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BENDING OF THIN UNIAXIAL CURVED ELEMENTS WITH INCONSTANT RIGIDITY AND BIG DISPLACEMENTS

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Symbols used

h	height of the element
$\varkappa$	change of curvature
ε_{el}	strain the elasticity limit
$\varepsilon_{\max}$	maximum strain of the element
ds	element of the curve length lying on the shell surface
v_i, v_i	covariant and/or contravariant tensor of the dislacement in the element of shell S
w	displacement in the direction of the normal to the element of shell S
$,i$	covariant derivative
ν	Poisson's ratio
D, G	constants depending on the rigidity and material of the shell
D'	inverse value of shell rigidity
J	inertia moment of the cross section
E	Young's module
i^1, i^2, i^3	unit vectors of the Cartesian coordinate system (y^1, y^2, y^3)
$q(\beta)$	continuous load which is parallel to line $\overline{P_1 P_2}$ joining the element boundaries
p	width of the element
σ_φ	normal stress in the cross section

1. Introduction

Thin elements subjected to external loads are characterized by big displacements. This is a characteristic of all thin springs which operate in the elastic region. Such elements are used in different fields of technics as bearing elements or even more frequently as basic elements in various measuring instruments. Therefore they are of very various forms and sizes and are manufactured with great accuracy from high quality spring materials. These elements are widely used also in the field of electronics in the manufacture of microswitches. They are of very small size and frequently operate in pairs according to the principle of system jump. They have to operate perfectly in very different climatic conditions with a foreseen service life. These requirements can be fulfilled only by the elements manufactured from high quality materials or alloys which are very expensive and which have optimal geometric characteristics.

Some solutions of the bending of thin uniaxial curved elements with constant rigidity were given by S. D. Ponomarev and L. E. Andreeva (1), but general solutions of arbitrary uniaxial elements with inconstant rigidity are less known. The paper presents a new contribution to the determination of deformation and stress states in thin arbitrarily curved uniaxial elements the rigidity of which can change along their axis.

The mathematical model was made on the following assumptions:

a) The cross sections rectangular to the axis of the element, which were flat before the deformation remain so also after the deformation process. That means that the influence of tangential stresses is not taken into account and that the normal stresses are distributed linearly across the cross section.

b) The axis of the element representing a line connecting the centres of gravity of the cross sections is an axis of symmetry.

c) The cross sections along the element axis are very small compared to the length and/or curvature radius of the element axis.

d) The curvature of the element axis is arbitrary.

e) Rigidity along the element axis is arbitrary.

f) The direction of the external loads does not change during the displacement process of the element.

g) The element can be loaded at both boundaries by an external force lying on line $\overline{P_1 P_2}$, by the bending moment at the boundaries and continuous load along the element axis which is parallel to line $\overline{P_1 P_2}$. The bending moment acts in the direction of the second Gaussian coordinate.

h) The specific deformation appearing in the direction of the tangent to the element axis is equal to zero.

i) The specific deformation of the element with big displacements has to remain in the elastic region, therefore the following inequation has to be fulfilled.

$$\varepsilon_{\max} = \frac{h}{2} \varkappa \leqslant \varepsilon_{el}$$

2. Basic equation for the determination of the displacements

The basic equation for the determination of the displacement tensor of thin elastic uniaxial element can be obtained from the nonlinear theory of thin shells 2, 3, 4.

Let a shell be defined in the Euclidean space E_3, (5) by the orthogonal Cartesian coordinate system y^φ, $\varphi = 1, 2, 3$ in which the function

$$F(y^1, y^2, y^3) = 0 \tag{1}$$

represents a geometric place of points which enable an analytical expression of the shell surface S.

To simplify the formulations we introduce Gaussian curvilinear coordinates u^i, $i = 1,2$ on the shell surface S by which we can express the Cartesian coordinates.

$$y^\varphi = y^\varphi(u^1, u^2), \; \varphi = 1,2,3 \tag{2}$$

For a one-to-one transformation at least one of the two-lined Jacobian determinants has to be different from zero.

The first quadratic form on the shell surface

$$ds^2 = a_{ij}\, du^i\, du^j \tag{3}$$

yields the covariant metric tensor of the surface in E_3

$$a_{ij} = \frac{\partial y^\varphi}{\partial u^i} \frac{\partial y^\varphi}{\partial u^j} \tag{4}$$

Besides the curvilinear Gaussian coordinates also curvilinear space coordinates x^α, $\alpha = 1,2,3$ in the Euclidean space E_3 are suitable for the analysis of the shell surface. They are related to the transformation equations by coordinates y^φ, $\varphi = 1,2,3$, Fig. 1.

$$x^\alpha = x^\alpha(y^1, y^2, y^3) \tag{5}$$

the Jacobian determinant of which has to be different from zero.

The square of the linear element of the curve ds in E_3 written (defined) in the coordinate system x^α, $\alpha = 1,2,3$ is

$$ds^2 = g_{\alpha\beta}\, dx^\alpha\, dx^\beta \tag{6}$$

where

$$g_{\alpha\beta} = \frac{\partial y^\varphi}{\partial x^\alpha} \frac{\partial y^\varphi}{\partial x^\beta} \tag{7}$$

represents a metric tensor in E_3

The system of equations (2) defining the shell surface S, can be written also in the following form

$$x^\alpha = x^\alpha(u^1, u^2), \; \alpha = 1,2,3 \tag{8}$$

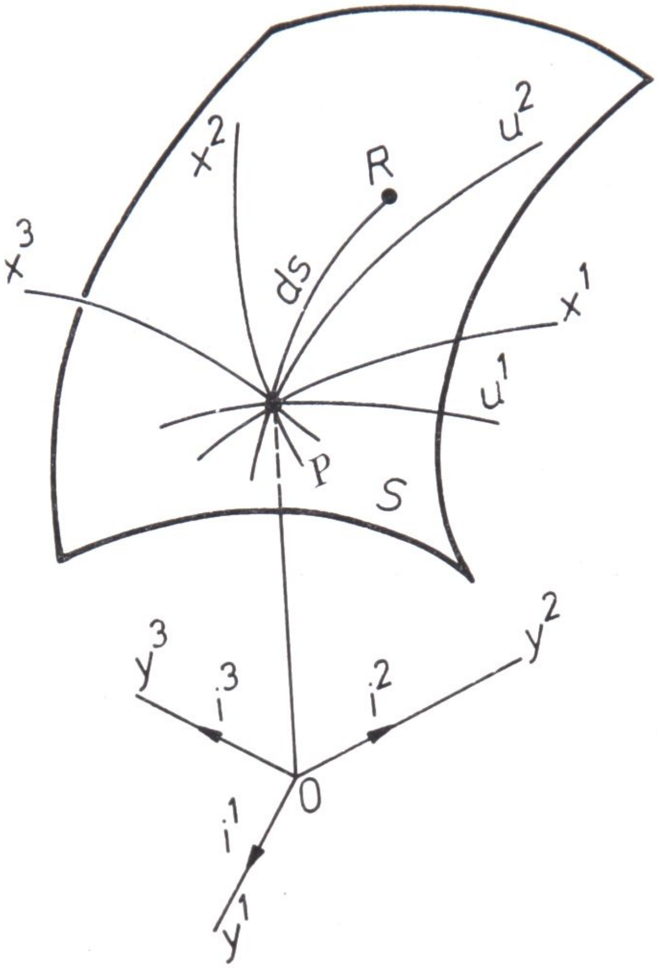

Fig. 1. Relationship between the coordinate systems.

Hence the relation between the metric tensors is

$$a_{ij} = g_{\alpha\beta} \frac{\partial x^\alpha}{\partial u^i} \frac{\partial x^\beta}{\partial x^j}, \qquad \alpha, \beta = 1,2,3 \quad i, j = 1,2 \tag{9}$$

where the indices denoted by Greek letters having the values from one to three refer to space E_3 in which there is shell surface S, while the indices denoted by Latin letters having the values from one to two refer to shell surface S lying in space E_3. Further the elements $g_{\alpha\beta}$ and the differential dx^α represent the tensors in the case of transformation into space coordinates x^α, and the element a_{ij} and the differential du^i the tensors in the case of transformation into Gaussian coordinates u^i.

The relation between the metric tensors (9) contains the elements with Greek and Latin indices, which means that the partial derivative

$$x_i^\beta = \frac{\partial x^\beta}{\partial u^i} \tag{10}$$

can be considered as a contravariant tensor of rank one in space E_3 or as a covariant tensor of rank one on surface S.

An arbitrary point P on surface S which is defined by Gaussian coordinantes u^i, $i = 1,2$ or by a space coordinate system x^α, $\alpha = 1,2,3$ can be defined also by a radius-vector

$$\Pi^\alpha = x^\alpha (u^1, u^2) = y^\alpha (x^1, x^2, x^3) \tag{11}$$

From equality (11) we can define by transformation in point P

$$\frac{\partial y^\varphi}{\partial u^i} = \frac{\partial y^\varphi}{\partial x^\beta} \frac{\partial x^\beta}{\partial u^i} = \frac{\partial y^\alpha}{\partial x^\beta} x_i^\beta \tag{12}$$

the fundametal covariant tensor of curvilinear coordinates x^α

$$h_\alpha = \frac{\partial y^\varphi}{\partial x^\alpha} \text{ or } h_\beta = \frac{\partial y^\varphi}{\partial x^\beta} \text{ respectively} \tag{13}$$

and the fundamental covariant tensor of Gaussian coordinates u^i

$$f_i = \frac{\partial y^\varphi}{\partial u^i} \text{ or } f_j = \frac{\partial y^\varphi}{\partial u^j}. \text{ respectively} \tag{14}$$

Considering tensors (13) and (14) the fundamental metric tensors (4) and (7) can be written in a simpler form

$$a_{ij} = f_i f_j \text{ or } g_{\alpha\beta} = h_\alpha h_\beta \text{ respectively} \tag{15}$$

The covariant unit tensor of normal n_φ in point P on shell surface S is defined in a curvilinear space coordinate system x^α, $\alpha = 1,2,3$

$$n_\varphi = \frac{1}{2} \chi^{ij} d_{\varphi\alpha\beta} x_i^\alpha x_j^\beta \tag{16}$$

where the absolute Ricci's anti-symmetric tensors

$$\chi^{ij} = \frac{1}{\sqrt{|a_{kl}|}} e^{ij} \quad \text{ or } d_{\varphi\alpha\beta} = \sqrt{|g_{\psi\vartheta}|}\, e_{\varphi\alpha\beta} \tag{17}$$

consist of the determinants of metric tensor a_{kl} and $g_{\psi\vartheta}$ and e-system representing the anti-symmetric relative tensors of weight $+1$, [5].

The covariant unit tensors of normal n_φ and the contravariant tensor of radius-vector $R^\varphi = x^\varphi$, are also functions of the Gaussian coordinates u^i, $i = 1,2$, hence their differentials are the same.

$$dn_\varphi = \frac{\partial n_\varphi}{\partial u^i} du^i \quad \text{or} \quad dy^\varphi = \frac{\partial y^\varphi}{\partial u^j} du^j \tag{18}$$

Their scalar product yields the second fundamental quadratic form on shell surface S

$$dn_\varphi \, dy^\varphi = \frac{\partial n_\varphi}{\partial u^i} \frac{\partial y^\varphi}{\partial u^j} du^i \, du^j \tag{19}$$

Defining the expression

$$b_{ij} = -\frac{1}{2}\left(\frac{\partial n_\varphi}{\partial u^i} \frac{\partial y^\varphi}{\partial u^j} + \frac{\partial n_\varphi}{\partial u^j} \frac{\partial y^\varphi}{\partial u^i}\right) \tag{20}$$

and taking into account equation (14) and the covariant vector

$$m_i = \frac{\partial n_\varphi}{\partial u^i} \tag{21}$$

we can simplify expressions (20) and (19)

$$b_{ij} = -\frac{1}{2}(m_i f_j + m_j f_i) \tag{22}$$

$$dn_\varphi \, dy^\varphi = b_{ij} \, du^i \, du^j \tag{23}$$

Under consideration of expressions (15) ad (22) the normal curvatures of the shell elements k_{ij} in the direction of the Gaussian coordinates u^i, $i = 1,2$ are

$$k_{ij} = -\frac{b_{ij}}{f_i f_j} \tag{24}$$

The covariant components of the tensor of strain ε_{ik} and the changes in curvature $\varkappa_{ik}$ of a thin shell element are, [2]:

$$2\,\varepsilon_{ik} = \overline{a}_{ik} - a_{ik}$$

$$\varkappa_{ik} = b_{ik} - \overline{b}_{ik} \tag{25}$$

where

$$\overline{a}_{ik} = c_{ik} + c_{ki} + a^{js} c_{ij} c_{ks} + \omega_i \omega_k + a_{ik}$$

$$\overline{b}_{ik} = \sqrt{I}\,\{E_3 \omega_{k,i} + E_j c^j_{k,i} + b_{ij} \overline{a}_{ks} [(2 + c^l_l) a^{js} - (\delta^s_t + c^s_t) a^{jt}]\}$$

$$c_{ik} = v_{k,i} - w\, b_{ik}$$

$$\omega_i = w_{,i} + b_{ik}\, v^k$$

$$E_i = \omega_k\, \overset{k}{c_i} - \omega_i\, (1 + \overset{l}{c_l})$$

$$E_3 = (1 + \overset{1}{c_1})\,(1 + \overset{2}{c_2}) - \overset{1}{c_2}\, \overset{2}{c_1}$$

$$I = 1 + 2\, \overset{k}{\varepsilon_k} + 4\, (\overset{1}{\varepsilon_1}\, \overset{2}{\varepsilon_2} - \overset{2}{\varepsilon_1}\, \overset{1}{\varepsilon_2})$$

$$\overset{i}{c_k} = a^{is}\, c_{ks} \tag{26}$$

$$a^{ik} = \chi^{ij}\, \chi^{ks}\, a_{sj}$$

$$a_{tk}\, a^{ks} = \delta^s_t$$

$$a^n_i = \chi_{is}\, a^{sn}$$

$$\overset{i}{\varepsilon_k} = a^{is}\, \varepsilon_{ks}$$

The covariant derivatives of tensors (14) and (21) in the above expressions can be written also in another form using the formulas of Gauss [5].

The relation between the tensors of moment M^{st}, strain ε_{ik} and the change in curvature $\varkappa_{ik}$ is defined on the basis of rheological equations [6] for a continuum following Hook's rheological model

$$M^{ij} = DG^{ijkl}\, \varkappa_{kl} + BH^{ijkl}\, \varepsilon_{kl} \tag{27}$$

where the contravariant tensors are

$$G^{ijkl} = a^{ik}\, a^{jl} + \nu\, \chi^{ik}_{\cdot}\, \chi^{jl} - \frac{1+\nu}{2}\, \chi^{\,ij}_{\cdot}\, \chi^{kl}$$

$$H^{ijkl} = -\, a^{st}\, b_{st}\, a^{ik}\, a^{jl} + 2\, a^{ik}\, b^{jl} - \frac{1+\nu}{2}\, (a^{ik}\, b^{jl} - a^{jl}\, b^{ik}) \tag{28}$$

Considering assumption c) we can neglect some terms in expressions (25), (27) and (28). The covariant tensors of strain and change in curvature are then

$$2 f_i f_k\, \varepsilon_{ik} = c_{ik} + c_{ki} \tag{29}$$

$$f_i\, f_k\, \varkappa_{ik} = -\, \omega_{k,i} - b_{ij}\, \overset{j}{e_k}$$

similarly under consideration of assumption c) we can simplify also expression (27)

$$M^{ij} = D\, (a^{ik}\, a^{jl} + \nu\, \chi^{ik}\, \chi^{jl})\, \varkappa_{kl} \tag{30}$$

or in inverse form

$$\varkappa_{ik} = D' (a_{is} a_{kj} - \nu \chi_{is} \chi_{kj}) M^{sj} \tag{31}$$

the relation between the elements of the tensor of deformation of the shell v_i and the contravariant or covariant tensor of the internal moment can be obtained making use of equation (29) in equation (31)

$$\omega_{k,i} + b_{ij} c^j_k = - \frac{f_i f_k}{E \mathcal{J}} (a_{is} a_{kj} - \nu \chi_{is} \chi_{kj}) M^{sj}$$

or (32)

$$\omega_{k,i} + b_{ij} c^j_k = - \frac{f_i f_k}{E \mathcal{J}} (M_{ki} - \nu a^s_i a^j_k M_{sj})$$

The physical coordinates of covariant components of the displacement tensor $v_{(i)}$ and of the bending moments $M_{(sj)}$ are defined from equations

$$v_{(i)} = \frac{v_i}{f_i} = (u, v)$$

$$M_{(sj)} = \frac{M_{sj}}{f_s f_j} = (M_{u1}, M_{u2}) \tag{32}$$

Considering the rules for a covariant derivative of a covariant and contravariant and contravariant tensor of rank one with respect to the Gaussian coordinate system

$$v_{i,k} = \frac{\partial v_i}{\partial u^k} - v_j \Gamma^j_{ik}$$

$$v^i_{,k} = \frac{\partial v^i}{\partial u^k} + v^j \Gamma^i_{jk} \tag{34}$$

$$\Gamma^k_{ij} = \frac{1}{2} a^{kh} \left(\frac{\partial a_{jh}}{\partial u^i} + \frac{\partial a_{hi}}{\partial u^j} - \frac{\partial a_{ij}}{\partial u^h} \right)$$

we can write the first equations (29) and the second equation (32) for the case $i = k = 1$ in the following form, [6]

$$\varepsilon_1 = \frac{1}{f_1} \frac{\partial u}{\partial u^1} + \frac{1}{f_1 f_2} \frac{\partial f_1}{\partial u^2} v + \frac{w^{\cdot}}{R_{11}} \tag{35}$$

$$\frac{1}{f_1} \frac{\partial}{\partial u^1} \left(- \frac{1}{f_1} \frac{\partial w}{\partial u^1} + \frac{u}{R_{11}} - \frac{v}{R_{12}} \right) + \frac{1}{f_1 f_2} \frac{\partial j_1}{\partial u^2} \left(\frac{1}{f_2} \frac{\partial w}{\partial u^2} + \frac{v}{R_{22}} - \frac{u}{R_{12}} \right) +$$

$$+ \frac{1}{2} \frac{1}{R_{12}} \frac{1}{f_1 f_2} \left[\frac{\partial}{\partial u^2} (f_1 u) - \frac{\partial}{\partial u^1} (f_2 v) \right] = \frac{1}{E \mathcal{J}} (M_u - \nu M_v) \tag{36}$$

Choosing the Gaussian coordinates $u^1 = \varphi$, $u^2 = \psi$, the function of the radius vector $\Pi^\alpha = y^\alpha$, $\alpha = 1,2,3$ defining the shell is, Fig. 2,

$$\Pi^\alpha = y^\alpha = i^1 R(\vartheta) \cos \vartheta \cos \psi + i^2 R(\vartheta) \sin \vartheta \cos \psi + i^3 R(\vartheta) \sin \psi \qquad (37)$$

where

$$\vartheta = \vartheta(\varphi)$$

Considering the assumptions c) and g) the stresses and displacement in the direction of the Gaussian axis ψ can be neglected. Then the relation between the curvature radius $R_{11} = r_\varphi$ and the radius-vector Π^α is the following

$$\frac{1}{R_{11}} = \frac{1}{r_\varphi} = -\frac{1}{f_1^2} \frac{\partial^2 \Pi^\alpha}{\partial \vartheta^2} n^\alpha = \frac{R^2(\vartheta) + 2\dot{R}^2(\vartheta) - \ddot{R}(\vartheta) R(\vartheta)}{[\dot{R}^2(\vartheta) + R^2(\vartheta)]^{3/2}} \qquad (38)$$

Since the value of the first element f_1 of the metric tensor a_{ij} is

$$f_1 = \vartheta'(\varphi) \sqrt{\dot{R}^2(\vartheta) + R^2(\vartheta)}, \quad \vartheta'(\varphi) = \frac{\dot{R}^2(\vartheta) + R^2(\vartheta)}{R^2(\vartheta) + 2\dot{R}^2(\vartheta) - R(\vartheta)\ddot{R}(\vartheta)} \qquad (39)$$

and hence

$$f_1 = R_{11} = r_\varphi = r(\sigma)$$

we can write taking account of assumption h), equations (35) and (36) determining the deformation

$$w = -\frac{\partial u(\sigma)}{\partial \varphi} = -u'(\varphi)$$

$$u'''(\varphi) + u'(\varphi) - \frac{r'(\varphi)}{r(\varphi)} [u''(\varphi) + u(\varphi)] = \frac{r_\varphi^2 M_\varphi}{E(\varphi) J(\varphi)} \qquad (40)$$

3. Solution of the basic equation of displacements

On the basis of the assumption and restrictions due to the geometry and external load we can define the element of the tensor of internal bending moments M_φ from the equilibrium condition for the internal static equilibrium state on a deformed system. In this case the effect of the element displacement in the direction of the coordinate ax is y^3 on the internal bending moment is taken into account, Fig. 2. To simplify the expression let us choose $y^1 = x$, $y^2 = y$, $y^2 = z$. Considering the relation of the elements of the displacement tensor in the direction of the normal n_φ and Gaussian coordinate $u^1 = \varphi$ to the displacements in the direction of the coordinate axes y and z, Fig. 2. and making use of the first equation under (40), we can write

$$\begin{aligned} v_z &= -u'(\varphi) \sin \varphi + u \cos \varphi \\ v_y &= +u'(\varphi) \cos \varphi + u \sin \varphi \end{aligned} \qquad (41)$$

then the internal bending moment $M_\varphi = M_\vartheta$ in consideration that $\varphi = \varphi(\beta)$ is

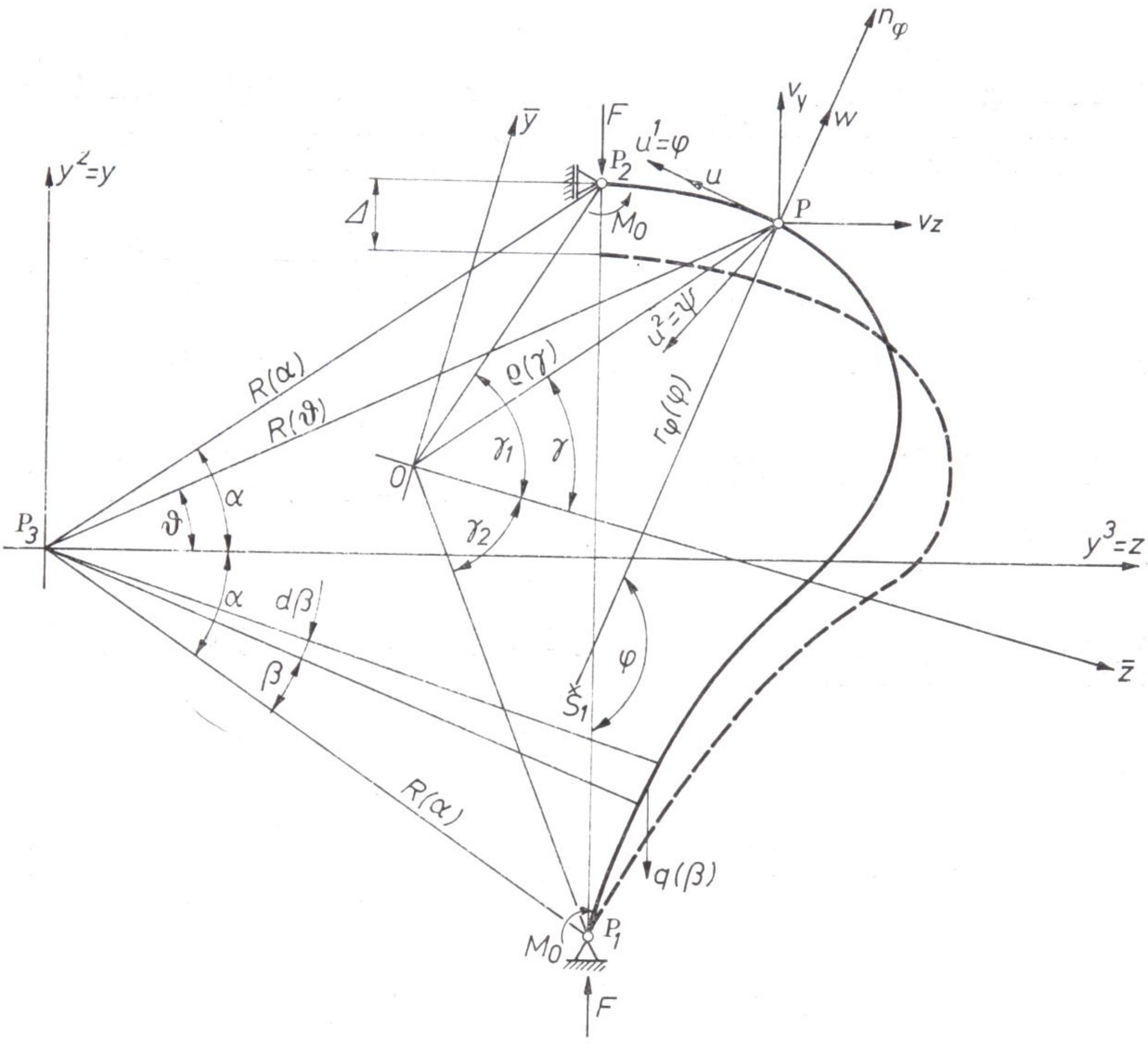

Fig. 2 Axis of a curvilinear element for a given clamping

$$M_\varphi = M_\vartheta = M_o + [R(\vartheta)\cos\vartheta - R(\alpha)\cos\alpha + v_z(\vartheta)]\ [F + \int_0^{\alpha+\vartheta} q(\beta)\, r(\beta)\, \varphi'(\beta)\, d\beta] -$$

$$- \int_0^{\alpha+\vartheta} q(\beta)\, r(\beta)\, [R(\vartheta)\cos\vartheta - R(\beta)\cos(\beta-\alpha) + v_z(\beta)]\ \varphi'(\beta)\, d(\beta) \qquad (42)$$

Now let us choose a new indepedent variable

$$\xi = \cos\varphi$$

Inserting it into equations (41), we then obtain

$$v_z = (1 - \xi^2)^{3/2} \frac{d}{d\,\xi} \left[\frac{u(\xi)}{\sqrt{1 - \xi^2}} \right]$$

$$v_y = - \xi^2 \sqrt{1 - \xi^{2}}\, \frac{d}{d\xi} \left[\frac{u(\xi)}{\xi} \right] \tag{44}$$

and the second equation under (40) in consideration of relations $\vartheta = \vartheta(\xi)$ $\varphi = \varphi(\xi)$ and $\beta = \beta(\xi)$ yields

$$v_z''(\xi) - Q(\xi)\, v'_z(\xi) + G[\xi,\ v_z(\xi)] = f(\xi) \tag{45}$$

where the functions are:

$$Q(\xi) = \frac{r'(\xi)}{r(\xi)}$$

$$G[\xi,\ v_z(\xi)] = \frac{A(\xi)}{\sin\varphi(\xi)} \{v_z(\xi)\,[F + \int_0^{\alpha+\vartheta(\xi)} q(\beta)\, r(\beta)\, \varphi'(\beta)\, d\beta] -$$

$$- \int_0^{\alpha+\vartheta(\xi)} p(\beta)\, r(\beta)\, v_z(\beta)\, \varphi'(\beta)\, d\beta\}$$

$$f(\xi) = - \frac{A(\xi)}{\sin\varphi(\xi)} \{M_0 + [E + \int_0^{\alpha+\vartheta(\xi)} q(\beta)\, r(\beta)\, \varphi'(\beta)\, d\beta]\ [R(\xi)\cos\vartheta(\xi) -$$

$$- R(\alpha)\cos\alpha] - \int_0^{\alpha+\vartheta(\xi)} q(\beta)\, r(\beta)\, [R(\xi)\cos\vartheta(\xi) - R(\beta)\cos(\beta - \alpha)]\, \varphi'(\beta)\, d\beta\}$$

$$A(\xi) = \frac{r\,\varphi^2(\xi)}{E(\xi)\, J(\xi)} \tag{46}$$

From functions (46) it can be seen that equation (45) represents an integro-differential equation with incostant coefficients. The possiblity of the analytical solution of this equation for a general case is probably very small. Therefore we shall limit only to the cases where no continuous load is present, $q(\beta) = 0$. In this case the integrodifferential equation changes into a differential equation of rank two with incostant coefficients

$$v_z''(\xi) - \frac{r'(\xi)}{r(\xi)}\, v_z'(\xi) + \frac{F\, A(\xi)}{\sin\varphi(\xi)}\, v_z(\xi) = - \frac{A(\psi)}{\sin\varphi(\xi)} \{M_0 + F\,[R(\xi)\,\cdot$$

$$\cdot \cos\vartheta(\xi) - R(\alpha)\cos\alpha)\} = f_1(\xi) \tag{47}$$

The solution of the homogeneous part of the differential equation (47) can be defined by the power series [7]

$$v_{zH} = \eta(\xi) = \sum_{i=0}^{i=\infty} a_i \xi^i \tag{48}$$

For this purpose we have to develop also the following functions in the power series

$$\frac{r'(\xi)}{r(\xi)} = \sum_{k=0}^{k=\infty} c_k \xi^k$$
$$\frac{F A(\xi)}{\sin\varphi(\xi)} = \sum_{n=0}^{n=\infty} b_n \xi^n \tag{49}$$

Hence the i-th element a_i of the searched homogeneous solution is

$$a_i = \frac{1}{i(i-1)} \{ \sum_{k=0}^{k=\infty} (i-k-1) c_k a_{i-k-1} - \sum_{n=0}^{n=\infty} b_n a_{i-n-2} \} \tag{50}$$

Choosing first as the value of the elements $a_0 = 1$, and $a_1 = 0$, and secondly vice versa $a_0 = 0$, and $a_1 = 1$, then the homogenous part of the solution of the differential equation (47) is

$$\eta(\xi) = c_1 \eta_1(\xi) + c_2 \eta_2(\xi) \tag{51}$$

where

$$\eta_1(\xi) = 1 + \frac{\xi^i}{i(i-1)} \sum_{i=2}^{i=\infty} [\sum_{k=0}^{k=\infty} (i-k-1) c_k a_{i-k-1} - \sum_{n=0}^{n=\infty} b_n a_{i-n-2}]$$
$$\eta_2(\xi) = \xi + \sum_{i=1}^{i=\infty} \frac{\xi^i}{i(i-2)} [\sum_{k=1}^{k=\infty} (i-k-1) c_k a_{i-k-1} - \sum_{n=0}^{n=\infty} b_n a_{i-n-2}] \tag{52}$$

The particular solution is determined on the basis of the known homogeneous solutions $\eta_1(\xi)$ and $\eta_2(\xi)$, [8]

$$v_{zp}(\xi) = -\eta_1(\xi) \int_{\xi_0}^{\xi} \frac{\eta_2(\xi)}{W(\xi)} f_1(\xi) d\xi + \eta_2(\xi) \int_{\xi_0}^{\xi} \frac{\eta_1(\xi)}{W(\xi)} f_1(\xi) d\xi \tag{53}$$

where Wronski's determinant is

$$W(\xi) = \begin{vmatrix} \eta_1(\xi), & \eta_2(\xi) \\ \eta_1'(\xi), & \eta_2'(\xi) \end{vmatrix} \tag{54}$$

The global solution for the displacement of the element in the direction of axis z is then

$$v_z(\xi) = c_1\, \eta_1(\xi) + c_2\, \eta_2(\xi) + v_{zp}(\xi) = v_{z1}[\xi(\varphi)] = v_{z1}(\varphi) \tag{55}$$

The displacements in the direction of the Gaussian coordinate φ, of the normal n_φ and the derivative are:

$$u(\varphi) = -\sin\varphi \int \frac{v_{z1}(\varphi)}{\sin^2\varphi}\, d\varphi + c_3 \sin\varphi$$

$$w(\varphi) = \cos\varphi \int \frac{v_{z1}(\varphi)}{\sin^2\varphi}\, d\varphi + \frac{v_{z1}(\varphi)}{\sin\varphi} - c_3\cos\varphi \tag{56}$$

$$w'(\varphi) = -\sin\varphi \int \frac{v_{z1}(\varphi)}{\sin^2\varphi}\, d\varphi + \frac{v_{z1}'(\varphi)}{\sin\varphi} + c_3 \sin\varphi$$

The displacement of the element in the direction of axis y can be defined from equation (41) or (44).

In the Cartesian coordinate system the displacements (55) and (56) contain two and in the Gaussian coordinate system three free integration constants. Hence on the both boundaries of the element we can prescribe three boundary conditions. An additional boundary condition can be set in the case when we are interested in the value of external load which causes the prescribed displacement of a definite point on the element.

The normal stress appearing in the cross section of a uniaxial element is

$$\sigma_\varphi = \frac{F}{p(\varphi)\, h(\varphi)} + \frac{h(\varphi)}{2\mathcal{J}(\varphi)} \{M_0 + F[R(\varphi)\cos\vartheta(\varphi) - R(\alpha)\cos\alpha] + v_{z1}(\varphi)\} \tag{57}$$

4. Selection of the most appropriate coordinate system

The determination of the elements of the displacement state tensor seems to be the easiest in the polar coordinate system $\rho = \rho(\gamma)$. In this coordinate system the expression of the shape of the element axis $\psi = 0$ is very simple. The Cartesian coordinates are

$$\begin{aligned} \bar{y}(\gamma) &= \rho(\gamma)\sin\gamma \\ \bar{z}(\gamma) &= \rho(\gamma)\cos\gamma \end{aligned} \tag{58}$$

The elasto-static problem of a uniaxial element with big displacements is here treated in the Cartesian coordinate system (y, z) or in the polar coordinate system $R = R(\vartheta)$. The coordinate axes are chosen so that axis y is parallel to the line $\overline{P_1 P_2}$, while axis z cuts this line in two. The point of origin of the coordinate system is chosen in point P_3, *Fig.* 3.

Considering coordinates (58) the polar coordinates of the chosen coordinate system are

$$R(\gamma) = [(\bar{y} - \bar{y}_3)^2 + (\bar{z} - \bar{z}_3)^2]^{1/2}$$

$$\vartheta(\gamma) = \arctan \frac{(\bar{y} - \bar{y}_3)(\bar{y}_2 - \bar{y}_1) + (\bar{z} - \bar{z}_3)(\bar{z}_2 - \bar{z}_1)}{(\bar{z} - \bar{z}_3)(\bar{y}_2 - \bar{y}_1) + (\bar{y} - \bar{y}_3)(\bar{z}_1 - \bar{z}_2)} \tag{59}$$

since also $\gamma = \gamma(\vartheta)$ and hence $R = R(\vartheta)$ we can express the curvature radius $r_\varphi(\vartheta)$ in the coordinate system (y, z) from expression (38) and with respect to the chosen points P_1, P_2, and P_3 also the Gaussian coordinate $\varphi(\vartheta)$

$$\varphi(\vartheta) = \arctan \frac{R(\vartheta)\sin\vartheta - \dot{R}(\vartheta)\cos\vartheta}{R(\vartheta)\cos\vartheta + \dot{R}(\vartheta)\sin\vartheta} + \frac{\pi}{2} \tag{60}$$

Example 1

We chose a thin curved element with a constant radius of curvature $\rho(\gamma) = R(\vartheta) = r_\varphi = 16{,}5$ mm having a rectangular cross section of constant thickness $h = 0{,}1$ mm, manufactured from a homogeneous isotropic alloy BERYLCO 251/2HT with a constant Young's module $E = 1{,}35 \cdot 10^5$. N/mm^2.

The width of the element cross section is a linear function of coordinate

$$p(\gamma) = -0{,}20879\,\gamma + 1{,}215 \text{ [mm]}$$

The element is subjected on the boundaries to a compressive force F [N], the lower boundary $\gamma = -\alpha$ is clamped, while the upper boundary is free. The range of the element is limited by the inequation

$$|\gamma| \leqslant 59^\circ$$

The physical coordinates of the displacement tensor in the coordinate systems (r_φ, φ) and (y, z) were calculated by the aid of a computer program using the programming language FORTRAN. The program is designed so that it can define displacements at an arbitrarv angle γ. Is also calculates the maximum normal stress σ_φ and angle γ_0 where this stress appears. The program enables also the treatment of these elements the width of which is a discontinuous function and is given in tabulated form.

The displacements of the free boundary of the element $\gamma = +\alpha$ for different values of force F can be seen from the following Table.

Table 1

$F\,[N]$	u [mm]	w [mm]	v_y [mm]	v_z [mm]	γ_0 [^{0}]	σ [N/mm^2]
0,01	0,90305	—1,24751	—0,59907	—1,42515	2,46	24,5
0,0311	2,69887	—3,51519	—1,62307	—4,12384	—4,91	69,5
0,0622	5,00613	—6,04032	—2,59918	—7,40208	—9,83	124,0
0,125	8,62876	—8,68387	—2,99934	—11,86881	—17,21	211,5

Example 2

A uniaxial curved element was chosen with inconstant radius of curvature, inconstant width of the element cross section, and with constant thickness and elasticity module. The element was subjected on both boundaries to bending moment $M_0 > 0$.

In this case the solution of the differential equation (47) writen in the coordinate system (y, z) with polar coordinates $\rho(\gamma)$ and γ is:

$$v_z(\gamma) = \int \left[\bar{A} + \int \frac{A(\gamma)\, M_0}{r(\gamma)} \varphi'(\vartheta)\, \vartheta'(\gamma)\, d\gamma \right] r(\gamma)\, \xi'(\varphi)\, \varphi'(\vartheta)\, \vartheta'(\gamma)\, d\gamma + B \tag{61}$$

where

$$\xi'(\varphi) = -\frac{R(\gamma)\cos\vartheta(\gamma) + R'(\gamma)\sin\vartheta(\gamma)}{[R^2(\gamma) + R'^2(\gamma)]^{1/2}}$$

$$\varphi'(\vartheta) = \frac{R^2(\gamma) + 2\,R'^2(\gamma) - R(\gamma)\,R''(\gamma)}{R^2(\gamma) + R'^2(\gamma)} \tag{62}$$

$$R'(\vartheta) = \frac{R'(\gamma)}{\vartheta'(\gamma)}$$

$$R''(\vartheta) = \frac{R''(\gamma)\,\vartheta'(\gamma) - R'(\gamma)\,\vartheta''(\gamma)}{\vartheta'^3(\gamma)}$$

Similarly also other displacements (56) can be defined where in the total differential $d\varphi = \varphi'(\vartheta)\,\vartheta'(\gamma)\cdot d\gamma$ we take account of expressions (62) and the expression for $\sin\varphi$ and $\cos\varphi$

$$\sin\varphi = -\,\xi'(\varphi)$$

$$\cos\varphi = \sqrt{1 - \xi'^2(\varphi)} \tag{63}$$

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BIEGUNG DUNNER EINACHSIGER KRUMMLINIGER ELEMETE MIT VERANDERLICHER BIEGEFESTIGKEIT UND GROSSEN VERSCHIEBUNGEN

Zusammenfassung

Dieser Beitrag behandelt die Biegung krummliniger einachsiger Elemente mit veränderlichen Biegefestigkeit, die wegen ihrer Dünnwandigkeit auch grosse Verschiebungen ausweisen. Der Werkstoff der Elemente, für die die grundsätzliche Differenzialgleichung für die Bestimmung von grossen Verscheibungen ausgeführt ist, folgt Hookeschen reologisches Modell. Der Einfluss der grossen Verschiebungen wurde bei der Bestimmung des Biegemomentes auf der Wieise berücksichtigt, dass der Gleichgewichtstand auf dem deformierten System behandelt wurde. Die Komponenten des Verschiebungsvektors werden allgemein in der Form von Potenzreihen bzw. für das spezifische Belastungsbeispiel in endlicher Form bestimmt. Zwei Zahlenbeispiele werden berechnet: das erste Beispiel wurde mit Hilfe des Rechners gelöst und betrifft ein Element mit konstanter' Krümmung, wo die Breite des rechteckigen Querrschnitts linear veränderlich ist. Das Element ist von einer Druckkraft belastet, und auf dem unteren Ende befestigt während das obere Ende frei ist. Das zweite Beispiel wird in der endlicher Form analytisch gelöst und betrifft ein Element mit veränderlicher Biegefestigkeit der Achse und Breite des rechteckigen Querrschnitts, das auf der beiden Enden von dem Moment M_0 belastet ist. Das untere Ende ist drehbar befestigt während das obere ist freiwillig bewegbar auf der Verbindungslinie der beiden Enden.

UPOGIB VITKIH UKRIVLENIH ENOOSNIH ELEMENTOV Z NEKONSTANTNO TOGOSTJO IN VELIKIMI PREMIKI

Povzetek

V tem prispevku je obravnavan upogib ukrivljenih enoosnih elementov z nekonstantno togostjo, ki imajo zaradi velike vitkosti tudi velike premike. Osnovna diferencialna enačba za določitev velikih premikov je izvedena za elemente izdelane iz gradiva, ki sledi Hook-ovemu reološkemu modelu. Vpliv velikih premikov je bil upoštevan pri določanju upogibnih momentov tako, da je bilo ravnotežno stanje obravnavano na deformiranem sistemu. Komponente vektorja premika so določene splošno v obliki potenčnih vrst, oziroma za poseben primer obremenitve v končni obliki.

Izračunana sta tudi dva številčna primera. Prvi primer je rešen z uporabo računalnika in predstavlja element konstantne ukrivljenosti, ki se mu širina pravokotnega prereza linearno spreminja. Obremenjen je s tlačno silo, vpet na spodnjem krajišču, zgornje krajišče pa je prosto.

Drugi primer je rešen analitično v končni obliki in predstavlja element nekonstantne ukrivljenosti osi in širine pravokotnega prereza. Obremenjen je z momentom M_0 na obeh krajiščih. Spodnje krajišče je vrtljivo vpeto, zgornje pa se lahko prosto premika po zveznici obeh krajišč.

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