fluid mechanics. First at all, experimentally (Wieselberger [1], Luthander & Rydberg [2]), and afterwords theoretically. Namely, to resolve the corresponding problem of incompressible laminar boundary layer, Shlichting [3], Truckenbrodt [4] and Parr [5], used procedure based on Karman--Pohlhaussen's method (1921), improved by Holstein & Bohler (1940).

Enlarging multi-parametric Loitskianskii's method [6] for the case of a sacious boundary layear, Bogdanova [7] chose, in more recent time, as one of the examples for obtained universal equations, this case of flow.

In this paper, complete universalisation of treated problem has done. Namely, in contrast to Bogdanova's [7] solution, for which practical application additional integration of corresponding momentum equation is indispensable, universal solutions (determined numerically) of the equations obtained in this paper, can be directly applied in a particular case. This methodical quality, important for its eficiency of application in tehnical practice, was attained introducing in the Bogdanova's [7] procedure, new, more appropriate Saljnikov's [8] variables, enlagred by Kukić [9] on the corresponding problems of axsymetric boundary layer.

2. Transformation of the governing equations

Examination of treated problem starts from basic equations of incompressible laminar flow in spacious boundary layer [7], evaluated for the usual hypothesis of the boundary layer theory. The equations are expressed in ortogonal curvilinear coordinates x_1, x_2, x_3 (x_3 is normal on the surface) and they contain corresponding:

— projections of the velocities $v_1(x_1, x_2, x_3)$, $v_2(x_1, x_2, x_3)$, $v_3(x_1, x_2\ x_3)$

— projections of the velocities of the outer flow U_1 (x_1, x_2), U_2 (x_1, x_2), and
— Lamé's coefficients h_1 (x_1, x_2), h_2 (x_1, x_2), $h_3 = 1$..

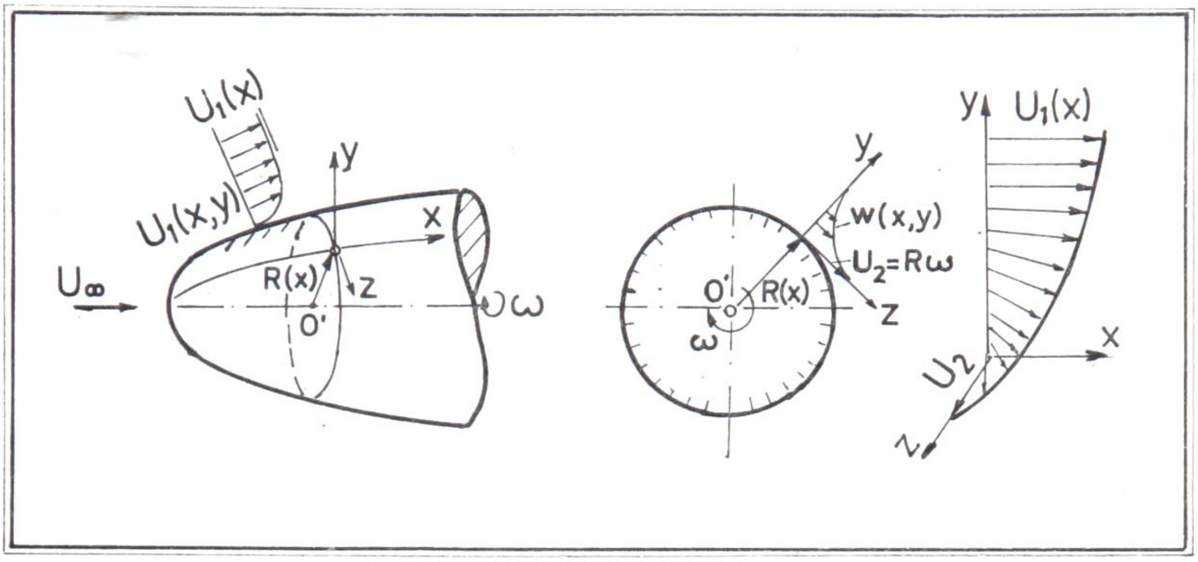

Fig. 1

Taking into cosideration addopted coordinative system, shown on the Fig. 1, the following notation can be done:

$$x_1 = x, \; x_2 = z, \; x_3 = y; \qquad v_1 = u, \; v_2 = w, \; v_3 = v, \tag{1}$$

with the corresponding Lamé's coeffieients:

$$h_1 = 1, \; h_2 = R\,(x), \; h_3 = 1. \tag{2}$$

Governing equations of treated problem, using (1) and (2) are:

$$\frac{\partial u}{\partial x} + \frac{u}{R}\,\frac{dR}{dx} + \frac{\partial v}{\partial y} = 0;$$

$$u\,\frac{\partial u}{\partial x} + v\,\frac{\partial u}{\partial y} - \frac{w^2}{R}\,\frac{dR}{dx} = U_1\,\frac{dU_1}{dx} + \nu\frac{\partial^2 u}{\partial y^2};$$

$$u\,\frac{\partial w}{\partial x} + v\,\frac{\partial w}{\partial y} + \frac{wu}{R}\,\frac{dR}{dx} = \nu\,\frac{\partial^2 w}{\partial y^2}.$$

Taking into consideration Fig. 1, boundary conditions are:

$$y = 0\colon \; u = v = 0, \; w = \; U_2 = R\,(x)\;\omega; \qquad y = \rightarrow \infty\colon \; u = U_1\,(x), \; w = 0. \tag{4}$$

The direct solution of the system of equations (3) with boundary conditions (4) presents considerable difficulties. Therefore, the Loitskianskii-Bogdanova's [7] teatement is used, improved by the more appropiate Saljnikov-Kukić's [9] variables.

Namely, first at all, following variabɩes:

$$x = x; \; \eta = U_1^{\frac{b_0}{2}} R(x)\, x \left(a_0 \nu \int_0^x U_1^{b_0-1} R(x)^2 dx\right)^{-\frac{1}{2}}, \tag{5}$$

and dimensioless flow functions

$$\Phi_1(x, \eta) = U_1^{\frac{b_0}{2}-1} \psi_1(x, y) \left(a_0 \nu \int_0^x U_1^{b_0-1} R(x)^2 dx\right)^{-\frac{1}{2}};$$

$$\Phi_2(x, \eta) = U_2^{-1} U_1^{\frac{b_0}{2}} R(x)^{-1} \psi_2(x, y) \left(a_0 \nu \int_0^x U_1^{b_0-1} R(x)^2 dx\right)^{-\frac{1}{2}}, \tag{6}$$

are introduced in the last two equations of the system (3). The equation of continuity is satisfied by:

$$u = \frac{1}{R(x)} \frac{\partial \psi_1}{\partial y}; \quad v = -\frac{1}{R(x)} \frac{\partial \psi_1}{\partial x}, \tag{7}$$

and funtion ψ_2 is defined as:

$$w = \frac{\partial \psi^2}{\partial y}. \tag{8}$$

In this way, transformed equations of the concerned problem are obtaned. They are not quoted here, since they are to large.

For their further universalisation, following expressions are introduced:

$$\eta = \frac{B}{l} y; \quad \bar{z} = \frac{l^2}{\nu} z. \tag{9}$$

Sense of these expressions in parametric methods is well known (6). Here is:

η — new varɩable, defined with (5).

For the linear proportion l, momentum thickness is addopted;

$$l = \delta_{11}^{**} = \int_0^\infty \frac{u}{U_1}\left(1 - \frac{u}{U_1}\right) dx, \tag{10}$$

as a distincive feature of the concerned problem.

Combining the expressions (9) and (10), variable B is defined:

$$B = \int_0^\infty \frac{\partial \Phi_1}{\partial \eta}\left(1 - \frac{\partial \Phi}{\partial \eta}\right) d\eta. \tag{11}$$

Introducing the variable η, defined with (5) in the exspression (9), relation:

$$\frac{\overline{z}}{B^2} = \frac{a_0 \int_0^x U_1^{b_0 - 1} R(x)^2 \, dx}{U_1^{b_0} R(x)^2} \tag{12}$$

indispensable for the further transformation of differential equations of treated problem, is obtained. Namely, after introducing the relation (12) and $\sigma = U_1/U_2$, they are:

$$\frac{\partial^3 \Phi_1}{\partial \eta^3} + \frac{\overline{z}}{B^2}\frac{d U_1}{dx}\left[1 - \left(\frac{\partial \Phi_1}{\partial \eta}\right)^2\right] + \sigma^2 \frac{\overline{z}}{B^2}\frac{U_1}{R(x)}\frac{d R(x)}{dx}\left(\frac{\partial \Phi_2}{\partial \eta}\right)^2 +$$

$$+ \frac{1}{2B^2}\left[(2 - b_0)\,\overline{z}\frac{d U_1}{d x} + a_0 B^2\right]\Phi_1 \frac{\partial^2 \Phi_1}{\partial \eta^2} = \frac{\overline{z}\, U_1}{B^2}\left(\frac{\partial \Phi_1}{\partial \eta}\frac{\partial^2 \Phi_1}{\partial x \partial \eta} - \frac{\partial \Phi_1}{\partial x}\frac{\partial^2 \Phi_1}{\partial \eta^2}\right); \tag{13}$$

$$\sigma \frac{\partial^3 \Phi_2}{\partial \eta^3} - \frac{\overline{z}}{B^2}\frac{d U_2}{dx}\frac{\partial \Phi_1}{\partial \eta}\frac{\partial \Phi_2}{\partial \eta} - \sigma \frac{\overline{z}}{B^2}\frac{U_1}{R(x)}\frac{d R(x)}{dx}\frac{\partial \Phi_1}{\partial \eta}\frac{\partial \Phi_2}{\partial \eta} +$$

$$+ \frac{\sigma}{2B^2}\left[(2 - b_0)\,\overline{z}\frac{d U_1}{dx} + a_0 B^2\right]\Phi_1 \frac{\partial^2 \Phi_1}{\partial \eta^2} = \sigma \frac{\overline{z}\, U_1}{B^2}\left(\frac{\partial \Phi_1}{\partial \eta}\frac{\partial^2 \Phi_2}{\partial x \partial \eta} - \frac{\partial \Phi_1}{\partial x}\frac{\partial^2 \Phi_1}{\partial \eta^2}\right).$$

It's easy to conclude that coefficients in the equtions (13) depend only on the variable x. Following the general Bogdanova's [7] examinations, and keeping her notation, following groups of parameters

$$f_{k0}^{[1]} = U_1^{k-1}\frac{d^k U_1}{dx^k}\,\overline{z}^k;$$

$$f_{k0}^{[2]} = 0; \tag{14}$$

$$f_{k0}^{[3]} = \frac{U_1^k}{R(x)}\frac{d^k R(x)}{d x^k}\,\overline{z}^k;$$

$$f_{k0}^{[4]} = U_1^{k-1}\frac{d^k U_2}{dx^k}\,\overline{z}^k,$$

can be introduced as a new variables. It is evident, that only three of them, i. e. $f_{k0}^{[1]}$, $f_{k0}^{[3]}$ $f_{k0}^{[4]}$ are indispensable.

Since the $U_2 = \omega R(x)$, following relation can be established:

$$f_{k0}^{[4]} = \sigma f_{k0}^{[3]}. \tag{15}$$

After introducing parameters (14) and relation (15), for $k = 1$, using more suitable notation:

$$f_{1_0}^{[1]} = f_1; \; f_{1_0}^{[3]} = g_1 \tag{16}$$

the system (13) is transformed, as follows:

$$\begin{aligned} &\frac{\partial^3 \Phi_1}{\partial \eta^3} + \frac{f_1}{B^2}\left[1 - \left(\frac{\partial \Phi_1}{\partial \eta}\right)^2\right] + \frac{1}{2 B^2}\left[(2 - b_0) f_1 + a_0 B^2\right] \Phi_1 \frac{\partial^2 \Phi_1}{\partial \eta^2} + \\ &\quad + \sigma \frac{g_1}{B^2}\left(\frac{\partial \Phi_2}{\partial \eta}\right)^2 = \frac{\bar{z}\, U_1}{B^2}\left(\frac{\partial \Phi_1}{\partial \eta} \frac{\partial^2 \Phi_1}{\partial x \partial \eta} - \frac{\partial \Phi_1}{\partial x} \cdot \frac{\partial^2 \Phi_1}{\partial \eta^2}\right) \\ &\frac{\partial^3 \Phi_1}{\partial \eta^3} - \frac{2 g_1}{B^2} \frac{\partial \Phi_1}{\partial \eta} \frac{\partial \Phi_2}{\partial \eta} + \frac{1}{2 B^2}\left[(2 - b_0) f_1 + a_0 B^2\right] \Phi_1 \frac{\partial^2 \Phi_2}{\partial \eta^2} = \\ &\quad = \frac{\bar{z}\, U_1}{B^2}\left[\frac{\partial \Phi_1}{\partial \eta} \frac{\partial^2 \Phi_2}{\partial x \partial \eta} - \frac{\partial \Phi_1}{\partial x} \frac{\partial^2 \Phi_2}{\partial \eta^2}\right], \end{aligned} \tag{17}$$

where:

$$\sigma^2 = \frac{\omega^2 R(x)^2}{U_1^2} = \lambda \tag{18}$$

is the "spinning factor".

Besides, by means of the relations (5), (8), (9) and (14), the following exspessions for the parameters f_1 and g_1 are obtained:

$$f_1 = \frac{a_0 B^2 U_1'}{U_1^{b_0} R(x)^2} \int_0^x U_1^{b_0 - 1} R(x)^2 \, dx; \tag{19}$$

$$g_1 = \frac{2 B^2 R'(x)}{U_1^{b_0 - 1} R(x)^3} \int_0^x U_1^{b_0 - 1} R(x)^2 \, dx, \tag{20}$$

where the variable B is defined with (11).

It's interesting to mention that for $a_0 = b_0 = 2$, relations (19) and (20) are reduced to the Görtler's "main function" of axisymetric boundary layer [10]:

$$\beta(x) = \frac{f_1}{B^2} = \frac{2 U_1' \int_0^x U_1 R(x)^2 \, dx}{U_1^2 R(x)^2}; \tag{21}$$

$$\gamma(x) = \frac{g_1}{B^1} = \frac{2 R'(x) \int_0^x U_1^{b_0} R(x)^2 \, dx}{U_1 R(x)^3}. \tag{22}$$

From the form of the equations (17) can be concluded that is indispensable to transform the factor $U_1 \bar{z}$ on the right side of the both equations for their complete universalisation. However, this can be done only after introduction of transformed momentum equation for x-direction.

3. Transformation of the corresponding momentum equation

Using usual treatement in the boundary layer theory, corresponding momentum equations can be derived from the system (3). Since the only one of them, which corresponds to meridional (x-) direction is neccessary for the further examination, only the first momentum equation is quoted:

$$\frac{\partial}{\partial x}\delta_{11}^{**} + \frac{U_1'}{U_1}(2\,\delta_{11}^{**} + \delta_1^{*}) + \frac{R'}{R}(\sigma^2\,\delta_{22}^{**} + \delta_{11}^{**}) = \frac{\tau_{0x}}{\rho\,U_1^2}. \tag{23}$$

Here are:

$$\delta_{11}^{**} = \int_0^\infty \frac{u}{U_1}\left(1 - \frac{u}{U_1}\right) dy; \quad \delta_{22}^{**} = \int_0^\infty \frac{w}{U_2}\frac{w}{U_2}\, dy \tag{24}$$

momentum thichness

$$\delta_1^{*} = \int_0^\infty \left(1 - \frac{u}{U_1}\right) dy \tag{25}$$

displacement thickness, and

$$\tau_{0x} = \mu\left(\frac{\partial u}{\partial y}\right)_0 \tag{26}$$

shear stress on the wall in meridional direction.

Combining the dimensionlless expession

$$\zeta_1 = \left[\frac{\partial\,(u/U_1)}{\partial\,(x/\delta_{11}^{**})}\right]_{y=0}, \tag{27}$$

which characterize shear stress on the wall in meridional direction with definition (26), right side of the equation (23) is transformed to:

$$\frac{\tau_{0x}}{\rho\,U_1^2} = \frac{\nu}{\delta_{11}^{**}\,U_1}\,\zeta_1. \tag{28}$$

Intoducing in the equation (23) expression (28) and relation:

$$\frac{d\,\delta_{11}^{**}}{d\,x} = \frac{\nu}{2\,\delta_{11}^{**}}\frac{d\,\bar{z}}{d\,x}, \tag{29}$$

which follows from (9), written as:

$$\bar{z} = \frac{\delta_{11}^{**2}}{\nu} \tag{30}$$

multiplying with $U_1 \delta_{11}^{**}/\nu$, momentum equation is obtained in the following form:

$$\frac{U_1}{2} \frac{d\bar{z}}{dx} = \zeta_1 - (2 + H_1^*) f_{10}^{[1]} - (\sigma^2 H_{22}^{**} + 1) f_{10}^{[1]} = \frac{F_1}{2} \tag{31}$$

where

$$H_{11}^{**} = \frac{\delta_{11}^{**}}{\delta_{11}^{**}} = 1; \quad H_{22}^{**} = \frac{\delta_{22}^{**}}{\delta_{11}^{**}}; \quad H_1^* = \frac{\delta_1^*}{\delta_{11}^{**}}. \tag{32}$$

On the other side, differentiating the groups of parameters (14), new relation is derived:

$$U_1 \bar{z} \frac{d f_{k0}^{[r]}}{dx} = N_{k+1}^{[r]} + k\, U_1 f_{k0}^{[r]} \frac{d\bar{z}}{dx}, \tag{33}$$

where

$$N_{k+1}^{[r]} = f_{k0}^{[r]} (k f_{10}^{[r]} - f_{10}^{[r]}) + f_{k+1}^{[r]}, \tag{34}$$

with r as the group index, i. e., in the most general case $r = 1, 2, 3, 4$.

Function $\theta_k^{[r]}$, indispensable for the definitive universalisation of the system of equations (17) is obtained introducing the momentum equation (31) in the relation (33):

$$U_1 \bar{z} \frac{d f_{k0}^{[r]}}{dx} = N_{k+1}^{[r]} + k f_{k0}^{[r]} F_1 = \theta_k^{[r]}. \tag{35}$$

4. Universal equations of treated problem

By means of relation (35) it is possible to derive differential operator:

$$U_1 \bar{z} \frac{\partial}{dx} = U_1 \bar{z} \sum_r \sum_k \frac{\partial}{\partial f_{k0}^{[r]}} = \sum_r \sum_k \theta_k^{[r]} \frac{\partial}{\partial f_{k0}^{[r]}}, \tag{36}$$

which transforms epuations (17) in universal form:

$$\frac{\partial^3 \Phi_1}{\partial \eta^3} + \frac{f_1}{B^2}\left[1 - \left(\frac{\partial \Phi_1}{\partial \eta}\right)^2\right] + \sigma_2 \frac{g_1}{B^2}\left(\frac{\partial \Phi_2}{\partial \eta}\right)^2 + \frac{1}{2B^2}[(2-b_0)f_1 + a_0 B^2]\Phi_1 \frac{\partial^2 \Phi_1}{\partial \eta^2} =$$

$$= \frac{1}{B^2}\sum_r \sum_k \left(\frac{\partial \Phi_1}{\partial \eta}\frac{\partial^2 \Phi_1}{\partial f_{k0}^{[r]}\partial \eta} - \frac{\partial \Phi_1}{\partial f_{k0}^{[r]}}\frac{\partial^2 \Phi}{\partial \eta^2}\right)\theta_k^{[r]}; \tag{37}$$

$$\frac{\partial^3 \Phi_2}{\partial \eta^3} - \frac{2g_1}{B^2}\frac{\partial \Phi_1}{\partial \eta}\frac{\partial \Phi_2}{\partial \eta} + \frac{1}{2B^2}[(2-b_0)f_1 + a_0 B^2]\Phi_1\frac{\partial^2 \Phi_2}{\partial \eta^2} =$$

$$= \frac{1}{B^2}\sum_r \sum_k \left(\frac{\partial \Phi_1}{\partial \eta}\frac{\partial^2 \Phi_2}{\partial f_{k0}^{[r]}\partial \eta} - \frac{\partial \Phi_1}{\partial f_{k0}^{[r]}}\frac{\partial^2 \Phi_2}{\partial \eta^2}\right)\theta_k^{[r]}$$

whith the corresponding boundary conditions:

$$\Phi_1 = \Phi_2 = \frac{\partial \Phi_1}{\partial \eta} = \frac{\partial \Phi_2}{\partial \eta} - 1 = 0 \text{ for } \eta = 0; \tag{38}$$

$$\frac{\partial \Phi_1}{\partial \eta} = 1; \ \frac{\partial \Phi_2}{\partial \eta} = 0 \text{ for } \eta = \infty.$$

Since in the treated case four original groups of parameters (14) are reduced to following two;

$$f_{k0}^{[3]} = f_k; \quad f_{k0}^{[4]} = g_k, \tag{39}$$

group index r has only two values: $r = 1$ and $r = 3$.

Separating the sums on the right sides of the system (37) according to r, and using the notation (39), definitive form of universal equations is:

$$\frac{\partial^3 \Phi_1}{\partial \eta^3} + \frac{f_1}{B_2}\left[1 - \left(\frac{\partial \Phi_1}{\partial \eta}\right)^2\right] + \sigma^2 \frac{g_1}{B_2}\left(\frac{\partial \Phi_2}{\partial \eta}\right)^2 + \frac{1}{2B^2}[(2-b_0)f_1 + a_0 B^2]\Phi_1\frac{\partial^2 \Phi_1}{\partial \eta^2} =$$

$$= \frac{1}{B^2}\sum_k \left[\theta_k^{[1]}\left(\frac{\partial \Phi_1}{\partial \eta}\frac{\partial^2 \Phi_1}{\partial f_k \partial \eta} - \frac{\partial \Phi_1}{\partial f_k}\frac{\partial^2 \Phi_1}{\partial \eta^2}\right) + \theta_k^{[3]}\left(\frac{\partial \Phi_1}{\partial \eta}\frac{\partial^2 \Phi_1}{\partial g_k \partial \eta} - \frac{\partial \Phi_1}{\partial g_k}\frac{\partial^2 \Phi_1}{\partial \eta^2}\right)\right]; \tag{40}$$

$$\frac{\partial^3 \Phi_2}{\partial \eta^3} - \frac{2g_1}{B^2}\frac{\partial \Phi_1}{\partial \eta}\frac{\partial \Phi_2}{\partial \eta} + \frac{1}{2B^2}[(2-b_0)f_1 + a_0 B^2]\Phi_1\frac{\partial^2 \Phi_1}{\partial \eta^2} =$$

$$= \frac{1}{B^2}\sum_k \left[\theta_k^{[1]}\left(\frac{\partial \Phi_1}{\partial \eta}\frac{\partial^2 \Phi_2}{\partial f_k \partial \eta} - \frac{\partial \Phi_1}{\partial f_k}\frac{\partial^2 \Phi_1}{\partial \eta^2}\right) + \theta_k^{[3]}\left(\frac{\partial \Phi_1}{\partial \eta}\frac{\partial^2 \Phi_1}{\partial g_k \partial \eta} - \frac{\partial \Phi_1}{\partial g_k}\frac{\partial^2 \Phi_2}{\partial \eta^2}\right)\right],$$

where, taking in consideration (34) and (35):

$$\theta_k^{[1]} = (k - 1) f_1 f_k + k\, F_1 f_k + f_{k+1};$$
$$\theta_k^{[3]} = (k f_1 - g_1)\, g_k + g_{k+1} + k\, F_1 g_k \tag{41}$$

while F_1, determined with (31), is:

$$F_1 = 2\, \xi_1 - 2 f_1 (2 + \overset{*}{H_1}) - 2 g_1 (\sigma^2 \overset{**}{H_{22}} + 1) \tag{42}$$

5. Solution of the universal equations

Taking in consideration limitation of computer (memory, speed) that was used for numerical integration, system (40) was solved in so-called two-parametric once localized approximation. That means that the only f_1 and g_1 parameters are kept, parameters with $k \geqslant 2$ are rejected. This is possible since the first members of the Loitskianskii's parametric groups (14) is "strong" — results obtained in such approximation is sufficiently good for practical application. Besides, all members with derivations of g_1 are rejected, that means that the solution is once localized by g_1. Betwen two possibilities: localization by f_1 or g_1, the second one is chosen. It was considered that parameter f_1, which characterize the influence of the outer velocity distribution and had implicitlly the influence of body shape on boundary layer developement, is more important than parameter g_1.

According to this, universal equations (40) in two parametric once localized approximation are:

$$\frac{\partial^3 \Phi_1}{\partial \eta^3} + \frac{f_1}{B^2}\left[1 - \left(\frac{\partial \Phi_1}{\partial \eta}\right)^2\right] + \sigma^2 \frac{g_1}{B^2}\left(\frac{\partial \Phi_2}{\partial \eta}\right)^2 + \frac{1}{2\, B^2}[(2 - b_0) f_1 + a_0 B^2]\, \Phi_1 \frac{\partial^2 \Phi_1}{\partial \eta^2} =$$
$$= \frac{1}{B^2} F_1 f_1 \left(\frac{\partial \Phi_1}{\partial \eta} \frac{\partial^2 \Phi_1}{\partial f_1 \partial \eta} - \frac{\partial \Phi_1}{\partial f_1} \frac{\partial^2 \Phi_1}{\partial \eta^2}\right);$$
$$\frac{\partial^3 \Phi_2}{\partial \eta^3} - 2 \frac{g_1}{B^2} \frac{\partial \Phi_1}{\partial \eta} \frac{\partial \Phi_2}{\partial \eta} + \frac{1}{2\, B^2}[(2 - b_0) f_1 + a_0 B^2]\, \Phi_1 \frac{\partial^2 \Phi_1}{\partial \eta^2} = \tag{43}$$
$$= \frac{1}{B^2} F_1 f_1 \left(\frac{\partial \Phi_1}{\partial \eta} \frac{\partial^2 \Phi_2}{\partial f_1 \partial \eta} - \frac{\partial \Phi_1}{\partial f_1} \frac{\partial^2 \Phi_2}{\partial \eta^2}\right),$$

with the corresponding boundary conditions:

$$\Phi_2 = \Phi_1 = \frac{\partial \Phi_1}{\partial \eta} = \frac{\partial \Phi_2}{\partial \eta} - 1 = 0 \quad \text{for } \eta = 0;$$
$$\frac{\partial \Phi_1}{\partial \eta} = 1; \; \frac{\partial \Phi_2}{\partial \eta} = 0 \quad \text{for } \eta \to \infty; \tag{44}$$
$$\Phi_1 = \Phi_1^{(0)}(\eta, g_1); \;\; \Phi_2 = \Phi_2^{(0)}(\eta, g_1) \quad \text{for } f_1 = 0.$$

This system of equations has been numerically integrated* by means of ınite difference method using an implicite scheme. Three diagonal alebraic system of equations, by which each of the original partial differential equations (43) has been substitued, have been solved using Gaussian elimination. Since the partial differential equations determing universal stream functions Φ_1 and Φ_2 are of third order, nonlinear and joined, the iterative procedure has been applied. The calculation has been realized with double precision, for four different values of spinning factor:

$$\sigma = 0 \;\; ; \;\; 0{,}5 \;\; ; \;\; 1 \;\; ; \;\; 10$$

and for the following values of the parameter:

$$g_1 = 0 \; ; \pm 0{,}02 \; ; \pm 0{,}04 \; ; \pm 0{,}06 \; ; \pm 0{,}08$$

The integration has been done with space inciements $\Delta f_1 = 0{,}001$ and $\Delta \eta = 0{,}1$, starting at the point $f_1 = 0$ (i. e. the point of the minimal pressure or maximal velocity in meridional direction), moving upstreams to the front stagnation point (positive domain portion) and downstveams to the separation point (negative domain portion). The cumulative calculation error, performer for both universal stream functions, was less than $1 \cdot 10^{-5}$

6. The results of the numerical integration and their practical application

Complicated nature of treated problem, which is described with great number of independant factors, parameters an variables, has, at the end of the numerical treatement, great number of the various results. The most interesting of them are quoted here.

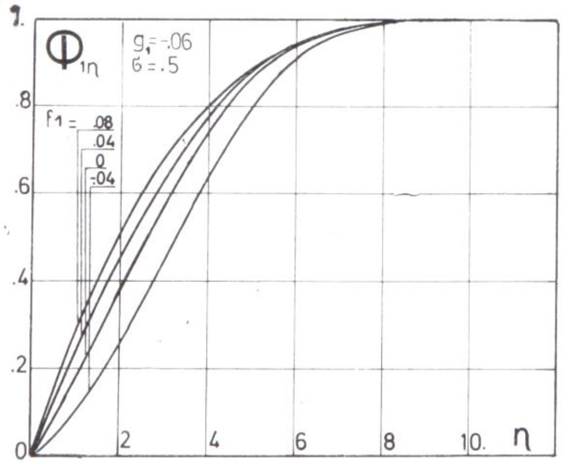

Fig. 1.

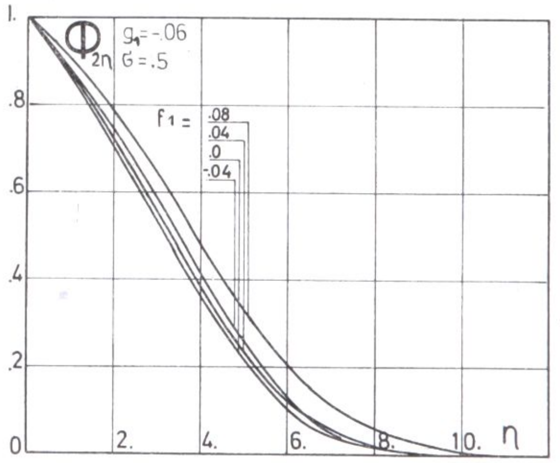

Fig. 2.

* The calculation has been done in C.I.R.C.E., France

On the figures 2. and 3. $\Phi_{1\eta}$ and $\Phi_{2\eta}$ are shown graphically for the $g_1 = -0{,}06$ and $\sigma = 0{,}5$. It can be mentioned characteristical evolution of the velocity profile, approaching the point og separation (f_1 decreasing).

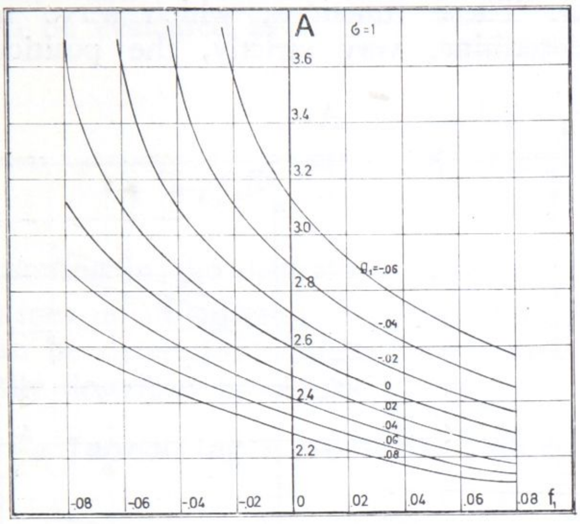

Fig. 5

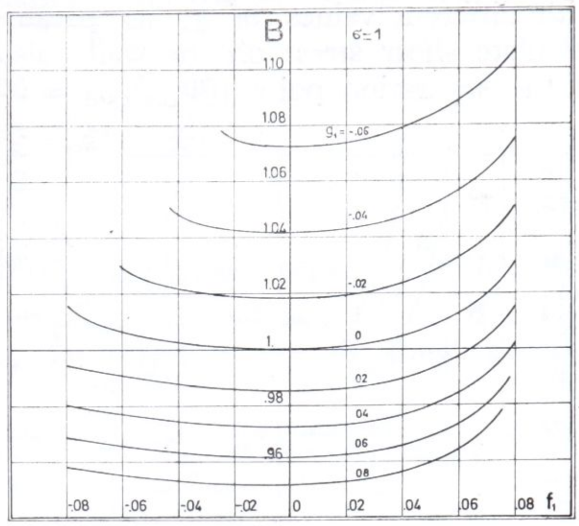

Fig. 6

Variables *A* and *B*, universal displacement and momentum thickness, are shown on the figures 4 and 5, for $\sigma = 1$ and different values of g_1 as parameter. As it can be expected, approaching the point of separation, chara-

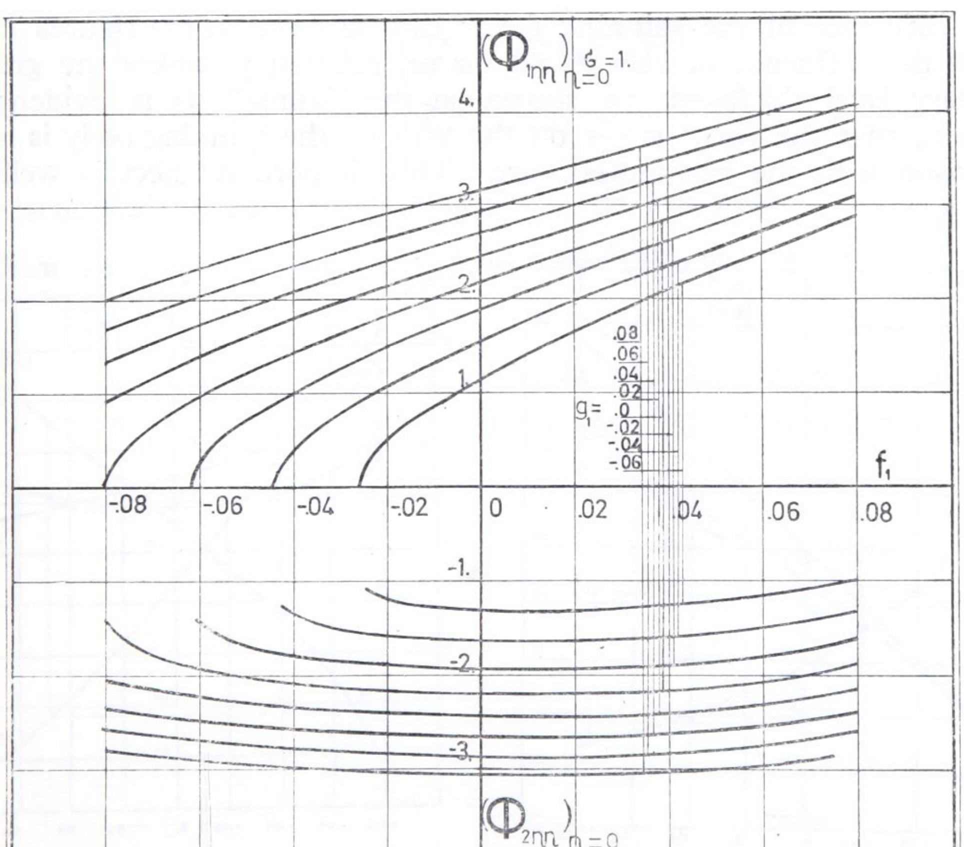

Fig. 6

cteristical increasing of the boundary layer thickness can be noted, particularly on the figure 5.

On the figure 6 functions $(\Phi_{1\eta\eta})_{\eta=0}$ and $(\Phi_{2\eta\eta})_{\eta=0}$ are shown for $\sigma = 1$, with different values of g_1 as parameter. These functions, which serve to calculate shear stress on the wall, also determine, very strictly, the position of the separation point $((\Phi_{1\eta\eta})_{\eta=0} = 0)$.

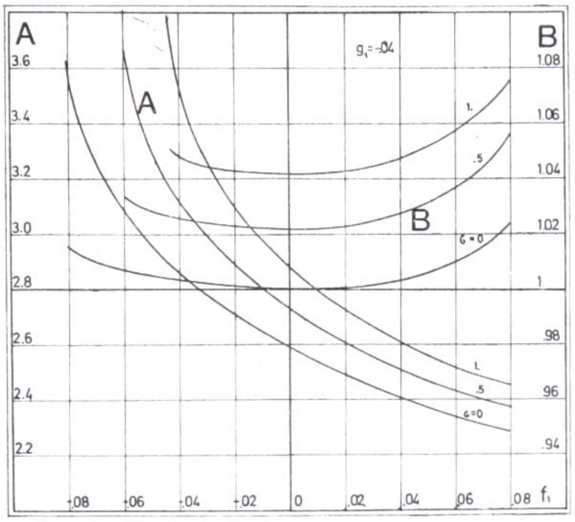

Fig. 7

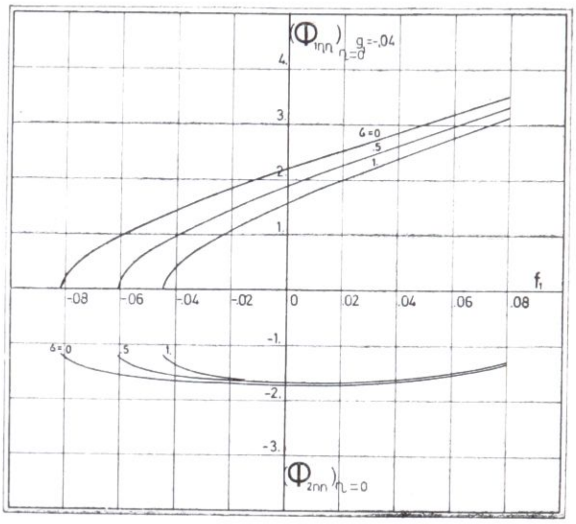

Fig. 8

The influence of the spinning factor can be seen on the figures 7. and 8. Because of the influence of centrifugal forces, rotation provokes the growth of the boundary layer thickness, as shown on the figure 7. It is evident, from the figure 8, that the shear stress on the wall of the spinning body is reduced, n comparison with the rotationless care. This important efect is well-known in balistics.

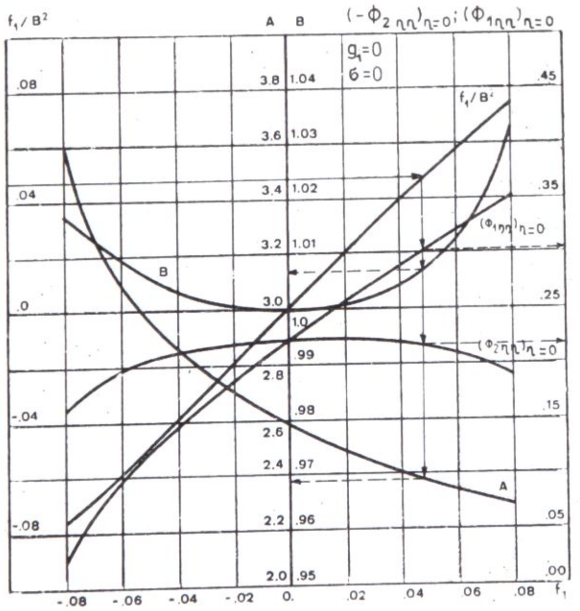

Fig. 9

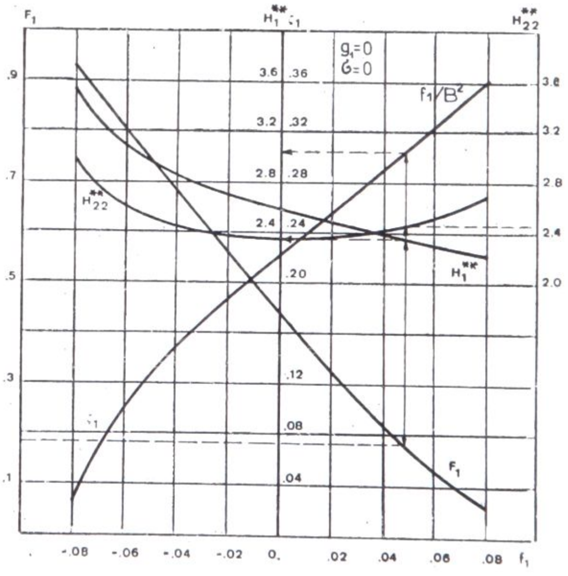

Fig. 10

On the figures 9. and 10., for $\sigma = 0$, all characteristical universal functions are shown. It can be mentioned that this rotationless case is chosen only as the example of application of given diagrams as nomograms. In the particular case, which is defined with the known $A_1(x)$, $R(x)$ and ω, using (19), f_1/B^2 can be evaluated as:

$$\frac{f_1}{B^2} = \frac{a_0 U_1'}{U_1^{b_0} R^2(x)} \int_0^x U_1^{b_0-1} R^2(x)\,dx.$$

Entering in the diagrams 9 and 10 with this particular value of f_1/B^2, the values of all universal functions (A, B $(\Phi_{1\eta\eta})_{\eta=0}$, $(\Phi_{1\eta\eta})_{\eta=0}$, ξ_1, F_1 $\overset{*}{H}_1$, $\overset{**}{H}_{22}$ can be determined. Lines with flashes on the figures 9 and 10 show graphi) cally described procedure.

Introducing the obtained values A, B and $\overset{**}{H}_{22}$ in the expressions:

$$\overset{**}{\delta}_{11} = \frac{(a_0 \nu \int_0^x U_1^{b_0-1} R^2(x)\,gx)^{1/2}}{U_1^{b_0/2} R(x)} B;$$

$$\overset{**}{\delta}_{22} = \overset{**}{H}_{22}\,\overset{**}{\delta}_{11};$$

$$\overset{*}{\delta}_1 = \frac{(a_0 \nu \int_0^x U_1^{b_0-1} R^2(x)\,gx)^{1/2}}{U_1^{b_0/2} R(x)} A$$

displacement and momentum thicknenss can be evaluated.

Shear stess on the wall, as follows from (28), is:

$$\tau_{0x} = \frac{\mu U_1^{\frac{b_0}{2}+1} R(x)}{(a_0 \nu \int_0^x U_1^{b_0-1} R^2(x)\,dx)^{1/2}} (\Phi_{1\eta\eta})_{\eta=0}.$$

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УНИВЕРСАЛЬНОЕ РЕШЕНИЕ НЕСЖИМАЕМОГО ЛАМИНАРНОГО ПОГРАНИЧНОГО СЛОЯ В СЛУЧАЕ ЗАКРУЧЕННОГО ДВИЖЕНИЯ ОСЕСИММЕТРИЧНОГО ТЕЛА ПРОИЗВОЛЬНОЙ ФОРМЫ

Резюме

Введением в метод Лойцянского-Богдановой [7] целесообразных переменых Сальникова-Кукичевой [9] достигнута полная универсализация уравнений лемирнаного несжимаемого пограничного слоя в случае закрученного движения осесимметричного тела произвольной форми. Полученная система уравнений нумерически решена в двухпраметрическом однажды локализованном приближении. Результаты интеграции представлены графически и анализированы. Показан способ для практического применения полученииых универсальниых функций посредством данных номограммов.

UNIVERZALNO REŠENJE NESTIŠLIIVOG LAMINARNOG GRANIČNOG SLOJA ZA SLUČAJ ZAVOJNOG KRETANJA OSNOSIMETRIČNOG TELA PROIZVOLJNOG OBLIKA

Rezime

Uvođenjem u metodu Lojcjanskog-Bogdanove [7] svrsishodnih promenljivih Saljnikov-Kukićeve [9], izvršena je potpuna univerzalizacija jednačina nestišljivog laminarnog graničnog sloja za slučaj zavojnog kretanja osnosimetričnog tela proizvojnog oblika. Izvedeni sistem jednačina je numerički rešen u dvoparametarsko jedared lokalizovanom približenju. Rezultati integracije su predstavleni grafički i analizirani. Prikazan je postupak za praktičnu primenu dobijenih univerzalnih funkcija posredstvom datih monograma.

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