

THE STRESS FUNCTIONS IN THE GENERALIZED BENDING THEORY OF ANISOTROPIC PLATES

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1. Introduction

In article [1] the generalized bending theory of anisotropic plates was introduced. It was shown that in order to describe the total stress-deformation state, six analytic functions are necessary:

$$\Omega(z_1), \omega(z_2), \Gamma(z_1), \gamma(z_2), \varphi(z_3), \psi(z_4) \quad (1.1)$$

where

$$z_k = x + \lambda_k y \quad (k = 1, 2, 3, 4) \quad (1.2)$$

and λ_k are complex numbers-roots of the characteristic equations (2.1)/[1] and (2.2)/[1]. (The term (2.1)/[1] stands for equation (2.1) found in article [1]).

In the present contribution the arbitrariness definition of functions (1.1) and their form in case of a multiply connected region, as well as the main force and the main moment will be analysed and formed out.

2. Arbitrariness of stress functions

Theorem 2.1 The components of stress tensor (2.7)/[1], the unit forces (3.4)/[1] and (3.5)/[1], as well as the unit moments (3.6)/[1] are not altered if one replaces

$$\begin{array}{ll} \Omega(z_1) & \text{by} \quad \Omega(z_1) = \frac{i}{2} \alpha^{(1)} C^{(1)} z_2 + \gamma^{(1)} z_1 + \tilde{\gamma}^{(1)} \\ \omega(z_2) & \text{by} \quad \omega_1(z_2) = \frac{i}{2} \alpha^{(2)} C^{(2)} z_2^2 + \gamma^{(2)} z_2 + \tilde{\gamma}^{(2)} \end{array}$$

$$\begin{aligned}
\Gamma(z_1) & \text{ by } \Gamma(z_1) = \frac{i}{2} \kappa^{(3)} C^{(3)} z_1^2 + \gamma^{(3)} z_1 + \gamma^{(3)} \\
\gamma(z_2) & \text{ by } \gamma_1(z_2) = \frac{i}{2} \kappa^{(4)} C^{(4)} z_2^2 + \gamma^{(4)} z_2 + \tilde{\gamma}^{(4)} \\
\varphi(z_3) & \text{ by } \varphi_1(z_3) = i \kappa^{(5)} C^{(5)} z_3 + \gamma^{(5)} \\
\psi(z_4) & \text{ by } \psi(z_4) = i \kappa^{(6)} C^{(6)} z_4 + \gamma^{(6)}
\end{aligned} \tag{2.1}$$

The complex constants κ^k ($k = 1, \dots, 6$) are defined by the coefficient of the Hook's matrix; all the remaining constants are free, whilst $C^{(k)}$ are real, and $\gamma^{(k)}$, $\tilde{\gamma}^{(k)}$ complex constants.

Theorem 2.2 If the displacements are defined as single valued functions, the constants $C^{(k)}$, $\gamma^{(k)}$, $\tilde{\gamma}^{(k)}$ appearing in the previous theorem are not free but must fulfill the following conditions:

$$R_e \gamma^{(i)} + R_e \gamma^{(k)} = 0 \quad (i = 1, k = 2 \text{ and } i = 3, k = 4) \tag{2.2}$$

$$I_m \lambda_1 I_m \gamma^{(i)} + I_m \lambda_2 I_m \gamma^{(k)} = R_e (\lambda_1 - \lambda_2) R_e \gamma^{(k)} \tag{2.3}$$

$$(i = 1, k = 2 \text{ and } i = 3, k = 4)$$

$$R_e \gamma^{(5)} + R_e \gamma^{(6)} = 0 \tag{2.4}$$

$$I_m \beta_o^{(3)} I_m \gamma^{(5)} + I_m \beta_o^{(4)} I_m \gamma^{(6)} = R_e (\beta_o^{(4)} - \beta_o^{(3)}) R_e \gamma^{(6)} \tag{2.5}$$

The proof of both theorems is done in the known manner [2]. Let us now analyse the example of the function $\varphi(z_3)$ only. From the equation (3.4) / [1] we shall eliminate the function $\psi(z_4)$. It can be established that the expression $\delta_1 \dot{\varphi}(z_3) + \overline{\delta_2 \dot{\varphi}(z_3)}$ is a single-valued function. If additionally the conjugated equation is taken into account, we get

$$\delta^{(5)} \dot{\varphi}(z_3) + \overline{\delta^{(5)} \dot{\varphi}(z_3)} = F(\Sigma_x, \Sigma_y, T_{xy}). \tag{2.6}$$

The further development is well known.

3. Form of stress functions for multiply connected regions]

The middle plane of plate should lie inside the contours c_o and outside the contours c_1, c_2, \dots, c_m and should include the region S of the plane $\{z: z = x + iy\}$. The equations

$$x_k = x + R_e(\lambda_k) y \tag{3.1}$$

$$y_k = I_m(\lambda_k) y$$

are defining the bijective mappig of the plane $\{z\}$ into the plane $\{z_k = x + \lambda_k y\}$. The contour c_j is mapped into contour $c_j^{(k)}$ ($j = 0, 1, \dots, m$; $k = 1, 2, 3, 4$); thus the orientation of the contour $c_j^{(k)}$ corresponds to the contour c_j if we require

$$\frac{\partial (x_k, y_k)}{\partial (x, y)} = \begin{vmatrix} 1 & R_e \lambda_k \\ 0 & I_m \lambda_k \end{vmatrix} = I_m \lambda_k > 0 \quad (3.3)$$

Theorem 3.1 In case of the finite plate with m holes, the stress functions have the following form:

$$\Omega(z_1) = \sum_{j=1}^m \kappa^{(1)} \left(\frac{1}{2} A_j^{(1)} z_1^2 + \gamma_j^{(1)} z_1 + \delta_j^{(1)} \right) \ln(z_1 - z_{1,j}) + \Omega^*(z_1) \quad (3.3)$$

$$\omega(z_2) = \sum_{j=1}^m \kappa^{(2)} \left(\frac{1}{2} A_j^{(2)} z_2^2 + \gamma_j^{(2)} z_2 + \delta_j^{(2)} \right) \ln(z_2 - z_{2,j}) + \omega^*(z_2) \quad (3.4)$$

$$\Gamma(z_1) = \sum_{j=1}^m \kappa^{(3)} \left(\frac{1}{2} A_j^{(3)} z_1^2 + \gamma_j^{(3)} z_1 + \delta_j^{(3)} \right) \ln(z_1 - z_{1,j}) + \Gamma^*(z_1) \quad (3.5)$$

$$\Upsilon(z_2) = \sum_{j=1}^m \kappa^{(4)} \left(\frac{1}{2} A_j^{(4)} z_2^2 + \gamma_j^{(4)} z_2 + \delta_j^{(4)} \right) \ln(z_2 - z_{2,j}) + \Upsilon^*(z_2) \quad (3.6)$$

$$\varphi(z_3) = \sum_{j=1}^m \kappa^{(5)} (A_j^{(5)} z_3 + \gamma_j^{(5)}) \ln(z_3 - z_{3,j}) + \varphi^*(z_3) \quad (3.7)$$

$$\psi(z_4) = \sum_{j=1}^m \kappa^{(6)} (A_j^{(6)} z_4 + \gamma_j^{(6)}) \ln(z_4 - z_{4,j}) + \psi(z_4) \quad (3.8)$$

$\kappa^{(i)}$ ($i = 1, \dots, 6$) are complex constants, defined by terms of the Hook's matrix

$\gamma_j^{(i)} \delta_j^{(i)}$ ($i = 1, \dots, 6$) are complex constants

$A_j^{(i)}$ ($i = 1, \dots, 6$) are real constants

$z_{k,j}$ ($k = 1, 2, 3, 4$) are fixed points inside the contours $c_j^{(k)}$

The functions marked with stars are holomorphic functions in the correspondent region $S^{(k)}$ which by means of the equations (3.1) represents the image of region S .

The proof of this theorem runs in known ways. Let us now study only the example of function φ . Due to the equation (2.6) it is evident that $R_e [\delta^{(5)} \varphi(z_3)]$ in the region $S^{(3)}$ is a single-valued function. In the known way it can be established that

$$\delta^{(5)} \dot{\phi}(z_3) = \sum_{j=1}^m A_j^{(5)} \ln(z_3 - z_{3,j}) + \Phi^*(z_3)$$

Due to this expression we get $\dot{\phi}(z_3)$. If $\kappa^{(5)} = 1/\delta^{(5)}$ is introduced and then integrated, we actually get (3.7).

Theorem 3.2 If the displacements defined by equations (2.6)/[1] are required to be single-valued functions, then the stress functions have the following form:

$$\Omega(z_1) = \sum_{j=1}^m \kappa^{(1)} \delta_j^{(1)} \ln(z_1 - z_{1,j}) + \Omega^*(z_1) \quad (3.9)$$

$$\omega(z_2) = \sum_{j=1}^m \kappa^{(2)} \delta_j^{(2)} \ln(z_2 - z_{2,j}) + \omega^*(z_2) \quad (3.10)$$

$$\Gamma(z_1) = \sum_{j=1}^m \kappa^{(3)} \delta_j^{(3)} \ln(z_1 - z_{1,j}) + \Gamma^*(z_1) \quad (3.11)$$

$$\gamma(z_2) = \sum_{j=1}^m \kappa^{(4)} \delta_j^{(4)} \ln(z_2 - z_{2,j}) + \gamma^*(z_2) \quad (3.12)$$

$$\phi(z_3) = \sum_{j=1}^m \kappa^{(5)} \gamma_j^{(5)} \ln(z_3 - z_{3,j}) + \phi^*(z_3) \quad (3.13)$$

$$\Psi(z_4) = \sum_{j=1}^m \kappa^{(6)} \gamma_j^{(6)} \ln(z_4 - z_{4,j}) + \Psi^*(z_4) \quad (3.14)$$

The complex constants $\delta_j^{(1)} \delta_j^{(2)} \delta_j^{(3)} \delta_j^{(4)} \gamma_j^{(5)} \gamma_j^{(6)}$ present in these equations and characteristic for contour c_j , are not entirely free but are connected with the following four equations:

$$I_m \kappa^{(2)} R_e \delta_j^{(2)} + R_e \kappa^{(2)} I_m \delta_j^{(2)} = I_m \lambda_1 R_e \delta_j^{(1)} + R_e \kappa^{(1)} I_m \delta_j^{(1)} \quad (3.15)$$

$$I_m \kappa^{(4)} R_e \delta_j^{(4)} + R_e \kappa^{(4)} I_m \delta_j^{(4)} = I_m \kappa^{(3)} R_e \delta_j^{(3)} + R_e \kappa^{(3)} I_m \delta_j^{(3)} \quad (3.16)$$

$$I_m \kappa^{(5)} R_e \gamma_j^{(5)} + R_e \kappa^{(5)} I_m \gamma_j^{(5)} + I_m \kappa^{(6)} R_e \gamma_j^{(6)} + R_e \kappa^{(6)} I_m \gamma_j^{(6)} = 0 \quad (3.17)$$

$$I_m (\beta_o^{(3)} \kappa^{(5)}) R_e \gamma_j^{(5)} + R_e (\beta_o^{(3)} \kappa^{(5)}) I_m \gamma_j^{(5)} + I_m (\beta_6^{(4)} \kappa^{(6)}) R_e \gamma_j^{(6)} + R_e (\beta_6^{(4)} \kappa^{(6)}) I_m \gamma_j^{(6)} = 0 \quad (3.18)$$

Proof. The coefficients formed at particular powers of the variable Z in equations (2.6)/[1] must be single-valued functions. If the mentioned equations (3.9) — (3.14) are taken into account and if we consider that on the counter-

-clockwise round of the contour c_j the function $\ln(z_k - z_{k,j})$ is increased by $2\pi i$, and $\ln(\bar{z}_k - \bar{z}_{k,j})$ is decreased by $2\pi i$, then the assertions of this theorem are established in the known way.

4. Main force and main moment

Definition 4.1 The main force along the arc \widehat{AB} is the vector

$$\vec{F}_{\widehat{AB}} = \frac{1}{2h} \int_{\widehat{AB}} \int_{-h}^{+h} (X_n, Y_n, Z_n) dZ ds \quad (4.1)$$

where

$$\begin{aligned} X_n &= \sigma_x \cos(n, x) + \tau_{xy} \cos(n, y) \\ Y_n &= \tau_{yx} \cos(n, x) + \sigma_y \cos(n, y) \\ Z_n &= \tau_{zx} \cos(n, x) + \tau_{zy} \cos(n, y) \end{aligned} \quad (4.2)$$

Let us take into consideration the unit force definition, i. e. the equations (3.1) – (3.2)/[1] and integrate it to the variable Z . If in addition the equations

$$\cos(n, x) ds = dy \quad \cos(n, y) = -dx \quad (4.3)$$

are taken into account, we get

$$\vec{F}_{\widehat{AB}} = \frac{1}{2h} \int_{\widehat{AB}} (\Sigma_x dy - T_{xy} dx, T_{xy} dy - \Sigma_y dx, N_x dy - N_y dx) \quad (4.4)$$

On the basis of the equations (3.4) – (3.5)/[1] it can quickly be appointed that under the integral sign in (4.4) there are total differentials.

If we consider

$$\begin{aligned} x &= \frac{i}{2 I_m \lambda_k} (\bar{\lambda}_k z_k - \lambda_k \bar{z}_k) \\ &(k = 1, 2, 3, 4) \end{aligned} \quad (4.5)$$

$$y = \frac{i}{2 I_m \lambda_k} (-z_k + \bar{z}_k)$$

resp.

$$dx = \frac{i}{2 I_m \lambda_k} (\bar{\lambda}_k dz_k - \lambda_k d\bar{z}_k) \quad (4.6)$$

$$dy = \frac{i}{2 I_m \lambda_k} (-dz_k + d\bar{z}_k)$$

the integral (4.4) can be calculated. Thus we obtain the following

Theorem 4.1 The components of the vector $[\vec{F}]_{\widehat{AB}}$ are the following:

$$\begin{aligned} [F_x]_{\widehat{AB}} &= -R_e [\beta_6^{(3)} \varphi(z_3) + \beta_6^{(4)} \Psi(z_4)]_B^A \\ [F_y]_{\widehat{AB}} &= -R_e [\beta_2^{(3)} \varphi(z_3) + \beta_2^{(4)} \Psi(z_4)]_B^A \end{aligned} \quad (4.7)$$

$$[F_z]_{\widehat{AB}} = h^2 R_e \left\{ \alpha_6^{(1)} \left[\frac{1}{3} \ddot{\Omega}(z_1) + \frac{1}{5} h^2 \ddot{\Gamma}(z_1) \right] + \alpha_5^{(2)} \left[\frac{1}{3} \ddot{\omega}(z_2) + \frac{1}{5} h^2 \ddot{\Upsilon}(z_2) \right] \right\}_B^A$$

On verifying, it must be taken into account that the equations

$$\beta_1^{(i)} + \lambda_i \beta_6^{(i)} = 0, \quad \beta_6^{(i)} + \lambda_i \beta_2^{(i)} = 0 \quad (i = 3, 4) \quad (4.8)$$

which are regarding to definitions (2.12) – (2.13)/[1] equivalent to equations (2.2) and (2.5)/[1], are in force.

Definition 4.2 The main moment along the arc \widehat{AB} is a vector:

$$\vec{M}_{\widehat{AB}} = \frac{1}{2h} \int_{\widehat{AB}} \int_{-h}^{+h} (Z_n y - Y_n z, X_n z - Z_n x, Y_n x - X_n y) dZ ds \quad (4.9)$$

Theorem 4.2 The coordinates of the main moment are by means of stress functions expressed as follows:

$$\begin{aligned} [M_x]_{\widehat{AB}} &= \frac{h^2}{6 I_m \lambda_1} I_m \{ \alpha_2^{(1)} (\bar{\lambda}_1 - \lambda_1) \dot{\Omega}(z_1) + \alpha_5^{(1)} (z_1 - \bar{z}_1) \dot{\Omega}(z_1) \}_B^A + \\ &+ \frac{h^2}{6 I_m \lambda_2} I_m \{ \alpha_2^{(2)} (\bar{\lambda}_2 - \lambda_2) \dot{\omega}(z_2) + \alpha_5^{(2)} (z_2 - \bar{z}_2) \dot{\omega}(z_2) \}_B^A + \\ &+ \frac{h^4}{10 I_m \lambda_1} I_m \{ \alpha_2^{(1)} (\bar{\lambda}_1 - \lambda_1) \dot{\Gamma}(z_1) + \alpha_5^{(1)} (z_1 - \bar{z}_1) \dot{\Gamma}(z_1) \}_B^A + \\ &+ \frac{h^4}{10 I_m \lambda_2} I_m \{ \alpha_2^{(2)} (\bar{\lambda}_2 - \lambda_2) \dot{\Upsilon}(z_2) + \alpha_5^{(2)} (z_2 - \bar{z}_2) \dot{\Upsilon}(z_2) \}_B^A \end{aligned} \quad (4.10)$$

$$\begin{aligned}
[M_y]_{AB} &= \\
&= \frac{h^2}{6 I_m \lambda_1} I_m \{(\lambda_1 - \bar{\lambda}_1) (2 \alpha_5^{(1)} - \lambda_1 \alpha_2^{(1)}) \dot{\Omega}(z_1) + \alpha_5^{(1)} (\bar{\lambda}_1 z_1 - \lambda_1 \bar{z}_1) \ddot{\Omega}(z_1)\}_B^A + \\
&+ \frac{h^2}{6 I_m \lambda_2} I_m \{(\lambda_2 - \bar{\lambda}_2) (2 \alpha_5^{(2)} - \lambda_2 \alpha_2^{(2)}) \dot{\omega}(z_2) + \alpha_5^{(2)} (\bar{\lambda}_2 z_2 - \lambda_2 \bar{z}_2) \ddot{\omega}(z_2)\}_B^A + \\
&+ \frac{h^4}{10 I_m \lambda_1} I_m \{(\lambda_1 - \bar{\lambda}_1) (2 \alpha_5^{(1)} - \lambda_1 \alpha_2^{(1)}) \dot{\Gamma}(z_1) + \alpha_5^{(1)} (\bar{\lambda}_1 z_1 - \lambda_1 \bar{z}_1) \ddot{\Gamma}(z_1)\}_B^A + \\
&\quad + \frac{h^4}{10 I_m \lambda_2} I_m \{(\lambda_2 - \bar{\lambda}_2) (2 \alpha_5^{(2)} - \lambda_2 \alpha_2^{(2)}) \dot{\gamma}(z_2) + \\
&\quad + \alpha_5^{(2)} (\bar{\lambda}_2 z_2 - \lambda_2 \bar{z}_2) \ddot{\gamma}(z_2)\}_B^A
\end{aligned} \tag{4.11}$$

$$\begin{aligned}
[M_z]_{AB} &= R_e \{ \beta_6^{(3)} \lambda_3^{-1} [z_3 \varphi(z_3) - \chi_1(z_3)] + \\
&\quad + \beta_6^{(4)} \lambda_4^{-1} [z_4 \Psi^o(z_4) - \chi_2(z_4)] \}_B^A
\end{aligned} \tag{4.12}$$

Here is

$$\chi_1(z_3) = \int \varphi(z_3) dz_3, \quad \chi_2(z_4) = \int \Psi^o(z_4) dz_4 \tag{4.13}$$

The proof of this theorem is similar to that of the preceding one. In controlling the fact that underneath the first two integrals (4.9) there is a total differential, we meet the equations

$$\begin{aligned}
\lambda_i \alpha_2^{(i)} + \alpha_3^{(i)} - \alpha_5^{(i)} &= 0 \quad (i = 1, 2) \\
\alpha_1^{(i)} + \lambda_i (\alpha_3^{(i)} + \alpha_5^{(i)}) &= 0 \quad (i = 1, 2)
\end{aligned} \tag{4.14}$$

which are filled up. This is due to (2.10)/[1] and (2.1) – (2.4)/[1]. In verifying the third formula we again meet the equation (4.8).

Theorem 4.3 The components of main force and main moment onto the contours c_j ($j = 1, \dots, m$) are:

$$\begin{aligned}
[F_x]_{c_j} &= -2\pi I_m [\beta_6^{(3)} \kappa^{(5)} \gamma_j^{(5)} + \beta_6^{(4)} \kappa^{(6)} \gamma_j^{(6)}] \\
[F_y]_{c_j} &= -2\pi I_m [\beta_2^{(3)} \kappa^{(5)} \gamma_j^{(5)} + \beta_2^{(4)} \kappa^{(6)} \gamma_j^{(6)}]
\end{aligned} \tag{4.15}$$

$$[F_z]_{c_j} = 0$$

$$[M_x]_{c_j} = [M_y]_{c_j} = 0$$

$$[M_z]_{c_j} = -2\pi I_m [\beta_6^{(3)} \lambda_3^{-1} \gamma_j^{(5)} + \beta_6^{(4)} \lambda_4^{-1} \gamma_j^{(6)}] \tag{4.16}$$

Proof. This theorem is the consequence of the preceding two theorems and of theorem 3.2. For the arc \widehat{AB} we take now the clockwise round of the contour c_j . At such a round, $\ln(z_k - z_{k,j})$ is decreased by $2\pi i$, and $\ln(\bar{z}_k - \bar{z}_{k,j})$ is increased by $2\pi i$.

The constants $\gamma_j^{(5)}$ and $\gamma_j^{(6)}$ are by means of the equations (4.15), (3.17) and (3.18) determined single-valued.

It can be ascertained that among the constants there is a fewer number of binding equations than constants. That means that several constants can be selected freely. This is due so the fact that instead of four functions used for describing the stress-deformation state we dispose with six of such functions.

R E F E R E N C E S

- [1] Krušič, B. and Brešar, F., *Some generalization of the theory of bending of anisotropic plates*, TAM, № . 8, 77-86, (1982).
- [2] Muskhelishvili, N. I., *Some basic problems of the mathematical theory of elasticity*, P. Noordhoff, Groningen, (1953).

SPÄNNUNGSFUNKTIONEN IN DER VERALLGEMEINERTEN BIEGUNGSTHEORIE DER ANISOTROPEN PLATTEN

Z u s a m m e n f a s s u n g

Der Beitrag ist die Fortsetzung der Veröffentlichung [1]. Dort haben wir den Spannungs-Deformationszustand der Platte mit sechs verallgemeinerten analytischen Funktionen behandelt. In diesem Beitrag erkennen wir den Definitionsgrad der gesamten Funktionen und der Form für vielfach zusammenhängendes Gebiet. Weiter beschreiben wir die Beziehung mit der Hauptkraft und Hauptmoment entlang der einzelnen Kontur.

NAPETOSTNE FUNKCIJE V POSPLOŠENI TEORIJI
UPOGIBA ANIZOTROPNIH PLOŠČ

P o v z e t e k

Ta prispevek pomeni nadaljevanje članka [1]. Tam smo napetostno-deformacijsko stanje plošče obravnavali s šestimi analitičnimi funkcijami. V pričujočem zapisu spoznamo najprej stopno definiranosti omenjenih funkcij in obliko teh funkcij za večkrat povezana območja. Pokažemo tudi kako se glavni vektor sile in glavni moment vzdolž posamezne konture izražata s temi funkcijami oz. z njihovimi parametri.

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