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THE APPLICATION OF GAMES IN THE FIELD OF HEAT FLOW

I. Šestak

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1. Introduction

In this work it is presented the application of games in the field of heat flow. A stationary process, with a constant coefficient of heat conducting and a constant coefficient of heat exchange with i. e. third limited condition, has been concerned. The controls are antagonistic. One of the controls has been distributed inside of the body and the other one outside of the body on the border. The cost function is of a square type.

2. Statement of the problem

We introduce the following hypothesis:

(a) body Ω — isotropic and homogenous,

(b) border Γ of the body Ω — smooth piece by piece,

(c) $f(M)$ — the density of heat source inside of the body — the function of the point position,

(d) k — coefficient of heat conduction — a constant,

(e) h — coefficient of heat exchange — a constant,

(f) the Fourier's law valid,

(g) heat conduction press — stationary.

Acording to the above hypothesis, the temperature in an arbitrary point M of the body Ω satisfy the following equation

$$h \, \Delta \, \theta \, (M) + f(M) = 0 \quad \text{in } \Omega, \tag{1}$$

where Δ is Laplacian.

In order to determine the temperature $\theta(M)$ $(M \in \Omega)$ in the unique manner let's see i. e. the third bordering condition [1]

$$k \frac{\partial \theta}{\partial n}(M) = h(\theta_0(M) - \theta(M)), M \in \Gamma \tag{2}$$

where $\theta_0(M)$ – the temperature of the outside of the $M \in \Gamma$ point, $\vec{n}$ – outside normal in the $M \in \Gamma$ point.

Let's introduce the following notation:

$$\left.\begin{array}{l}\theta(M) = y(M) = y(x)\\ f(M) = u_1(M) = u_1(x)\end{array}\right\} x \in \Omega,$$

$$\left.\begin{array}{l}h\,\theta_0(M) = u_2(M) = u_2(x)\\ h\,\theta(M) = g(M) = g(x).\end{array}\right\} x \in \Gamma.$$

As the temperature $y(x)$ depends on the quantities of heats u_1 and u_2 there is $y(x) = y(x; u_1, u_2)$. In order to write short, the functions will be written without appropriate variables. Therefore the equations (1) and (2) get the form:

$$-k\,\Delta y = u_1 \qquad \text{in } \Omega, \tag{3}$$

$$k \frac{\partial y}{\partial n} = -g + u_2 \qquad \text{on } \Gamma. \tag{4}$$

Let's see the following problem.

The first player (the sources inside of the body — Ω) emits a certain quantity of heat with an aim to reach the temperature $z_d(x)$ inside the body — Ω. Suppose that the outside temperature on the border is lower than $z_d(x)$ $(x \in \Gamma)$, then a spontaneous transmission of temperature on outer space is presented. The temperature $z_d(x)$ $(x \in \Gamma)$ does not respond to the other player (outer space) therefore it tends to make it lower. Both of the players tend to the minimal consumption of heat quantity. Therefore the cost function is in the form

$$J(v_1, v_2) = \int_\Omega |y - z|^2\,dx + \int_\Omega |v_1|^2\,dx - \int_\Gamma |v_2|^2\,d\Gamma. \tag{5}$$

Sets of admissible controls are presented in the form

$$U_{ad}^1 = \{v_1 | v_1 \in L^2(\Omega), 0 \leqslant v_1 \leqslant v_{1\,max} > \bar{z}_d\}, \tag{6}$$

$$U_{ad}^2 = \{v_2 | v_2 \in L^2(\Gamma), 0 \leqslant v_2 \leqslant v_{2\,max} < \bar{z}_d\}, \tag{7}$$

where $\bar{z}_d$ is the quantity of heat necessary to realize the temperature z_d. It is demanded that

$$z_d < min\,\{y|_\Omega, y|_\Gamma, \frac{u_2}{h}|_\Gamma\}$$

because on the contrary k and h would depend of temperature.

3. An example

In work [2] the following example has been cencerned

$$Ay = f + u_2 \qquad \text{in } \Omega, \tag{8}$$

$$\frac{\partial y}{\partial \nu_A} = g + u_1 \qquad \text{on } \Gamma, \tag{9}$$

where operator A has the form

$$Ay = - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(a_{ij} \frac{\partial y}{\partial x_j} \right) + a\, y, \tag{10}$$

$$\frac{\partial y}{\partial \nu_A} = \sum_{i,j=1}^{n} a_{ij} \frac{\partial y}{\partial x_j} \cos\,(\vec{n}, x_i), \tag{11}$$

$y \in H^1(\Omega)$,

Controls: $u_1 \in L^2(\Gamma)$, $u_2 \in L^2(\Omega)$,

functions: $g \in H^{-1/2}(\Gamma)$ — dual space of the space $H^{\frac{1}{2}}(\Gamma)$,

$f \in L^2(\Omega)$.

The cost function is of following form

$$J(v_1, v_2) = \int_\Omega |y - z|^2\, dx + (N_1 v_1, v_1) - (N_2 v_2, v_2) \tag{12}$$

where $N_i\ (i = 1, 2)$ — symetric, linear operator from U^i_{ad} on U^i_{ad} and statisfy the condition $(N_i v_i, v_i) \geqslant \nu_i\, ||\upsilon_i||^2, \nu_i \geqslant 0$ $((.\,,.)\ ||\cdot||$ — the scalar product and norm in $U^i_{ad})$.

Adjoint system is presented with

$$A^* p = y - z \qquad \text{in } \Omega, \tag{13}$$

$$\frac{\partial p}{\partial \nu_{A^*}} = 0 \qquad \text{on } \Gamma. \tag{14}$$

The saddle point is characterized by conditions

$$\int_{\Gamma} (p + N_1 u_1)(v_1 - u_1)\, d\Gamma \geqslant 0, \ \forall\ v_1 \in U^1_{ad},\ u_1 \in U^1_{ad}, \tag{15}$$

$$\int_{\Omega} (p + N_2 u_2)(v_2 - u_2)\, dx \leqslant 0, \ \forall\ v_2 \in U^2_{ad},\ u_2 \in U^2_{ad}, \tag{16}$$

where v_i are admissible and u_i optimal controls.

4. Evaluation of the optimal control

The equation (8) becomes equation (3) if in (10) we put: $a_{ij} = k\,\delta_{ij}$, (δ_{ij} — Kronecker delta symbol), $a_o = 0$, $f = 0$, and instead of u_2, the value u_1. For $a_{ij} = k\,\delta_{ij}$ (11) becomes the left side of the equation (4), and instead of $(g + u_1)$, $(-g + u_2)$ should be put in (9) in order to obtain the right side in (4). The cost function (12) becomes (5) for $N_1 = N_2 = I$ — operator identity.

The adjoint operator is $A^* = -k\,\Delta$, [4]. The inequalities (15) and (16) become

$$\int_{\Omega} (p + u_1)(v_1 - u_1)\, dx \geqslant 0, \ \forall\ v_1 \in U^1_{ad},\ u_1 \in U^1_{ad}, \tag{17}$$

$$\int_{\Gamma} (p - u_2)(v_2 - u_2)\, d\Gamma \leqslant 0, \ \forall\ v_2 \in U^2_{ad},\ u_2 \in U^2_{ad}. \tag{18}$$

The integral (17) is nonnegative if

$$(p + u_1)(v_1 - u_1) \geqslant 0, \ \forall\ v_1 \in U^1_{ad},\ u_1 \in U^1_{ad}. \tag{19}$$

The integral (18) is nonpositive if

$$(p - u_2)(v_2 - u_2) \leqslant 0, \ \forall\ v_2 \in U^2_{ad},\ u_2 \in U^2_{ad}. \tag{20}$$

The inequality (19) is valid if $(p + u_1 > 0)$ and $(v_1 - u_1 > 0)$, or $(p + u_1 < 0)$ and $(v_1 - u_1 < 0)$.

For $u_1 < v_1$ $(p + u_1 > 0)$, v_1 can take the value 0, so it will be $u_1 < 0$, but this is not permited, therefore $u_1 = 0$

$$\begin{cases} \text{if } p + u_1 > 0 \\ \text{then } u_1 = 0. \end{cases} \tag{21}$$

For $v_1 < u_1$ $(p + u_1 < 0)$, v_1 can take the value $v_{1\,max}$, so it will be $u_1 > v_{1\,max}$ but this is not permited, therefore $u_1 = v_{1\,max}$:

$$\begin{cases} \text{if } p + u_1 < 0 \\ \text{then } u_1 = v_{1\,max} \end{cases} \tag{22}$$

It is possible that following case presents

$$\begin{cases}\text{if } p + u_1 = 0 \\ \text{then } u_1 = -p.\end{cases} \tag{23}$$

The inequality (20) is valid if $(p - u_2 > 0)$ and $(v_2 - u_2 < 0)$ or $(p - u_2 < 0)$ and $(v_2 - u_2 > 0)$.

For $v_2 < u_2$ $(p - u_2 > 0)$, v_2 can take the value $v_{2\,max}$, so it will be $v_{2\,max} < u_2$, but this is not permitted, therefore $u_2 = v_{2\,max}$:

$$\begin{cases}\text{if } p - u_2 > 0 \\ \text{then } u_2 = v_{2\,max}.\end{cases} \tag{24}$$

For $u_2 < v_2$ $(p - u_2 < 0)$, v_2 can take the value 0, so it will be $u_2 < 0$, but this is not permitted, therefore $u_2 = 0$:

$$\begin{cases}\text{if } p - u_2 < 0 \\ \text{then } u_2 = 0.\end{cases} \tag{25}$$

It is possible that following case is presented

$$\begin{cases}\text{if } p - u_2 = 0 \\ \text{then } u_2 = p.\end{cases} \tag{26}$$

The expressions (21) — (23) can be presented in the form [3]

$$\Phi_1\,(0,\, v_{1max})\; p = \begin{cases} -p, & 0 \leqslant -p \leqslant v_{1max} \\ 0, & -p < 0 \\ v_{1max}, & -p > v_{1max}. \end{cases} \tag{27}$$

The expressions (24) — (26) can be presented in the form

$$\Phi_2\,(0,\, v_{2max})\; p|_\Gamma = \begin{cases} p|_\Gamma, & 0 \leqslant p|_\Gamma \leqslant v_{2max} \\ 0, & p|_\Gamma > v_{2max} \\ v_{2max}, & p|_\Gamma < 0 \end{cases} \tag{28}$$

Therefore the optimal controls u_1 and u_2 have the form

$$u_1 = \Phi_1\,(0,\, v_{1max})\, p, \qquad u_2 = \Phi_2\,(0,\, v_{2max})\, p|_\Gamma,$$

where p is determined from the equations

$$k\,\Delta\, y + \Phi_1\,(0,\, v_{1max})\, p = 0 \text{ in } \Omega,$$

$$k \frac{\partial y}{\partial n} = -g + \Phi_2 (0, v_{2max})\, p|_\Gamma \quad \text{on } \Gamma,$$

$$k \Delta p + y = z_d \ \text{in } \Omega,$$

$$\frac{\partial p}{\partial n} = 0 \ \text{on } \Gamma.$$

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ПРИМЕНЕНИЕ ТЕОРИИ ИГР В ОБЛАСТИ РАСПРОСТРАНЕНИЯ ТЕПЛА

СОДЕРЖЕНИЕ

Показано применение теории игр в области распространения тепла. Рассматривается стационираный процесс с постоянным коэффициентом теплопроводности и постоянным коэффициентом теплобмена c, так называемым, третьим краевым условием. Управления являются антигонистическими. Одно распределение внутри тела, а другое — вне тела на границе. Критерий оптимизации квадратного типа.

PRIMENA TEORIJE IGARA U OBLASTI PROSTIRANJA TOPLOTE

Izvod

Prikazana je primena igara u oblasti prostiranja toplote. Posmatran je stacioniran proces sa konstantnim koeficijentom toplotne razmene uz tz. treći igranični uslov. Upravljanja su antagonistička. Jedno je raspodeljeno unutar tela a drugo izvan tela na granici. Kriterijum optimizacije je kvadratnog tipa.

Ivan Šestak
Tehnički fakultet
19210 Bor, JNA 12