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EXPANSION TENSOR IN THE METRICS OF EINSTEIN AND SCHWARZSCHILD

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In this paper we investigate the expansion tensor in the cosmological models of Einstein and Schwarzschild.

We shall consider the exspansion tensor defined as [1].

$$\vartheta_{\alpha\beta} = \mathfrak{L}_\xi \left(g_{\alpha\beta} + \xi_\alpha \xi_\beta\right) \tag{1}$$

where ξ_α is the unit time-like vektor pointed into the future, tangent on the world lines, $g_{\alpha\beta}$ is the metric tensor of the considered space, and $\mathfrak{L}_\xi$ denotes Lie derivative with respect to ξ^α. This tensor satisfies

$$\vartheta_{\alpha\beta}\, \xi^\beta = 0 \tag{2}$$

Since ξ_α is time-like, we can choose a coordinate system in which ξ_α is the unit vector of proper time. In this paper we try to find out the field of expansion relative to a system of radially oriented four velocities. The vector ξ_α has the form

$$\xi_\alpha = \{\xi_1, 0, 0, \xi_4\}$$

We shall suppose that coordinate transformations are of form

$$\begin{aligned}
\overline{x}^1 &= \overline{r} = \psi(r, \tau) = \psi\, x^1, x^4) \\
\overline{x}^2 &= \overline{\theta} = \theta = x^2 \\
\overline{x}^3 &= \overline{\varphi} = \varphi = x^3 \\
\overline{x}^4 &= \overline{\tau} = \tau + \mu(r) = t + \mu(x^1)
\end{aligned} \tag{3}$$

The components of the vector ξ_α could be exspressed as

$$\xi_1 = \frac{\frac{\partial \bar{\tau}}{\partial r}}{\sqrt{-g^{11}\left(\frac{\partial \bar{\tau}}{\partial r}\right)^2 - g^{44}\left(\frac{\partial \bar{\tau}}{\partial \tau}\right)^2}} = \frac{\dot{\mu}}{\sqrt{-g^{11}\dot{\mu}^2 - g^{44}}}$$

$$\xi_4 = \frac{1}{\sqrt{-g^{11}\left(\frac{\partial \bar{\tau}}{\partial r}\right)^2 - g^{44}\left(\frac{\partial \bar{\tau}}{\partial \tau}\right)^2}} = \frac{1}{\sqrt{-g^{11}\dot{\mu}^2 - g^{44}}} \qquad (4)$$

The sistem of coordinates $\bar{x}^\alpha$ is ortogonal also, and thus

$$g^{\alpha\beta}\xi_\alpha \frac{\partial \psi}{\partial x^\beta} = 0 \qquad (5)$$

In order to find out the function μ relevant for tranformation (3) we shall use the relations having $\frac{\partial \bar{x}^\alpha}{\partial x^\beta}$.

First, we find the components of tensor $\vartheta_{\alpha\beta}$ from definition (1); (the metrics of Einstein and Schwarzschild are both diagonal).

$$\vartheta_{11} = 2(1 + 2\xi_1\xi^1)\frac{\partial \xi_1}{\partial r} + 2\xi_1\xi^4\frac{\partial \xi_4}{\partial r} - \xi^1(1+\xi_1\xi^1)\frac{\partial g_{11}}{\partial r} - 2\xi_1\xi^4\xi^4\frac{\partial g_{44}}{\partial r}$$

$$\vartheta_{14} = 2\xi_1\xi^1\frac{\partial \xi_1}{\partial r} + \xi_1\xi_1\xi_4\frac{\partial g^{11}}{\partial r} - \xi^4(1+\xi_4\xi^4)\frac{\partial g_{44}}{\partial r}$$

$$\vartheta_{22} = \xi^1\frac{\partial g_{22}}{\partial r} \qquad (6)$$

$$\vartheta_{33} = \xi^1\frac{\partial g_{33}}{\partial r}$$

$$\vartheta_{44} = 2\xi^1\left(\xi_4\frac{\partial \xi_4}{\partial r} + \frac{\partial g^{44}}{\partial r}\right)$$

Tensor $\vartheta_{\alpha\beta}$ is symmetric. Other components are zero.

The conravariant components of the expansion tensor could be obtained by

$$\vartheta^{\alpha\beta} = g^{\alpha\rho}g^{\beta\varphi}\vartheta_{\rho\varphi} \qquad (7)$$

We shall use the contravariant components of the expansion tensor in the coordinate system $\bar{x}^\alpha$ also

$$\bar{\vartheta}^{\alpha\beta} = \vartheta^{\sigma\nu} \frac{\partial \bar{x}^{\alpha}}{\partial x^{\sigma}} \frac{\partial \bar{x}^{\beta}}{\partial x^{\nu}} \tag{8}$$

Further, we need the mixed components of the expansion tensor, but we shall express them in the folowing form

$$\bar{\vartheta}^{\alpha\beta} = \bar{g}^{\alpha\rho}\, \bar{\vartheta}^{\beta}_{\rho} \tag{9}$$

which enable us to find $\bar{\vartheta}^{\alpha}_{\beta}$. Using (3), (5), (8) and (9) we obtain

$$\bar{\vartheta}^{1}_{1} = \frac{1}{g^{11}} \left[\vartheta^{11} - 2\frac{\xi^{1}}{\xi^{4}}\,\vartheta^{14} + \left(\frac{\xi^{1}}{\xi^{4}}\right)^{1} \vartheta^{44}\right] \left(\frac{\partial \psi}{\partial r}\right)^{2}$$

$$\bar{\vartheta}^{2}_{2} = \frac{1}{g^{22}}\,\vartheta^{22}$$

$$\bar{\vartheta}^{3}_{3} = \frac{1}{g^{33}}\,\vartheta^{33} \tag{10}$$

In the coordinate system $\bar{x}^{\alpha}$ other components are zero.
Using (2), (5) and (7) we obtain $\bar{\vartheta}^{1}_{1}$ depending on ϑ_{11} only

$$\bar{\vartheta}^{1}_{1} = g^{11}\, g^{11} \frac{[1 + (\xi^{1}/\xi^{4})^{2}]^{2}}{g^{11} + (\xi^{1}/\xi^{4})^{2} g^{44}}\, \vartheta_{11} \tag{11}$$

We shall investigate, first, the exspansion tensor in the metric of Einstein, which can be expressed in the form [2]

$$d s^{2} = \frac{d r^{2}}{1 - \frac{r^{2}}{R^{2}}} + r^{2} (d \theta^{2} + \sin^{2}\theta\, d \varphi^{2}) - d t^{2}$$

We get the components of the expansion tensor from (10) and (11)

$$\bar{\vartheta}^{1}_{1} = g^{11}\, g^{11} \frac{[1 + (\xi^{1}/\xi^{4})^{2}]^{2}}{g^{11} + (\xi^{1}/\xi^{4})^{2}\, g^{44}}\, \vartheta_{11}$$

$$\bar{\vartheta}^{2}_{2} = \frac{2}{r}\left(1 - \frac{r^{2}}{R^{2}}\right) \xi_{1}$$

$$\bar{\vartheta}^{2}_{3} = \frac{2}{r}\left(1 - \frac{r^{2}}{R^{2}}\right) \xi_{1}$$

We note that in the coordinate system $\bar{x}^\alpha$ we have $\bar{\vartheta}_2^2 = \bar{\vartheta}_3^3$.

We shall investigate only the case

$$\bar{\vartheta}_2^2 = \bar{\vartheta}_3^3 = E = \text{const} \tag{12}$$

It implies the following equation for μ appearing in (3)

$$\dot{\mu} = \frac{E}{2} \frac{r}{\sqrt{\left(1 - A\frac{r^2}{R^2}\right)\left(1 - \frac{r^2}{R^2}\right)}}, \quad A = 1 - \frac{E^2 R^2}{4} \tag{13}$$

This equation has the solution

$$\mu = \frac{-E R^2}{2\sqrt{1-A}} \, ln\left\{\sqrt{1 - \frac{r^2}{R^2}} + \sqrt{\frac{1}{A} - \frac{r^2}{R^2}}\right\}$$

This solution means that in Einstein metric the world lines could exist analogous to the world lines of Lemaître transformation [2]

$$\bar{t} = t + k \, ln\left(1 - \frac{r^2}{R^2}\right)$$

The solution (13) means also that there exist the transformations realizing the condition (12). Than it is meaningfull to consider a component $\bar{\vartheta}_1^1$. We find that $\bar{\vartheta}_1^1$ is positive as well. In Einstein metric the expansion is positive but not isotopic, i. e. equal in every space-like direction.

We consider now the expansion tensor in Schwarzschild metric. This metric can be expressed in the form (3)

$$d s^2 = \frac{d r^2}{1 - \frac{2m}{r}} + r^2 (d\theta^2 + \sin^2\theta \, d\varphi^2) - \left(1 - \frac{2m}{r}\right) d t^2$$

Using the formulas (10) and (11) again, we obtain the folloving components of the expansion tensor, different from zero

$$\bar{\vartheta}_1^1 = g^{11} g^{11} \frac{[1 + (\xi^1/\xi^4)^2]^2}{g^{11} + (\xi^1/\xi^4)^2 \, g^{44}}$$

$$\bar{\vartheta}_2^2 = \frac{2}{r}\left(1 - \frac{2m}{r}\right)\xi_1$$

$$\bar{\vartheta}_3^3 = \frac{2}{r}\left(1 - \frac{2m}{r}\right)\xi_1$$

We have again $\bar{\vartheta}_2^2 = \bar{\vartheta}_3^3$. We investigate the case

$$\bar{\vartheta}_2^2 = \bar{\vartheta}_3^3 = S = \text{const}$$

The function μ from (3) is determined by the equation

$$\dot{\mu} = \frac{r\sqrt{r}}{\left(1 - \frac{2m}{r}\right)\sqrt{r^2 + \frac{4}{S^2}r - \frac{8m}{S^2}}}$$

The exact solution of this equation is expressed by the elliptic functions However, we shall analyse only the region near Schvarzschild "horizon", i. e. find an approximative solution if $2m - \varepsilon < r < 2m + \varepsilon$ where ε means a small quantity. In this approximation we shall neglect r and $\frac{8m}{S^2}$ relative to r^3. This case adopts a (considerable) simplification up to

$$\mu = \int \frac{dr}{1 - \frac{2m}{r}}$$

Out of "horizon", $r = 2m + \varepsilon$, we find the solution

$$\mu = r - 2m + 2m \ln(r - 2m)$$

On the other side of "horizon", where $r = 2m - \varepsilon$ we find

$$\mu = r - 2m + 2m \ln(2m - r)$$

There is a possibility of finding the world lines analogous to those in Lemaître metric, in Schwarzschild metric also.

The component $\bar{\vartheta}_1^1$ is positive in this region. The expansion is positive but not isotropic.

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LE TENSEUR D'EXPANSION DANS LA METRIQUE D'EINSTEIN ET DE SCHWARZSCHILD

Résumé

On considère ici le tenseur d'expansion dans la métrique d'Einstein et de Schwarzschild. Le tenseur d'expansion est défini par

$$\vartheta_{\alpha\beta} = \mathfrak{L}_\xi (g_{\alpha\beta} + \xi_\alpha \xi_\beta)$$

où ξ_α est le vecteur orienté dans le temps, $g_{\alpha\beta}$ est le tenseur métrique de l'espace correspondant et où $\mathfrak{L}_\xi$ est la derivation de Lie.

Dans la métrique de Schwarzschild on n'analyse que la région autour de "l'horizon". L'exansion est considerée par rapport à un champ de quadrivitesse radial. Dans les deux métriques, on obtient l'expansion positive, mais pas isotrope, i. e. elle n'est pas égale dans toutes les directions.

TENZOR EKSPANZIJE U AJNŠTAJNOVOJ I ŠVARCŠILDOVOJ METRICI

Izvod

U ovom radu ispituje se tenzor ekspanzije u Ajnštajnovoj i Švarcšildovoj metrici. Tenzor ekspanzije definiše se kao

$$\vartheta_{\alpha\beta} = \mathfrak{L}_\xi (g_{\alpha\beta} + \xi_\alpha \xi_\beta)$$

gde je ξ_α jedinični vektor vremenskog tipa, $g_{\alpha\beta}$ je metrički tenzor odgovarajućeg prostora, a $\mathfrak{L}_\xi$ je Liov izvod. U odnosu na polje radijalnih četvorobrzina (posmatrača) dobija se da je ekspanzija pozitivna i u jednoj i u drugoj metrici. U Švarcšilendovoj metrici koordinatne transformacije posmatrača određene su u linearnoj aproksimaciji.

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