

## SOLUTION OF GENERALIZED BOUNDARY — VALUE PROBLEMS FOR ELASTIC PLATE WITH CUTS IN ONE STRAIGHT LINE

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### 1. Introduction

For more exact presentation of stress-strain state in the plate, which also gives information about circumstances outside the middle plane of the plate, we need five analytic functions:

$$\Omega(z), \omega(z), \varphi(z), \psi(z), P(z).$$

The function  $P(z)$  is determined by transverse loading  $q(x,y) = Re(P''(z))$ , the other four of them can be determined if we know the circumstances on the boundary of the plate. If the external stresses acting on boundaries of the plate are given, we have the first boundary-value problem; but if the displacements of the points of the boundary of a plate are given, the second boundary-value problem is obtained, respectively.

The complete definition of generalized boundary-value problems are found in [1]. There one can also find the meaning of all those notations which are not especially described in this paper.

In this paper both boundary-value problems for a plate with cuts on a straight line are solved by the theory introduced in [1].

### 2. Stress functions for an infinite plate with cuts on the x-axis

Let the cuts  $[a_k, b_k]$  lie along  $x$ -axis ( $k = 1, 2, \dots, m$ )

and

let us introduce the notations

$$c = \sum_{k=1}^m c_k, \quad c_k = [a_k, b_k], \quad a_k < b_k < a_{k+1}. \quad (2.1)$$

Then:

*Theorem 1.* In the case of an infinite plate with cuts along  $x$ -axis the stress-functions have the following form:

$$\varphi(z) = \sum_{k=1}^m \gamma_k I_k(z) + \Gamma_1 z + K_1 + \varphi_0(z) \quad (2.2)$$

$$\psi(z) = -\kappa \sum_{k=1}^m \bar{\gamma}_k I_k(z) + \Gamma'_1 z + K'_1 + \psi_0(z) \quad (2.3)$$

$$\Omega(z) = \sum_{k=1}^m (A_{k,1} z + \gamma_{k,1}) I_k(z) + \Gamma_2 z + K_2 + \Omega_0(z) \quad (2.4)$$

$$\omega'(z) = \sum_{k=1}^m \bar{\gamma}_{k,1} I_k(z) + \Gamma'_2 z + K'_2 + \omega'_0(z) \quad (2.5)$$

Here is

$$I_k(z) = \frac{1}{s_k} \int_{a_k}^{b_k} \ln(z-x) dx, \quad (2.6)$$

$$s_k = b_k - a_k, \quad (2.7)$$

$$A_{1,1} + A_{2,1} + \dots + A_{m,1} = 0, \quad (2.8)$$

and  $K_1, K'_1, K_2, K'_2$  are arbitrary complex constants.

For large  $|z|$ , ( $|z| > R$ ), we get

$$\varphi(z) = B_1 \ln z + \Gamma_1 z + K_1 + O(z^{-1}) \quad (2.9)$$

$$\psi(z) = -\kappa \bar{B}_1 \ln z + \Gamma'_1 z + K'_1 + O(z^{-1}) \quad (2.10)$$

$$\Omega(z) = B_2 \ln z + \Gamma_2 z + K_2 + O(z^{-1}) \quad (2.11)$$

$$\omega'(z) = \bar{B}_2 \ln z + \Gamma'_2 z + K'_2 + O(z^{-1}). \quad (2.12)$$

These expressions assure that the stresses at  $z = \infty$  are bounded. The displacements are however bounded if we additionally demand:

$$B_1 = 0, \quad \Gamma_1 = 0, \quad \Gamma'_1 = 0 \quad (2.13)$$

$$B_2 = 0, \quad \Gamma_2 = 0, \quad \Gamma'_2 = 0, \quad K_2 = K'_2 = 0, \quad (2.14)$$

$$\omega'(z) = 0(z^{-2})$$

This theorem is the consequence of analogical theorems which are found in [1]. The use of the function  $I_k(z)$  instead of  $\ln(z-x)$  was introduced by Krušić [2]. Its preference is that it is bounded when the point  $z$  is approached to the cut.

### 3. Solution of the Problem I

We must find the elastic equilibrium of a plate if the external stresses acting on its boundary are given. We know [1] that the problem is reduced to the following boundary equation ( $t \in c_k$ ):

$$\varphi(t) + t \overline{\varphi'(t)} + \overline{\psi(t)} - \frac{2}{3} \sigma h^2 \overline{\varphi''(t)} = f(t) + Q_1(t) + \alpha_k \quad (3.1)$$

$$x' \Omega(t) - t \overline{\Omega'(t)} - \overline{\omega'(t)} + x'' h^2 \overline{\Omega''(t)} = \tilde{f}(t) + \\ + \tilde{Q}_1(t) + i r_k t + \beta_k \quad (3.2)$$

$$\sigma = \frac{v}{v+1}, \quad x' = \frac{3+v}{1-v}, \quad x'' = \frac{2-v}{1-v} \quad (3.3)$$

At first we discuss the problem (3.1), and later the problem (3.2).

#### 3.1 Determination of the functions $\Phi$ and $\Psi$

*Theorem 2.* If we introduce the functions

$$\Phi_0(z) = \frac{1}{2\pi i X(z)} \int_c \frac{X(x) p_1(x)}{x-z} dx + \frac{1}{2\pi i} \int_c \frac{p_2(x)}{x-z} dx, \quad (3.4)$$

$$\Lambda_{1,0}(z) = \frac{1}{2\pi i X(z)} \int_c \frac{X(x) p_1(x)}{x-z} dx - \frac{1}{2\pi i} \int_c \frac{p_2(x)}{x-z} dx, \quad (3.5)$$

$$\Lambda_1(z) = \Lambda_{1,0}(z) + P_1(z) X^{-1}(z) + \frac{1}{2} \Gamma'_1, \quad (3.6)$$

where the following notations is used:

$$X(z) = \prod_{k=1}^m (z - a_k)^{1/2} (z - b_k)^{1/2}, \quad (3.7)$$

$$X(x) = X^+(x) = \lim_{z \rightarrow x} X(z), \quad Im(z) > 0, \quad (3.8)$$

$$hp_1(x) = (\Sigma_y - \Sigma_y^0)^+ + (\Sigma_y - \Sigma_y^0)^- - i(T_{xy} - T_{xy}^0)^+ - i(T_{xy} - T_{xy}^0)^-, \quad (3.9)$$

$$hp_2(x) = (\Sigma_y - \Sigma_y^0)^+ - (\Sigma_y - \Sigma_y^0)^- - i(T_{xy} - T_{xy}^0)^+ - i(T_{xy} - T_{xy}^0)^-, \quad (3.10)$$

$$P_1(z) = A_0 z^m + A_1 z^{m-1} + \dots + A_m, \quad (3.11)$$

$$A_0 = \Gamma_1 + \frac{1}{2} \bar{\Gamma}'_1, \quad (3.12)$$

$$\Gamma_1 = \bar{\Gamma}_1 = \frac{1}{2h} (\Sigma_x^\infty + \Sigma_y^\infty - \Sigma_x^{0,\infty} - \Sigma_y^{0,\infty}), \quad (3.13)$$

$$\Gamma'_1 = \frac{1}{h} (\Sigma_y^\infty - \Sigma_x^\infty + 2i T_{xy} - \Sigma_y^{0,\infty} + \Sigma_x^{0,\infty} - 2i T_{xy}^{0,\infty}), \quad (3.14)$$

then the stress-functions  $\Phi(z) = \varphi'(z)$ ,  $\Psi(z) = \psi'(z)$  take the following expression:

$$\Phi(z) = \Phi_0(z) + P_1(z) X^{-1}(z) - \frac{1}{2} \Gamma'_1, \quad (3.15)$$

$$\Psi(z) = \bar{\Lambda}'_1(z) - \Phi(z) - z \Phi'(z) + \frac{2}{3} \sigma h^2 \Phi''(z), \quad (3.16)$$

The coefficients  $A_1, A_2, \dots, A_m$  of the polynomial  $P_1(z)$  we find from the following (Cramer's) system of the linear equations:

$$\int_{a_k}^{b_k} \{2(\kappa + 1) P_1(x) X^{-1}(x) + \kappa [\Phi_0^+(x) - \Phi_0^-(x)] + [\Lambda_{1,0}^+(x) - \Lambda_{1,0}^-(x)]\} dx = 0, \quad (3.17)$$

$$(k = 1, 2, \dots, m).$$

*Proof.* This problem is solved by the method of Muskhelishvili [4] therefore we mention only some basic phases of the proof.

In the sense of the definition  $\bar{f}(z) = \overline{f(z)}$  we introduce the function

$$\Lambda_1(z) = \bar{\Phi}(z) + z \bar{\Phi}'(z) + \bar{\Psi}(z) - \frac{2}{3} \sigma h^2 \bar{\Phi}''(z).$$

At once we can see that the expression

$$\Phi(z) + \Lambda_1(\bar{z}) + (z - \bar{z}) \overline{\Phi'(z)} = L$$

is the derivative of the left side of the Eq. (3.1). From the expression of unit forces  $\Sigma_y, T_{xy}$  [1] we get

$$L = \frac{2}{h} [(\Sigma_y - \Sigma_y^0) - i(T_{xy} - T_{xy}^0)].$$

If we add the conjugate equation and put down the requirement

$$\lim_{z \rightarrow x} (z - \bar{z}) \Phi'(z) = 0, \quad (\forall x \in c, x \neq a_k, x \neq b_k),$$

we get

$$\Phi^+(x) + \Lambda_1^-(x) = \frac{2}{h} [(\Sigma_y - \Sigma_y^0)^+ - i(T_{xy} - T_{xy}^0)^+].$$

The rest of the solution is well known [2], [3], [4]. The system (3.17) is derived from the requirement that the function  $D_0 = u_0 + i v_0$  is unique.

### 3.2. Determination of the functions $\Omega$ and $\omega'$

*Theorem 3.* If we introduce the functions

$$\Omega_0(z) = \frac{X(z)}{2\pi i \kappa'} \int_c \frac{p_3(x) dx}{X(x)(x-z)} + \frac{1}{2\pi i \kappa'} \int_c \frac{p_4(x) dx}{x-z}, \quad (3.18)$$

$$\bar{\Lambda}_2(z) = -\frac{X(z)}{2\pi i} \int_c \frac{p_3(x) dx}{X(x)(x-z)} + \frac{1}{2\pi i} \int_c \frac{p_4(x) dx}{x-z}, \quad (3.19)$$

$$\omega'_0(z) = \bar{\Lambda}_2(z) - z \Omega'_0(z) + \kappa'' h^2 \Omega''_0(z), \quad (3.20)$$

then we obtain

$$\begin{aligned} \Omega(z) &= \sum_{k=1}^m (A_{k,1}z + \gamma_{k,1}) \cdot \frac{1}{s_k} [(z-a_k) \ln(z-a_k) - \\ &\quad - (z-b_k) \ln(z-b_k) - s_k] + \Gamma_2 z + K_2 + \Omega_0(z), \end{aligned} \quad (3.21)$$

$$\begin{aligned} \omega'(z) &= \sum_{k=1}^m \bar{\gamma}_{k,1} \cdot \frac{1}{s_k} [(z-a_k) \ln(z-a_k) - \\ &\quad - (z-b_k) \ln(z-b_k) - s_k] + \Gamma'_2 z + K'_2 + \omega'_0(z). \end{aligned} \quad (3.22)$$

The constants  $K_2$  and  $K'_2$  are arbitrary, the meaning of the remaining constants is found in [1].

It must be fulfilled (2.8).

Functions  $p_3$ ,  $p_4$  are defined by

$$p_3(x) = \frac{1}{2} [f_0^+(x) + f_0^-(x)] + i r_k x + \beta_k, \quad (3.23)$$

$$p_4(x) = \frac{1}{2} [f_0^+(x) - f_0^-(x)] \quad (3.24)$$

$$\begin{aligned} f_0(z) &= \tilde{f}(z) + \tilde{Q}_1(z) - \kappa' \sum_{k=1}^m (A_{k,1}z + \gamma_{k,1}) E_k(z) - \\ &\quad - \kappa' (\Gamma_2 z + K'_2) + z \sum_{k=1}^m (A_{k,1}\bar{z} + \bar{\gamma}_{k,1}) \cdot \frac{1}{s_k} \cdot \ln \frac{\bar{z}-a_k}{\bar{z}-b_k} + \\ &\quad + A_{k,1} \cdot E_k(\bar{z})] + z \bar{\Gamma}_2 + \sum_{k=1}^m \gamma_{k,1} E_k(z) + \end{aligned} \quad (3.25)$$

$$+ \bar{\Gamma}'_2 \bar{z} + \bar{K}'_2 - \kappa'' h^2 \sum_{k=1}^m \left[ 2 A_{k,1} \cdot \frac{1}{s_k} \ln \frac{\bar{z}-a_k}{\bar{z}-b_k} - \frac{A_{k,1} \bar{z} + \bar{\gamma}_{k,1}}{(\bar{z}-a_k)(\bar{z}-b_k)} \right],$$

$$E_k(z) = \frac{1}{s_k} [(z - a_k) \ln(z - a_k) - (z - b_k) \ln(z - b_k) - s_k]. \quad (3.26)$$

If we choose  $\beta_1$ , then we must still determine  $3m - 2$  real constants, i.e.  $m$  real constants  $r_k$  and  $m - 1$  complex constants  $\beta_k$ . We find them from the following (Cramer's) system:

$$\int_c^b \frac{x^{k-1} p_3(x)}{X(x)} dx = 0, \quad (k = 1, \dots, m-1), \quad (3.27)$$

$$Re \left\{ \int_{a_k}^{b_k} [\Lambda_2^+(x) - \Lambda_2^-(x)] dx \right\} = 0, \quad (k = 1, \dots, m). \quad (3.28)$$

*Proof.* If we introduce

$$\Lambda_2(z) = z \Omega'_0(z) + \bar{\omega}'_0(z) - \kappa'' h^2 \bar{\Omega}''_0(z),$$

and require

$$\lim_{z \rightarrow \infty} (\bar{z} - z) \Omega'_0(z) = 0,$$

the boundary-value problem (3.2) acquires the form

$$\kappa' \Omega_0^+(x) - \Lambda_2^-(x) = f_0^+(x) + i r_k x + \beta_k,$$

$$\kappa' \Omega_0^-(x) - \Lambda_2^+(x) = f_0^-(x) + i r_k x + \beta_k.$$

The further way to find the solution is known. Eq. (3.27) are deduced from the requirement that the solution is bounded in the neighbourhood of points  $a_k$ ,  $b_k$ . Eq. (3.28) are deduced from the requirement of uniqueness of bending  $w_0$ .

#### 4. Solution of the Problem II

In this case we have the following boundary equations [1]:

$$\kappa \varphi(t) - t \overline{\varphi'(t)} - \overline{\psi(t)} = g(t) + Q_2(t), \quad (4.1)$$

$$\Omega(t) + t \overline{\Omega'(t)} + \overline{\omega'(t)} - \frac{4 h^2}{1 - \nu} \overline{\Omega''(t)} = \tilde{q}(t) + \tilde{Q}_2(t), \quad (4.2)$$

$$w_0 = h(t). \quad (4.3)$$

Also here we shall first look for the solution of the problem (4.1), and for the problem (4.2) later.

#### 4.1. Determination of the functions $\Phi$ and $\Psi$

*Theorem 4.* The solution of the problem (4.1) are the functions  $\varphi'(z) = \Phi(z)$ ,  $\psi'(z) = \Psi(z)$ , which are expressed the following way:

$$\kappa\Phi(z) = \Phi_1(z) + P_3(z)X^{-1}(z) + 2^{-1}(\kappa\Gamma_1 + \bar{\Gamma}_1 + \bar{\Gamma}'_1), \quad (4.4)$$

$$\Psi(z) = -\Phi(z) + \bar{\Lambda}_3(z) - z\Phi'(z). \quad (4.5)$$

Here we denoted

$$\Lambda_3(z) = \frac{-1}{2\pi i X(z)} \int_c^{\infty} \frac{X(x)f_1(x)}{x-z} dx + \frac{1}{2\pi i} \int_c^{\infty} \frac{f_2(x)}{x-z} dx \quad (4.6)$$

$$- P_3(z)X^{-1}(z) + 2^{-1}(\kappa\Gamma_1 + \bar{\Gamma}_1 + \bar{\Gamma}'_1),$$

$$\Phi_1(z) = -\frac{1}{2\pi i X(z)} \int_c^{\infty} \frac{X(x)f_1(x)}{x-z} dx + \frac{1}{2\pi i} \int_c^{\infty} \frac{f_2(x)}{x-z} dx, \quad (4.7)$$

$$2f_1(x) = [g'(x) + Q'_2(x)]^+ + [g'(x) + Q'_2(x)]^-, \quad (4.8)$$

$$2f_2(x) = [g'(x) + Q'_2(x)]^+ - [g'(x) + Q'_2(x)]^-, \quad (4.9)$$

$$P_3(z) = C_0 z^m + C_1 z^{m-1} + \dots + C_m, \quad (4.10)$$

$$C_0 = 2^{-1}(\kappa\Gamma_1 - \bar{\Gamma}_1 - \bar{\Gamma}'_1),$$

$$C_1 = \kappa B_1 - 2^{-1}C_0(a_1 + b_1 + a_2 + b_2 + \dots + a_m + b_m).$$

The coefficients  $C_2, C_3, \dots, C_m$  can be found from the (Cramer's) system:

$$\begin{aligned} \int_{b_k}^{a_{k+1}} [\kappa\Phi(x) - \Lambda_3(x)] dx &= g(a_{k+1}) + Q_2(a_{k+1}), \\ &\quad - g(b_k) - Q_2(b_k), \\ (k &= 1, 2, \dots, m-1). \end{aligned} \quad (4.11)$$

*Proof.* The Eq. (4.1) is now differentiated also with respect to the variable  $x$  and by the introduction of the function

$$\Lambda_3(z) = \bar{\Phi}(z) + z\bar{\Phi}'(z) - \Psi(z),$$

the problem is transformed to

$$[\kappa\Phi(x) - \Lambda_3(x)]^+ + [\kappa\Phi(x) - \Lambda_3(x)]^- = 2f_1(x),$$

$$[\kappa\Phi(x) + \Lambda_3(x)]^+ - [\kappa\Phi(x) + \Lambda_3(x)]^- = 2f_2(x).$$

From

$$D_0^+(a_k) = D_0^-(a_k), \quad D_0^+(b_k) = D_0^-(b_k),$$

$$(k = 1, \dots, m, \quad D_0 = u_0 + i v_0),$$

we get the conditions (4.11).

### 3.2. Determination of the functions $\Omega$ and $\omega'$ .

*Theorem 5.* The function  $\Omega$  and  $\omega'$  are in this case also expressed by the formulae (3.21) and (3.22), but now the functions  $\Omega_0$ ,  $\omega_0$  have the following meaning:

$$\Omega_0(z) = \frac{X(z)}{4\pi i} \int_c \frac{H_0^+(x) + H_0^-(x)}{X(x)(x-z)} dx + \frac{1}{4\pi i} \int_c \frac{H_0^+(x) + H_0^-(x)}{x-z} dx, \quad (4.12)$$

$$\omega'_0(z) = -\bar{\Lambda}_4(z) - z\Omega'_0(z) + \frac{4h^2}{1-\nu} \Omega''_0(z), \quad (4.13)$$

$$\Lambda_4(z) = -\frac{X(z)}{4\pi i} \int_c \frac{H_0^+(x) + H_0^-(x)}{X(x)(x-z)} dx + \frac{1}{4\pi i} \int_c \frac{H_0^+(x) - H_0^-(x)}{x-z} dx, \quad (4.14)$$

$$H_0(x) = \tilde{g}(x) + \tilde{Q}_2(x) - \tilde{G}(A_{k,1}, \gamma_{k,1}), \quad (4.15)$$

$$\begin{aligned} \tilde{G}(A_{k,1}, \gamma_{k,1}) &= (\Gamma_1 + \bar{\Gamma}_2)z + \bar{\Gamma}'_2 \bar{z} + K_2 + \bar{K}'_2 + \\ &+ \sum_{k=1}^m \{(A_{k,1}z + \gamma_{k,1})E_k(z) + \gamma_{k,1}E_k(\bar{z}) - \\ &- \frac{4h^2}{1-\nu} \left[ 2A_{k,1} \cdot \frac{1}{s_k} \cdot \ln \frac{z-a_k}{z-b_k} - \frac{A_{k,1}z + \gamma_{k,1}}{(z-a_k)(z-b_k)} \right] + \\ &+ z \sum_{k=1}^m \left\{ (A_{k,1}\bar{z} + \bar{\gamma}_{k,1}) \cdot \frac{1}{s_k} \cdot \ln \frac{\bar{z}-a_k}{\bar{z}-b_k} + A_{k,1}E_k(\bar{z}) \right\}. \end{aligned} \quad (4.16)$$

The constants  $A_{k,1}$ ,  $\gamma_{k,1}$  are found from the system

$$\sum_{k=1}^m A_{k,1} = 0, \quad \sum_{k=1}^m \gamma_{k,1} = 0, \quad (4.17)$$

$$w_0(a_{k+1}) - w_0(a_k) = h(a_{k+1}) - h(a_k). \quad k = 1, \dots, m-1. \quad (4.18)$$

$$\int \frac{x^k [H_0^+(x) + H_0^-(x)]}{X(x)} dx = 0, \quad k = 1, \dots, m-1. \quad (4.19)$$

*Proof.* Now we introduce the function

$$\Lambda_4(z) = -z\bar{\Omega}'_0(z) - \bar{\omega}'_0(z) + \frac{4h^2}{1-\nu} \bar{\Omega}''_0(z),$$

and find

$$[\Omega_0(x) - \Lambda_4(x)]^+ + [\Omega_0(x) - \Lambda_4(x)]^- = H_0^+(x) + H_0^-(x),$$

$$[\Omega_0(x) + \Lambda_4(x)]^+ - [\Omega_0(x) + \Lambda_4(x)]^- = H_0^+(x) - H_0^-(x).$$

The conditions (4.18) follow from (4.3), however, we get the conditions (4.17) from the requirement that the solution stays bounded in the neighbourhood of the points  $a_k, b_k$ .

The obtained solution  $w_0$  is determined up to an additive constant, [2]. The real solution is found by considering (4.3).

#### R E F E R E N C E S

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## DIE LÖSUNG DER VERALLGEMEINTEN RANDWERTPROBLEME FÜR EINE ELASTISCHE PLATTE MIT SCHNITTEN AUF EINER GERADEN

### Z u s a m m e n f a s s u n g

Dieser Beitrag ist die Fortsetzung einer früheren Veröffentlichung [1]. Diese frühere Veröffentlichung ergibt die allgemeine Biegungstheorie der Platten und als besonderen Fall die Lösung einer Kreisplatte.

In der folgenden Arbeit sind die beiden Randwertprobleme für die Platte mit Schnitten längs einer Geraden behandelt. Da die Verschiebungen und Spannungen in der Mittelebene berücksichtigt sind, stellt dieser Beitrag eine Neuigkeit in der Theorie der Platten.

## REŠITEV POSPLOŠENIH ROBNIH PROBLEMOV ZA ELASTIČNO PLOŠČO Z ZAREZAMI NA ENI PREMICI

### P o v z e t k

Prispevek predstavlja nadaljevanje in dopolnilo članka [1]-ta prinaša splošnp teorijo upogiba plošč in kot poseben primer je v njem rešena samo krožna plošča.

Pričujoči članek prinaša rešitev obeh osnovnih robnih problemov za ploščo, ki ima zarezo na eni premici. Ker so upoštevani pomiki in napetosti v srednji ravnini plošče, prispevek pomeni novost v teoriji plošč.

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