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DEFORMATION OF AN INFINITE MEDIUM WITH A RIGID ELLIPTICAL INCLUSION DUE TO A FINITE ROTATION OF THE INCLUSION

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1. Introduction

In infinitesimal plane theory of elasticity the problem of an infinite medium bonded to a rigid elliptical inclusion and which is deformed by giving the inclusion a rotation through a small angle has been solved by England [1] with the conformal trasformation method. A treatment of the above problem has not as far as I know been given in the literature for finite plane theory of elasticity. This is the purpose of the present paper concerning an infinite medium of homogeneous compressible isotropic hyperelastic material, of harmonic type, containing a rigid elliptical inclusion, and which is deformed by giving the inclusion a rotation through a finite angle. To solve this boundary-value problem, we list below the general solution for the displacement and stress field of the finite plane strain problems with an elliptical boundary for such material and define a function, as they have been derived in [2]. It is thown that this problem is solvable in integral form by means of a conformal iransformation, asymptotic analysis and an appropriate Laurent series. They are, sn that notation,

a) General solution:

$$z=\frac{1}{2\overline{C}(\overline{\zeta})}\int^{\zeta}\Phi^{-1}\left(C(\eta)\,\overline{C}(\overline{\zeta})\right)C(\eta)\,(1-\mathrm{ma}^2\eta^{-2})\,d\eta+\overline{\overline{B}}(\overline{\zeta}) \tag{1}$$

and

$$i\mu^{-1}\varphi=\frac{1}{\overline{C}(\overline{\zeta})}\int^{\zeta}\Phi^{-1}\left(C(\eta)\,\overline{C}(\overline{\zeta})\right)C(\eta)\,(1-\mathrm{ma}^2\eta^{-2})\,\mathrm{d}\eta-\widetilde{A}(\zeta)+2\overline{\overline{B}}(\overline{\zeta}). \tag{2}$$

b) Define the function

$$C(\zeta)=\left\{\frac{\zeta^2\widetilde{A}'(\zeta)}{\zeta^2-\mathrm{ma}^2}\right\}^{1/2}. \tag{3}$$

All introduced quantities are defined in [2].

2. Mathematical Formulation and Solution of the Problem

We consider an infinite medium of compressible isotropic hyperelastic material, of harmonic type, bonded to a rigid elliptical inclusion and suppo se the medium is deformed by rotating the inclusion through a finite angle Ω, the stresses and rotation at infinity being zero.

Taking into account the state of the problem, the subsidiary conditions are written in the form

$$z = e^{i\Omega} Z \quad \text{at the ellipse} \tag{4}$$

and

$$z \sim Z \quad \text{as } |Z| \to \infty . \tag{5}$$

With the conformal transformation [2]

$$Z = \zeta + \mathrm{m}a^2 \zeta^{-1}, \tag{6}$$

which transforms the outside of the ellipse of the outside of the circle, $\xi \bar{\zeta} = a^2$ the above conditions become

$$z = e^{i\Omega} \zeta + me^{i\Omega} \bar{\zeta} \quad \text{at } \zeta \bar{\zeta} = a^2 \tag{7}$$

and

$$z \sim \zeta \quad \text{as } |\zeta| \to \infty \tag{8}$$

On substituting (7) into (1) we get

$$\bar{\tilde{B}}(\bar{\zeta}) = -\frac{1}{2\bar{C}(\bar{\zeta})} \int^{a^2 \bar{\zeta}^1} \Phi^{-1}(C(\eta)\bar{C}(\bar{\zeta}))\, C(\eta)(1 - \mathrm{m}a^2 \eta^{-2})\, d\eta + \\ + e^{i\Omega} a^2 \bar{\zeta}^{-1} + \mathrm{m}e^{i\Omega} \bar{\zeta}. \tag{9}$$

This relation holds on the ellipse given by $\zeta \bar{\zeta} = a^2$. However, $\tilde{B}$ is analytic in a domain which includes this curve and extends to the entire ζ-plane apart from isolated singularities. By the principle of analytic continuation, equation (9) holds throughout the domain of analyticity of $\tilde{B}$ and we can eleminate $\tilde{B}$ between (1) and (9) to obtain

$$z = \frac{1}{2\bar{C}(\bar{\zeta})} \int_{a^2 \zeta^{-1}}^{\zeta} \Phi^{-1}(C(\eta)\bar{C}(\bar{\zeta}))\, C(\eta)\, (1 - \mathrm{m}a^2 \eta^{-2}) d\eta + \\ + e^{i\Omega} a^2 \bar{\zeta}^{-1} + \mathrm{m}e^{i\Omega} \bar{\zeta}. \tag{10}$$

We now show that the condition (8) at infinity can be satisfied by adopting for the analytical function $\tilde{A}(\zeta)$ the Laurent series

$$\tilde{A}(\zeta) = 2\zeta + \frac{1}{2} \mathrm{m}a^2 \beta^2 e^{2i\alpha} \zeta^{-1}, \tag{11}$$

where α and β are real constants which are to be determined. Then, from equation (3) we get

$$C(\zeta) = \left\{ \frac{2\zeta^2 - \frac{1}{2} m a^2 \beta^2 e^{2id}}{\zeta^2 - m a^2} \right\}^{1/2} \tag{12}$$

and

$$C(\zeta) \to 2^{1/2} \text{ as } \zeta \to \infty, \quad C(\zeta) \to 2^{-1/2} \beta e^{i\alpha} \text{ as } \zeta \to 0. \tag{13}$$

The integrand in (10) is evidently asymptotic to

$$\left. \begin{array}{l} 2^{1/2} \phi^{-1}(2^{1/2} \bar{C}(\bar{\zeta})) \quad \text{as } \eta \to \infty \\ \text{and} \\ -2^{-1/2} \beta e^{i\alpha} m a^2 \phi^{-1}(2^{-1/2} \beta e^{i\alpha} \bar{C}(\bar{\zeta})) \eta^{-2} \quad \text{as } \eta \to 0. \end{array} \right\} \tag{14}$$

We deduce from (10), (12), (13) and (14) that

$$z \sim \zeta + \left\{ m e^{i\Omega} - \frac{1}{4} m \beta e^{i\alpha} \phi^{-1}(\beta e^{i\alpha}) \right\} \bar{\zeta} \quad \text{as } |\zeta| \to \infty. \tag{15}$$

The condition (8) is therefore satisfied if

$$\beta e^{i\alpha} = \phi(4\beta^{-1} e^{i(\Omega - \alpha)}). \tag{16}$$

The real and imaginary parts of this equation determine, in general, the constants α and β. Then, equation (11) determines completely the function $\tilde{A}(\zeta)$ and (9) the function $\tilde{B}(\zeta)$. The solution of this problem is thus given by (2) and (10) in the form of integral, through (9), (11) and (12).

REFERENCES

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[2] Perdikis, C., *On finite plane deformations of compressible isotropic hyperelastic solids*, Acta Mechanica, 43, 159—168, (1982).

DEFORMATION D'UN MILIEU INFINI A INCLUSION ELLIPTIQUE RIGIDE DUE A UNE ROTATION FINIE DE L'INCLUSION

R é s u m é

Le problème de déformation d'un mulieu infini à inclusion élliptique rigide due à une rotation finie de l'inclusion peut être resolu au moyen d'une transformation conforme, de l'analyse asymptotique et d'une série de Laurent appropriée. La solution est obtenue en forme intégrale.

DEFORMACIJA NEOGRANIČENE SREDINE SA KRUTOM ELIPTIČKOM INKLUZIJOM IZAZVANA KONAČNOM ROTACIJOM INKLUZIJE

I z v o d

Problem deformacije neograničene sredine sa krutom eliptičkom inkluzijom nastale usled konačne rotacije inkluzije se može rešiti primenom konfomne transformacije, asimtotske analize i odgovarajućeg Loranovog reda. Rešenje se dobija u integralnom obliku.

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