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HARMONIC ACCELERATION METHOD FOR DYNAMIC STRUCTURAL ANALYSIS

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1. Introduction

Nowadays the structural dynamic problems are usually solved by the finite element technique. A structure is modeled by the finite elements which are connected together in a certain number of nodes. The physical characteristics of structure are transfered through the finite elements to the nodes of model forming in such a way a discretized system. The dynamic equilibrium of the nodal forces leads to the matrix differential equation of the nodal displacements. This equation may be solved by direct integration or it may be previously transformed into the set of modal equations. In the case of transient vibration the integration can be performed only numericaly. In order to obtain the reliable results concerning the stability and accuracy, by a reasonable computation, a number of numerical integration methods has been established. Among these the most commonly used are the Houbolt, the Newmark and the Wilson Θ method, [1—5]. These mothods have been applied with success in linear as well as in nonlinear dynamic structural analysis [6, 8].

Recently, a more mathematically oriented method has been in use for general physical dynamic problems, [9]. That is so called stiffly stable method, in which the second order dynamic equation is transformed into a first order equation, [10, 11].

Since the dynamic response of a structure can be expanded into the harmonic components, the idea of harmonic acceleration approximation in each time step of the integration is worked out in detail in this paper, [12]. This assumption is used for direct integration of the governing equilibrium equation as well as for its modal transformation.

2. Dynamic Equilibrium Equation

In the finite element method the dynamic equilibrium equation may be written in the matrix notation, [13],

$$[K]\{\delta\} + [C]\{\dot{\delta}\} + [M]\{\ddot{\delta}\} = \{F(t)\}, \tag{1}$$

where $[K]$, $[C]$ and $[M]$ are the stiffness, damping and mass matrices, respectively; $\{\delta\}$, $\{\dot{\delta}\}$ and $\{\ddot{\delta}\}$ are the displacement, velocity and acceleration vectors, respectively; and $\{F(t)\}$ is the force vector. The above matrices are symmetrical as a result of the energy approach to the finite element method.

Equation (1) may be transformed in space into the modal coordinates assuming the displacement vector in the form

$$\{\delta\} = [\Phi]\{X\}, \tag{2}$$

where $[\Phi] = [\{\phi\}_1, \{\phi\}_2 \dots \{\phi\}_n]$ is the undamped mode matrix and $\{X\}$ is the generalized displacement vector. The natural modes $\{\phi\}$; and the correspoding natural frequencies ω_j are obtained by solving the eigenproblem of free undamped vibration,

$$([K] - \omega^2 [M])\{\phi\} = \{0\}. \tag{3}$$

Equation (3) is derived from (1) assuming $\{\delta\} = \{\phi\} \cos \omega t$.

Substituting (2) into (1) and premultiplying (1) by $[\Phi]^T$, we obtain the modal equation

$$\lceil k \rfloor \{X\} + [c]\{\dot{X}\} + \lceil m \rfloor \{\ddot{X}\} = \{f(t)\}, \tag{4}$$

where

$$\lceil k \rfloor = [\Phi]^T [K] [\Phi] \text{ — modal stiffness matrix,}$$

$$[c] = [\Phi]^T [C] [\Phi] \text{ — modal damping matrix,}$$

$$\lceil m \rfloor = [\Phi]^T [M] [\Phi] \text{ — modal mass matrix,}$$

$$\{f(t)\} = [\Phi]^T \{F(t)\} \text{ — modal load vector.} \tag{5}$$

Matrices $\lceil k \rfloor$ and $\lceil m \rfloor$ are diagonal, while $[c]$ is not diagonal in a general case.

From equation (4) for undamped natural vibration, we find relation $\lceil k \rfloor = \lceil \omega^2 m \rfloor$. By its backward substitution into (4) the final form of the modal equation is obtained

$$\lceil \omega^2 \rfloor \{X\} + 2 \lceil \omega \rfloor [\xi] \{\dot{X}\} + \{\ddot{X}\} = \{\varphi(t)\}, \tag{6}$$

where

$$\lceil \omega \rfloor = \left\lceil \sqrt{\frac{k_{ii}}{m_{ii}}} \right\rfloor \text{ — matrix of natural frequencies,}$$

$$[\xi] = \left[\frac{c_{ij}}{2\sqrt{k_{ii}\, m_{ii}}} \right] \text{ — relative damping matrix,}$$

$$\{\varphi(t)\} = \left\{ \frac{f_i(t)}{m_{ii}} \right\} \text{ — relative load vector.} \tag{7}$$

If matrix $[\xi]$ is diagonal, the matrix equation (4) and (6) respectively are split into a set of uncoupled mode equations. This happens in some special cases of definition of damping matrix $[C]$, [14].

3. Direct Integration of Equilibrium Equation

For given initial conditions $\{\delta\}_0$ and $\{\dot{\delta}\}_0$ equation (1) may be integrated step-by-step over the time intervals $\Delta t = t_{i+1} - t_i$. Starting from the fact that at time $t = t_i + \tau$

$$\{\dot{\delta}\} = \{\dot{\delta}\}_i + \int_0^{\tau} \{\ddot{\delta}\}\, d\tau, \tag{8}$$

and further

$$\{\delta\} = \{\delta\}_i + \int_0^{\tau} \{\dot{\delta}\}\, d\tau, \tag{9}$$

it follows that the displacement vector $\{\delta\}$ can be determined at any time t if the acceleration vector $\{\ddot{\delta}\}$ is known within the time interval Δt. For that purpose let us assume the acceleration vector in the harmonic form

$$\{\ddot{\delta}\} = \{u\} \cos \lambda t + \{v\} \sin \lambda t, \tag{10}$$

where $\{u\}$ and $\{v\}$ are unknown vectors, and λ is an assumed interpolation frequency, all constant within the time interval Δt. If $\{\ddot{\delta}\}$ is known at time t_i and t_{i+1}, than equation (10) written at these two time points,

$$\begin{bmatrix} \cos \lambda t_i & \sin \lambda t_i \\ \cos \lambda t_{i+1} & \sin \lambda t_{i+1} \end{bmatrix} \begin{Bmatrix} \{u\} \\ \{v\} \end{Bmatrix} = \begin{Bmatrix} \{\ddot{\delta}\}_i \\ \{\ddot{\delta}\}_{i+1} \end{Bmatrix}. \tag{11}$$

makes determination of $\{u\}$ and $\{v\}$ possible. Hence,

$$\begin{Bmatrix} \{u\} \\ \{v\} \end{Bmatrix} = \frac{1}{\sin \lambda\Delta t} \begin{bmatrix} \sin \lambda t_{i+1} & -\sin \lambda t_i \\ -\cos \lambda t_{i+1} & \cos \lambda t_i \end{bmatrix} \begin{Bmatrix} \{\ddot{\delta}\}_i \\ \{\ddot{\delta}\}_{i+1} \end{Bmatrix}. \tag{12}$$

Furthermore, substituting (12) and $t = t_i + \tau$ into (10) and utilizing some of the fundametal trigometric identies, we finally obtain the interpolation acceleration vector

$$\{\ddot{\delta}\} = \cos \lambda\Delta t \left(\frac{\cos \lambda\tau}{\cos \lambda\Delta t} - \frac{\sin \lambda\tau}{\sin \lambda\Delta t} \right) \{\ddot{\delta}\}_i + \frac{\sin \lambda\tau}{\sin \lambda\Delta t} \{\ddot{\delta}\}_{i+1}\,. \tag{13}$$

In that way the integration of equations (8) and (9) is made possible, that results in

$$\{\dot{\delta}\} = \{\dot{\delta}\}_i + \frac{\cos \lambda\Delta t}{\lambda} \left(\frac{\sin \lambda\tau}{\cos \lambda\Delta t} - \frac{1 - \cos \lambda\tau}{\sin \lambda\Delta t} \right) \{\ddot{\delta}\}_i + \frac{1 - \cos \lambda\tau}{\lambda \sin \lambda\Delta t} \{\ddot{\delta}\}_{i+1},$$

$$\{\delta\} = \{\delta\}_i + \tau \{\dot{\delta}\}_i + \frac{\cos \lambda\Delta t}{\lambda^2} \left(\frac{1 - \cos \lambda\tau}{\cos \lambda\Delta t} + \frac{\sin \lambda\tau - \lambda\tau}{\sin \lambda\Delta t} \right) \{\ddot{\delta}\}_i + \frac{\lambda\tau - \sin \lambda\tau}{\lambda^2 \sin \lambda\Delta t} \{\ddot{\delta}\}_{i+1}. \tag{14}$$

At the end of Δt, expressions (14) yield

$$\{\dot{\delta}\}_{i+1} = \{\dot{\delta}\}_i + \frac{1-\cos\lambda\Delta t}{\lambda \sin\lambda\Delta t}\{\ddot{\delta}\}_i + \frac{1-\cos\lambda\Delta t}{\lambda\sin\lambda\Delta t}\{\ddot{\delta}\}_{i+1},$$

$$\{\delta\}_{i+1} = \{\delta\}_i + \Delta t\{\dot{\delta}\}_i + \frac{1}{\lambda^2}(1-\lambda\Delta t \operatorname{ctg}\lambda\Delta t)\{\ddot{\delta}\}_i + \\ + \frac{1}{\lambda^2}\left(\frac{\lambda\Delta t}{\sin\lambda\Delta t} - 1\right)\{\ddot{\delta}\}_{i+1}. \tag{15}$$

Now, $\{\dot{\delta}\}_{i+1}$ and $\{\ddot{\delta}\}_{i+1}$ can be determined explicitly from (15)

$$\{\dot{\delta}\}_{i+1} = \frac{a}{\Delta t}(\{\delta\}_{i+1} - \{\delta\}_i) - c\{\dot{\delta}\}_i - d\,\Delta t\{\ddot{\delta}\}_i,$$
$$\{\ddot{\delta}\}_{i+1} = \frac{b}{\Delta t^2}(\{\delta\}_{i+1} - \{\delta\}_i) - \frac{b}{\Delta t}\{\dot{\delta}\}_i - c\{\ddot{\delta}\}_i, \tag{16}$$

where

$$a = \frac{\lambda\Delta t}{w}(1-\cos\lambda\Delta t),$$

$$b = \frac{\lambda^2\Delta t^2}{w}\sin\lambda\Delta t,$$

$$c = \frac{1}{w}(\sin\lambda\Delta t - \lambda\Delta t\cos\lambda\Delta t),$$

$$d = \frac{1}{\lambda\Delta t w}(2 - 2\cos\lambda\Delta t - \lambda\Delta t\sin\lambda\Delta t),$$

$$w = \lambda\Delta t - \sin\lambda\Delta t. \tag{17}$$

Furthemore, substituing (16) into differential equation (1) at time t_{i+1} we obtain the following algebraic equation for determination of $\{\delta\}_{i+1}$ depending on $\{\delta\}_i$, $\{\dot{\delta}\}_i$ and $\{\ddot{\delta}\}_i$:

$$[S]\{\delta\}_{i+1} = \{f\}_{i+1}, \tag{18}$$

where

$$[S] = [K] + \frac{a}{\Delta t}[C] + \frac{b}{\Delta t^2}[M],$$
$$\{f\}_{i+1} = \{F(t)\}_{i+1} + [P]\{\delta\}_i + [Q]\{\dot{\delta}\}_i + [R]\{\ddot{\delta}\}_i,$$
$$[P] = \frac{a}{\Delta t}[C] + \frac{b}{\Delta t^2}[M],$$
$$[Q] = c[C] + \frac{b}{\Delta t}[M], \tag{19}$$
$$[R] = d\,\Delta t[C] + c[M].$$

Equations (18) and (16) represent the algorithm for determination of dynamic response.

4. Integration of Modal Equation

Modal equation (6) can also be integrated by the harmonic acceleration method if the generalized acceleration vector $\{\ddot{X}\}$ at time $t = t_i + \tau$ is interpolated between $\{\ddot{X}\}_i$ and $\{\ddot{X}\}_{i+1}$ at time t_i and $t_{i+1} = t_i + \Delta t$ respectively. Thus, according to (13)

$$\{\ddot{X}\} = \left[\cos\omega\Delta t\left(\frac{\cos\omega\tau}{\cos\omega\Delta t} - \frac{\sin\omega\tau}{\sin\omega\Delta t}\right)\right]\{\ddot{X}\}_i + \left[\frac{\sin\omega\tau}{\sin\omega\Delta t}\right]\{\ddot{X}\}_{i+1}, \quad (20)$$

where ω_j are the natural frequencies of system.

Following the procedure from the previous chapter we obtain further, analogously to (16),

$$\{\dot{X}\}_{i+1} = \frac{1}{\Delta t}[a](\{X\}_{i+1} - \{X\}_i) - [c]\{\dot{X}\}_i - \Delta t\,[d]\{\ddot{X}\}_i,$$

$$\{\ddot{X}\}_{i+1} = \frac{1}{\Delta t^2}[b](\{X\}_{i+1} - \{X\}_i) - \frac{1}{\Delta t}[b]\{\dot{X}\}_i - [c]\{\ddot{X}\}_i, \quad (21)$$

where

$$[a] = \left[\frac{\omega\Delta t}{w}(1 - \cos\omega\Delta t)\right],$$

$$[b] = \left[\frac{\omega^2\Delta t^2}{w}\sin\omega\Delta t\right],$$

$$[c] = \left[\frac{1}{w}(\sin\omega\Delta t - \omega\Delta t\cos\omega\Delta t)\right],$$

$$[d] = \left[\frac{1}{\omega\Delta t w}(2 - 2\cos\omega\Delta t - \omega\Delta t\sin\omega\Delta t)\right],$$

$$[w] = [\omega\Delta t - \sin\omega\Delta t]. \quad (22)$$

By substitution (21) into (6) at time t_{i+1} the following matrix equation is obtain for determination of $\{X\}_{i+1}$:

$$[S]\{X\}_{i+1} = \{\psi\}_{i+1}, \quad (23)$$

where

$$[S] = \frac{1}{\Delta t^2}([e] + 2[\omega\Delta t\, a][\xi]), \quad [e] = \left[\frac{\omega^3\Delta t^3}{w}\right],$$

$$\{\psi\}_{i+1} = \{\varphi(t)\}_{i+1} + [P]\{X\}_i + [Q]\{\dot{X}\}_i + [R]\{\ddot{X}\}_i,$$

$$[P] = \frac{1}{\Delta t^2}([b] + 2[\omega\Delta t\, a][\xi]),$$

$$[Q] = \frac{1}{\Delta t}\left([b] + 2\,[\omega\Delta t\, c]\,[\xi]\right),$$

$$[\bar{R}] = [c] + 2\,[\omega\Delta t\, d]\,[\xi]. \tag{24}$$

Expressions (23) and (21) represent the algorithm for calculation of dynamic response by the mode superposition.

If the relative damping matrix $[\xi]$ is diagonal, the mode equations are uncoupled and the above algorithm can be presented in the explicit form. This leads to the following recurrent formula for each mode:

$$\{Y\}_{i+1} = [T]\,\{Y\}_i + \Delta t^2\,\{L\}\,\varphi(t)_{i+1}, \tag{25}$$

where

$$\{Y\} = \begin{Bmatrix} X \\ \Delta t\,\dot{X} \\ \Delta t^2\,\ddot{X} \end{Bmatrix},\ \{L\} = \frac{1}{e+2\,\xi\omega\Delta t\, a}\begin{Bmatrix} 1 \\ a \\ b \end{Bmatrix},\ f = \frac{1}{w}\,\omega^3\,\Delta t^3\cos\omega\Delta t,$$

$$[T] = \frac{1}{e+2\,\xi\omega\,\Delta ta}\begin{bmatrix} b+2\,\xi\omega\Delta ta, & b+2\,\xi\omega\Delta t\,c, & c+2\,\xi\omega\Delta td \\ -\omega^2\,\Delta t^2\,a, & f & b-a \\ -\omega^2\,\Delta t^2\,b, & -\omega^2\,\Delta t^2\,a - 2\,\xi\omega\Delta tb, & -\omega^2\,\Delta t^2\,c - 2\,\xi\omega\Delta ta \end{bmatrix}. \tag{26}$$

The transfer matrix $[T]$ and vector $\{L\}$ are the integration and load operators respectively, which depend only on argument $\omega\Delta t = 2\,\pi\,\frac{\Delta t}{T}$ and damping ratio ξ.

In the case of an undamped system the integration and load operators are reduced to the simple form,

$$[T] = \begin{bmatrix} \frac{\sin\omega\Delta t}{\omega\Delta t} & \frac{\sin\omega\Delta t}{\omega\Delta t} & \frac{\sin\omega\Delta t}{\omega^3\,\Delta t^3} - \frac{\cos\omega\Delta t}{\omega^2\,\Delta t^2} \\ \cos\omega\Delta t - 1 & \cos\omega\Delta t & \frac{\cos\omega\Delta t - 1}{\omega^2\,\Delta t^2} + \frac{\sin\omega\Delta t}{\omega\Delta t} \\ -\,\omega\Delta t\sin\omega\Delta t & -\,\omega\Delta t\sin\omega\Delta t & -\,\frac{\sin\omega\Delta t}{\omega\Delta t} + \cos\omega\Delta t \end{bmatrix},$$

$$\{L\} = \begin{Bmatrix} \frac{1}{\omega^2\,\Delta t^2} - \frac{\sin\omega\Delta t}{\omega^3\,\Delta t^3} \\ \frac{1-\cos\omega\Delta t}{\omega^2\,\Delta t^2} \\ \frac{\sin\omega\Delta t}{\omega\Delta t} \end{Bmatrix}. \tag{27}$$

The same operators are found in this special case directly by the analytical solution of the mode equation, [15].

5. Stability and Accuracy Analysis

5.1. General

Reliability of the predicted structural response depends on the stability and accuracy of the applied integration method. Stability may be unconditional or conditional. An integration method is unconditionally stable if the solution for any problem of initial coditions is bounded for the increasing time step Δt from 0 to ∞. The method is conditionally stable if the above only holds provided the time step within a certain lower and upper critical value, i.e. $\Delta t_l < \Delta t < \Delta t_u$.

The direct integration and the mode superposition utilizing the harmonic acceleration are two different methods and therefore the stability for each of them has to be examined. In the former case frequency λ and time step Δt have to be chosen, while in the latter the natural frequencies ω_j are known and only a choice for Δt has to be made.

5.2. Problem of small time step

Let us firstly consider the stability of the methods for small values of arguments $\lambda\Delta t$ and $\omega\Delta t$ respectively. If the values of these arguments approach zero, the stability of the methods seems to become uncertain, because coefficients (17) and (22) are led to the indeterminate form 0/0. Since this is the same problem of the both methods it is sufficiently to analyse it in the case of direct integration. The indeterminate form is an apparent singularity and can be avoided by exanding the trigonometric functions into the exponentiel series, i.e.

$$\sin \lambda\Delta t = \sum_{n=0}^{\infty} (-1)^n \frac{(\lambda\Delta t)^{2n+1}}{(2n+1)!},$$

$$\cos \lambda\Delta t = \sum_{n=0}^{\infty} (-1)^n \frac{(\lambda\Delta t)^{2n}}{(2n)!}. \tag{28}$$

Thus, coefficients (17) yield

$$a = \frac{1}{w} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(\lambda\Delta t)^{2n+1}}{(2n)!},$$

$$b = \frac{1}{w} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(\lambda\Delta t)^{2n+1}}{(2n-1)!},$$

$$c = \frac{1}{w} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2n\,(\lambda\Delta t)^{2n+1}}{(2n+1)!},$$

$$d = \frac{1}{w} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2n\,(\lambda\Delta t)^{2n+1}}{(2n+2)!},$$

$$w = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(\lambda\Delta t)^{2n+1}}{(2n+1)!}. \tag{29}$$

Their limit values are finite,

$$\lim_{\lambda\Delta t\to 0}\begin{Bmatrix} a \\ b \\ c \\ d \end{Bmatrix} = \begin{Bmatrix} 3 \\ 6 \\ 2 \\ 1/2 \end{Bmatrix}. \tag{30}$$

Hence, the harmonic acceleration method is stable for small values of time step Δt.

The limit values (30) are equal to the coefficients in the linear acceleration method which is conditionally stable, (16). This is obviously since the limit of harmonic acceleration vector (13) is the linear vector when $\lambda\Delta t$ approaches 0 due to λ, i.e.

$$\lim_{\lambda\to 0}\ \{\delta\} = \left(1 - \frac{\tau}{\Delta t}\right)\{\delta\}_i + \frac{\tau}{\Delta t}\{\delta\}_{i+1}. \tag{31}$$

Hence, the linear acceleration method is only an asymptotic approximation of he general harmonic acceleration method.

5.3. Mode superposition method

In order to examine the stability of the methods for high values of arguments $\lambda\Delta t$ and $\omega\Delta t$ respectively, we may consider problem of arbitrary initial conditions, that also includes the case when no load and damping are specified. These parameters do not influence the overall stability of the methods, but their omission simplifies the analysis. Since the stability of the mode superposition method depends only on the chosen value of time step Δt, let us firstly consider this problem. For a mode the problem is specified as

$$\omega^2 x + \ddot{x} = 0,\quad x(0) = x_0,\quad \dot{x}(0) = \dot{x}_0,\quad \ddot{x}(0) = -\,\omega^2 x_0, \tag{32}$$

with the analytical solution of free vibration

$$x = x_0 \cos\omega t + \frac{\dot{x}_0}{\omega}\sin\omega t. \tag{33}$$

If the integration method is stable than bounded difference between results of the long time step integration, $\Delta t_n = n\,\Delta t$, and n integrations for short time Δt should be obtained when $n\to\infty$. According to (25) we can write for these two cases

$$\{Y_n\}_n = [T_n]\{Y_n\}_0, \tag{34}$$

and respectively

$$\{Y\}_n = [T]^n\{Y\}_0. \tag{35}$$

Since

$$\{Y_n\}_0 = [N]\{Y\}_0, \tag{36}$$

and

$$\{Y_n\}_n = \Lambda^n [N] \{Y\}_n, \tag{37}$$

where Λ is the correlation factor and

$$[N] = \begin{bmatrix} 1 & & \\ & n & \\ & & n^2 \end{bmatrix}, \tag{38}$$

substitutions (36) into (34), and (35) into (37) and further into (34) lead to the eigenvalue problem for determination of Λ

$$([T_n] - \Lambda^n [\theta]_n) \{Y\}_0 = \{0\}, \tag{39}$$

where

$$[\theta]_n = [N] [T]^n [N]^{-1}. \tag{40}$$

Depending on the values of Λ_k, $k = 1, 2, 3$, we may conclude the following:

- if $\Lambda_k = 1$, than $[T]_n \equiv [\theta]_n$; the results are exact and consequently the method is unconditionally stable,
- if the spectal radius $\rho = \max |\Lambda_k| \leqslant 1$, matrix $[T_n]$ is bounded and the method is unconditionally stable,
- if $\rho > 1$, $[T_n]$ is not bounded and the method is conditionally stable.

In the considered problem (32) $\ddot{x} = -\omega^2 x$ and, in order to simplify the analysis, this relationship may be used for the linear transformation of matrix $[T]$ defined by (27). Thus, we obtain

$$[T] = \begin{bmatrix} \cos \omega\Delta t & \dfrac{\sin \omega\Delta t}{\omega\Delta t} & 0 \\ -\omega\Delta t \sin \omega\Delta t & \cos \omega\Delta t & 0 \\ -\omega^2 \Delta t^2 \cos \omega\Delta t & -\omega\Delta t \sin \omega\Delta t & 0 \end{bmatrix}. \tag{41}$$

As a result of this transformation, the eigenproblem (39) is reduced to its minor

$$([T_n]' - \Lambda^n [\theta]_n') \{Y\}_0' = \{0\}, \tag{42}$$

where

$$[T_n]' = \begin{bmatrix} \cos \omega\, n\, \Delta t & \dfrac{\sin \omega\, n\, \Delta t}{\omega\, n \Delta t} \\ -\omega n\, \Delta t \sin \omega n\, \Delta t, & \cos \omega\, n\, \Delta t \end{bmatrix}, \tag{43}$$

and

$$[\theta]_n' = \begin{bmatrix} 1 & 0 \\ 0 & n \end{bmatrix} \begin{bmatrix} \cos \omega\Delta t & \dfrac{\sin \omega\Delta t}{\omega\Delta t} \\ -\omega\Delta t \sin \omega\Delta t & \cos \omega\Delta t \end{bmatrix}^n \begin{bmatrix} 1 & 0 \\ 0 & \dfrac{1}{n} \end{bmatrix}. \tag{44}$$

Furthermore, it is easy to prove $[T_n]' \equiv [\theta]_n'$ utilizing the fundamental identities of the trigonometric functions of a single argument and its multipliers. Consequently, $\Lambda_{1,2} = 1$ and the method is exact in the considered case and also unconditionally stable in general.

The above conclusion may be substantiated by the fact that exactly the same matrix $[T]$ given by [41] can be derived directly utilizing the exact solution (33) of the problem. Extending problem (32) to the case of constant load, i.e.

$$\omega^2 x + \ddot{x} = 1, \quad x(0) = x_0, \quad \dot{x}(0) = \dot{x}_0, \quad \ddot{x}(0) = 1 - \omega^2 x_0, \tag{45}$$

with the analytical solution

$$x = x_0 \cos \omega t + \frac{\dot{x}_0}{\omega} \sin \omega t + \frac{1}{\omega^2}, \tag{46}$$

where $1/\omega^2$ is the static displacement, the load operator (27) may be transformed into the form

$$\{L\} = \begin{Bmatrix} \dfrac{1}{\omega^2 \Delta t^2} - \dfrac{\cos \omega \Delta t}{\omega^2 \Delta t^2} \\ \dfrac{\sin \omega \Delta t}{\omega \Delta t} \\ \cos \omega \Delta t \end{Bmatrix}. \tag{47}$$

The same operator is obtained by the analytical solution (46), and therefore the method gives the exact results also in this case.

This means that in a general case solution of the problem is approximate in each time step by the interpolation function, which represents solution of the free undamped vibration and static displacement for the constant value of load in the considered time step. As a result of this, the accuracy of the method dependes primarily on an adequate description of the load history by the time subdivision.

5.4. Direct integration method

In the direct integration method the interpolation frequency λ and time step Δt are chosen and therefore it is necessary to determine for which rations ω_j/λ and $\Delta t/T_\lambda$ the method is stable and accurate enough. Since $\lambda \neq \omega_j$ this problem may be analysed taking a single d. o f. system into consideration and assuming $\lambda \neq \omega$. Therefore, let us consider the same problem as in the modal analysis, i.e.

$$\omega^2 \delta + \ddot{\delta} = 0, \quad \delta(0) = \delta_0, \quad \dot{\delta}(0) = \dot{\delta}_0, \quad \ddot{\delta}(0) = \omega^2 \delta_0 \tag{48}$$

According to the algorithm in Section 3, the following reccurent formula is derived in this case:

$$\{\Delta\}_{i+1} = [T_\lambda] \{\Delta\}_i, \{\Delta\} = \begin{Bmatrix} \delta \\ \Delta t \, \dot{\delta} \\ \Delta t^2 \, \ddot{\delta} \end{Bmatrix}, \tag{49}$$

where

$$[T_\lambda] = \frac{1}{\omega^2 \Delta t^2 + b} \begin{bmatrix} b & b & c \\ -\omega^2 \Delta t^2 a & b - \omega^2 \Delta t^2 c & a - \omega^2 \Delta t^2 d \\ -\omega^2 \Delta t^2 b & -\omega^2 \Delta t^2 b & -\omega^2 \Delta t^2 c \end{bmatrix}. \tag{50}$$

Employing the linear dependence $\ddot{\delta} = -\omega^2 \delta$, the integration operator is transformed and reduced to the minor

$$[T_\lambda]' = \frac{1}{\omega^2 \Delta t^2 + b} \begin{bmatrix} b - \omega^2 \Delta t^2 c & b \\ \omega^2 \Delta t^2 (\omega^2 \Delta t^2 d - 2a) & b - \omega^2 \Delta t^2 c \end{bmatrix}. \tag{51}$$

According to (42) the eigenproblem for the stability examination is formulated as

$$([T_{\lambda n}]' - \Lambda^n [\theta]_n') \{\Delta\}_0' = \{0\}, \tag{52}$$

where $[T_{\lambda n}]'$ is represented by (51) for $\Delta t_n = n \Delta t$ while $[\theta]_n'$ is given by (44) and also by (43) since $[\theta]_n' \equiv [T_n]'$. The characteristic polynomial of (52) is

$$\Lambda^{2n} - 2 \frac{p}{q} \Lambda^n + 1 = 0, \tag{53}$$

and its roots

$$\Lambda_{1,2} = \left[\frac{p}{q} \pm i \sqrt{1 - \frac{p^2}{q^2}} \right]^{1/n}, \tag{54}$$

where

$$p = \alpha^2 \beta^2 \cos(\alpha - \beta) + \frac{1}{2} \alpha\beta (\alpha - \beta)^2 \sin\alpha \sin\beta +$$

$$+ (\alpha^2 - \beta^2) [\alpha \sin\alpha \cos\beta + \beta \sin\beta (1 - \cos\alpha)],$$

$$q = \alpha^2 \beta^2 + \alpha (\alpha^2 - \beta^2) \sin\alpha,$$

$$\alpha = \lambda n \Delta t, \quad \beta = \omega n \Delta t. \tag{55}$$

Since $\frac{p^2}{q^2} \leqslant 1$ for any value of α and β within 0 and ∞, we find further

$$\Lambda_{1,2} = \cos \frac{\varphi + 2k\pi}{n} + i \sin \frac{\varphi + 2k\pi}{n}, \tag{56}$$

where

$$k = 0, \pm 1, \pm 2 \ldots,$$

$$\varphi = \operatorname{arc\,tg} \frac{\sqrt{q^2 - p^2}}{p}. \tag{57}$$

Hence, $\rho = |\Lambda_{1,2}| \equiv 1$ and the method is unconditionally stable for any value of ω_j/λ and $\Delta t/T_\lambda$. If $\lambda = \omega$ than $\Lambda_{1,2} \equiv 1$ and the method gives the exact solution in the considered case.

The accuracy analysis has been performed integrating equation (48) in a period $T_\omega = \frac{2\pi}{\omega}$ for different values of parametres ω_j/λ and $\Delta t/T_\lambda$, which are of practical significance. The difference between the exact and approximate displacement at the end of period T_ω is defined as error, Fig. 1. It holds both the period and amplitude decays, which are of the same order. The ampitude decay is result of the unconditional stability of the method. For $\omega/\lambda = 1$ the exact solution is obtained. The error increases by difference $|1 - \omega/\lambda|$ and as well as by the value of $\Delta t/T_\lambda$. However, the error converges by time to some finite value due to the unconditional stability of the method.

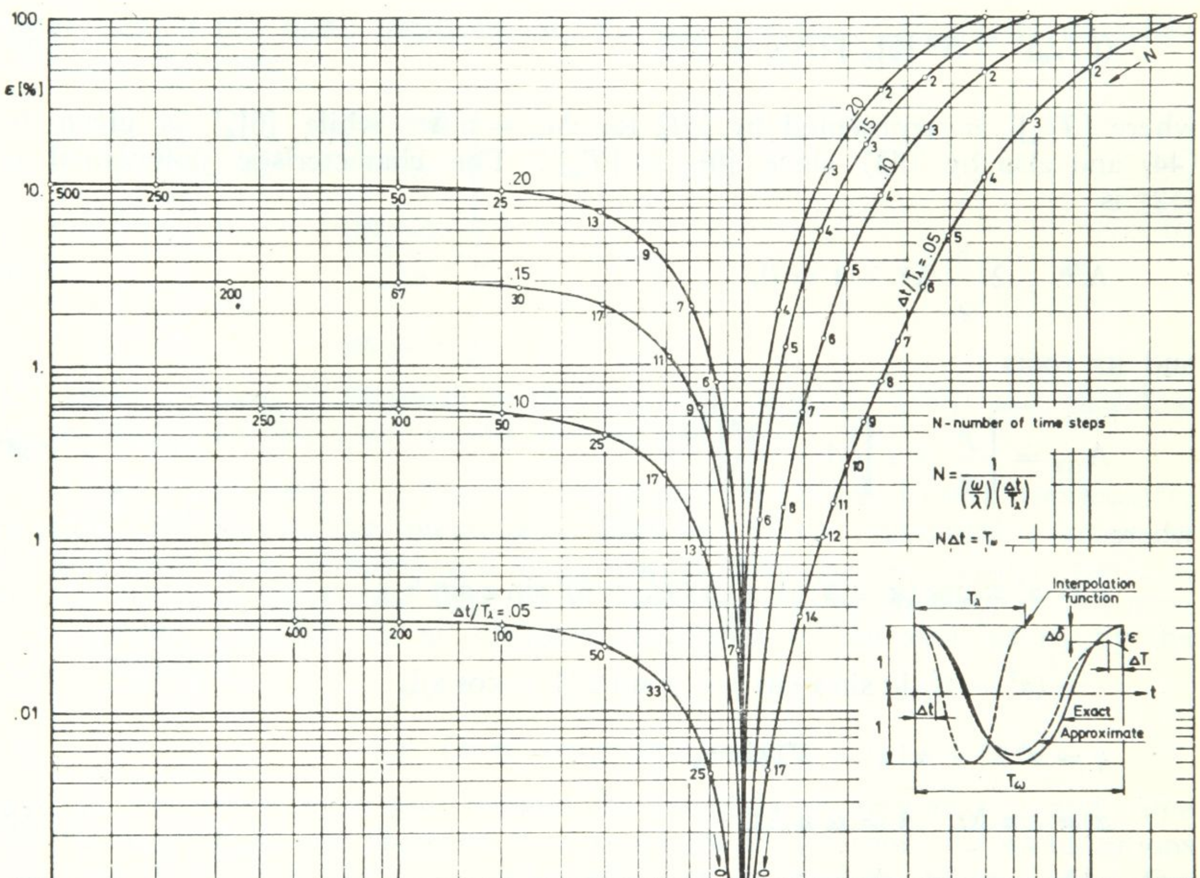

Fig. 1. Error of the direct integration method

Applying the previous results of the accuracy analysis for the single d. o f. system to the case of a multi d. o f. system, the most accurate response is obtained if the interpolation frequency λ is chosen equal to the natural ω_p of the predominant mode. The predominant mode may be predicted according to the load history or comparing the initial deformation of the structure to the natural modes. If the predominant mode is variable, different value of λ and Δt may be chosen, for each time step.

Looking through the modal analysis, the predominant mode is integrated very reliably (or exactly if constant load and no damping are specified), while the integration of the other modes includes some error. According to Fig. 1., the mode error ε_j increases by difference $|1 - \omega_j/\lambda|$. On the other hand, contribution of each mode to the total response, defined by the maximum value of displacement for a period of time, $A_j = \max |x_j|$, decreases by the above difference. Therefore, resulting error ε of the response is smaller than the maximum mode error in the frequency domain, Fig. 2,

$$\varepsilon = \frac{\sum_{j=1}^{n} A_j \varepsilon_j}{\sum_{j=1}^{n} A_j}. \tag{58}$$

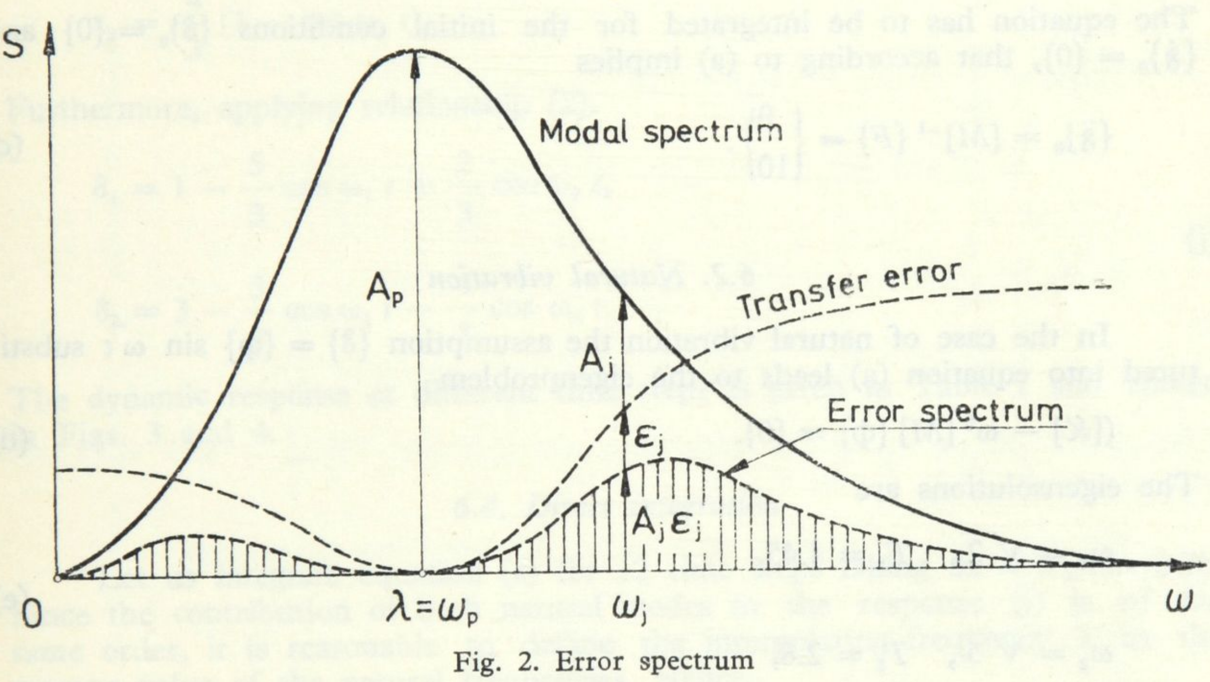

Fig. 2. Error spectrum

In the time domain the mode error accumalates. In each period $T_{\omega j}$ its value increases for ε_j, (Fig. 1. if no load and damping are specified). At arbitrary time, we may write,

$$\varepsilon_j(t) = 1 - (1 - \varepsilon_j)^{t/T\omega j}.$$

Hence, the total error (58) is time dependent. Therefore, the modal spectrum decreases by time and becomes more narrow. When $t \to \infty$ than $\varepsilon_j(t) \to 1$ and all modes disappear besides the static displacement of the predominant mode which surely remains if no damping is specified.

In the mode superposition method a limited number of the first modes are taken into account. The highest natural frequency of the chosen modes usually exceeds the frequency of the predominant mode for double. In that case an optimal integration concerning the accuracy and computation, may be performed taking $\Delta t = T_\lambda/10$, that causes the maximum mode error of 3,5%, Fig. 1.

6. Illustrative Example

6.1 Formulation of the problem

Application of the presented integration methods assuming harmonic acceleration is illustrated by the example of a two d. o f. system without damping, which is solved in (5) analytically and by the most commonly used integration methods. The differential equation of the system is

$$[K]\{\delta\} + [M]\{\ddot{\delta}\} = \{F\}, \tag{a}$$

where

$$[K] = \begin{bmatrix} 6 & -2 \\ -2 & 4 \end{bmatrix}, \quad [M] = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \quad \{F\} = \begin{Bmatrix} 0 \\ 10 \end{Bmatrix}. \tag{b}$$

The equation has to be integrated for the initial conditions $\{\delta\}_0 = \{0\}$ and $\{\dot{\delta}\}_0 = \{0\}$, that according to (a) implies

$$\{\ddot{\delta}\}_0 = [M]^{-1}\{F\} = \begin{Bmatrix} 0 \\ 10 \end{Bmatrix}. \tag{c}$$

6.2. Natural vibration

In the case of natural vibration the assumption $\{\delta\} = \{\phi\} \sin \omega t$ substituted into equation (a) leads to the eigenproblem

$$([K] - \omega^2 [M] \{\phi\} = \{0\}. \tag{d}$$

The eigensolutions are

$$\begin{aligned} \omega_1 &= \sqrt{2}, \quad T_1 = 4.45, \\ \omega_2 &= \sqrt{5}, \quad T_2 = 2.8, \end{aligned} \tag{e}$$

$$\{\phi\} = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}, \quad \{\phi\}_2 = \begin{Bmatrix} -\frac{1}{2} \\ 1 \end{Bmatrix}. \tag{f}$$

6.3. Exact solution of the dynamic response

The exact solution of the dynamic response may be obtained analytically by the mode superposition method. The corresponding mode equations are uncoupled

$$\begin{aligned} 2x_1 + \ddot{x}_1 &= \frac{10}{3}, \\ 5x_2 + \ddot{x}_2 &= \frac{20}{3}. \end{aligned} \tag{g}$$

Their solution consist of the homogeneous and particular integrals, i.e.

$$x_1 = A_{11} \sin \omega_1 t + A_{12} \cos \omega_1 t + \frac{5}{3},$$
$$x_2 = A_{21} \sin \omega_2 t + A_{22} \cos \omega_2 t + \frac{4}{3}. \tag{h}$$

Coefficients A_{ij} are determined satisfying the initial conditions, which for the generalized coordinates yield $\{X\}_0 = \{0\}$ and $\{\dot{X}\}_0 = \{0\}$. Thus, we find

$$x_1 = \frac{5}{3}(1 - \cos \omega_1 t),$$
$$x_2 = \frac{4}{3}(1 - \cos \omega_2 t). \tag{i}$$

Furthermore, applying relationship (2),

$$\delta_1 = 1 - \frac{5}{3} \cos \omega_1 t + \frac{2}{3} \cos \omega_2 t,$$
$$\delta_2 = 3 - \frac{5}{3} \cos \omega_1 t - \frac{4}{3} \cos \omega_2 t. \tag{j}$$

The dynamic response at different time steps is given in Table 1 and shown in Figs. 3 and 4.

6.4. Direct integration

Let us integrate equation (a) for 12 time steps taking $\Delta t = T_2/10 = 0.28$. Since the contribution of both natural modes to the response (j) is of the same order, it is reasonable to define the interpolating frequency λ as the average value of the natural frequencies. Hence,

$$\lambda = \frac{1}{2} (\omega_1 + \omega_2) = 1.825.$$

For the integration parameters the following values are obtained:

$$\lambda \Delta t = 0.511, \qquad a = 2.973,$$
$$\sin \lambda \Delta t = 0.489, \qquad b = 5.804,$$
$$\cos \lambda \Delta t = 0.872, \qquad c = 1.973,$$
$$w = 0.022, \qquad d = 0.534,$$
$$[S] = [K] + \frac{b}{\Delta t^2} [M] = \begin{bmatrix} 154.062 & -2 \\ -2 & 78.031 \end{bmatrix}.$$

The values of a, b, c and d are close to the limit values (30).

The algorithm for the response calculation is the following:

$$\{\delta\}_0 = \{0\}, \quad \{\dot{\delta}\}_0 = \{0\}, \quad \{\ddot{\delta}\}_0 = \begin{Bmatrix} 0 \\ 10 \end{Bmatrix},$$

$$\{f\}_{i+1} = \{F(t)\}_{i+1} + \frac{b}{\Delta t^2}[M]\,\delta_i + \frac{b}{\Delta t}[M]\,\dot{\delta}_i + c\,[M]\,\ddot{\delta}_i =$$

$$= \begin{Bmatrix} 0 \\ 10 \end{Bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} (74.031\,\{\delta\}_i + 20.729\,\{\dot{\delta}\}_i + 1.973\,\{\ddot{\delta}\}_i,$$

$$\{\delta\}_{i+1} = [S]^{-1}\{f\}_{i+1} = 10^{-3} \begin{bmatrix} 6.493 & 0.1664 \\ 0.1664 & 12.819 \end{bmatrix} \{f\}_{i+1},$$

$$\{\dot{\delta}\}_{i+1} = \frac{a}{\Delta t}(\{\delta\}_{i+1} - \{\delta\}_i) - c\,\{\dot{\delta}\}_i - d\,\Delta t\,\{\ddot{\delta}\}_i =$$

$$= 10.618\,(\{\delta\}_{i+1} - \{\delta\}_i) - 1.973\,\{\dot{\delta}\}_i - 0.149\,\{\ddot{\delta}\}_i,$$

$$\{\ddot{\delta}\}_{i+1} = \frac{b}{\Delta t^2}(\{\delta\}_{i+1} - \{\delta\}_i) - \frac{b}{\Delta t}\{\dot{\delta}\}_i - c\,\{\ddot{\delta}\}_i =$$

$$= 74.031\,(\{\delta\}_{i+1} - \{\delta\}_i) - 20.729\,\{\dot{\delta}\}_i - 1.973\,\{\ddot{\delta}\}_i.$$

The results of the numerical calculation are given in Table 1 and they are also shown in Figs. 3 and 4.

The relative mode errors depend on the following parameters:

$$\frac{\omega_1}{\lambda} = 0.773, \quad \frac{\omega_2}{\lambda} = 1.23, \quad \frac{\Delta t}{T_\lambda} = 0.0814.$$

From the diagram shown in Fig. 1, we find $\varepsilon_1 = 1.5\%$ and $\varepsilon_2 = 1.4\%$ for a period of time T_1 and T_2 respectively. The integration is performed up to $t_{12} = 0.76\,T_1 = 1.2\,T_2$, that gives $\varepsilon_{1,12} = 1.14\%$ and $\varepsilon_{2,12} = 1.68\%$ respectively. According to the analytical solution (i), the modal contributions to the response are $A_1 = 5/3$ and $A_2 = 4/3$. Using (58) we find the resulting response error, $\varepsilon = 1.4\%$. The error caused by the load integration is omitted.

6.5. Mode superposition

Modal equations (g) are integrated for the same time subdivision. The procedure is similar to the previous for the direct integration. Due to large number of numerical operations, the calculation has been perfomed by means of computer. The results are also given in Table 1 and shown in Figs. 3 and 4.

Table 1. Response of two d. o f. system, $\Delta t = 0.28$

Method	Step	1	2	3	4	5	6	7	8	9	10	11	12	Error ε%
Exact and Senjanović MS	δ_1	0.00251	0.0381	0.1756	0.4860	0.9963	1.657	2.338	2.861	3.052	2.806	2,131	1.157	0
	δ_2	0.3819	1.412	2.781	4.094	4.996	5.290	4.986	4.277	3.457	2.806	2.484	2.489	
Senjanović DI	δ_1	0.0049	0.0465	0.1905	0.5042	1.013	1.666	2.336	2.846	3.029	2.773	2.081	1.092	1.6
	δ_2	0.3818	1.410	2.785	4.110	5.016	5.309	5.013	4.301	3.477	2,772	2.435	2.521	
Finite differences	δ_1	0	0.031	0.168	0.487	1.02	1.70	2.40	2.91	3.07	2.77	2.04	1.02	2.5
	δ_2	0.392	1.45	2.83	4.14	5.02	5.26	4.90	4.17	3.37	2.78	2.54	2.60	
Newmark	δ_1	0.007	0.050	0.189	0.485	0.961	1.58	2.23	2.76	3.00	2.85	2.28	1.40	4.4
	δ_2	0.364	1.35	2.68	4.00	4.95	5.34	5.13	4.48	3.64	2.90	2.44	2.31	
Wilson θ	δ_1	0.006	0.052	0.196	0.490	0.952	1.54	2.16	2.67	2.92	2.82	2.33	1.54	6.9
	δ_2	0.366	1.34	2.64	3.92	4.88	5.31	5.18	4.61	3.82	3.06	2.52	2.29	
Houbolt	δ_1	0	0.031	0.167	0.461	0.923	1.50	2.11	2.60	2.86	2.80	2.40	1.72	9.1
	δ_2	0.392	1.45	2.80	4.08	5.02	5.43	5.31	4.77	4.01	3.24	2.63	2.28	

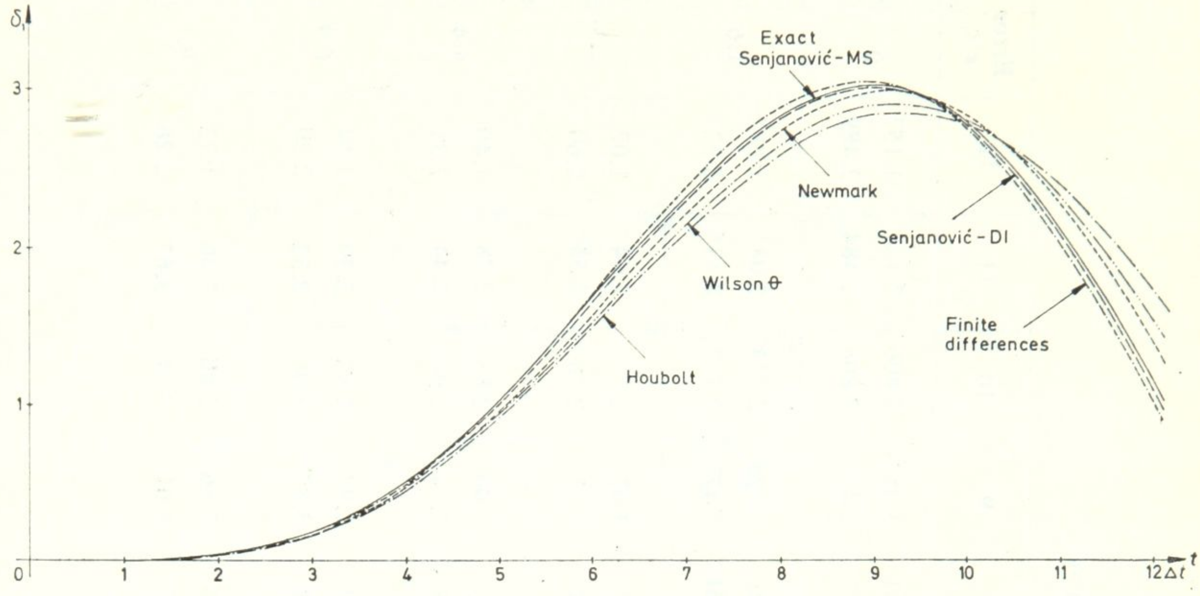

Fig. 3. Response δ_1 of two d. o f. system

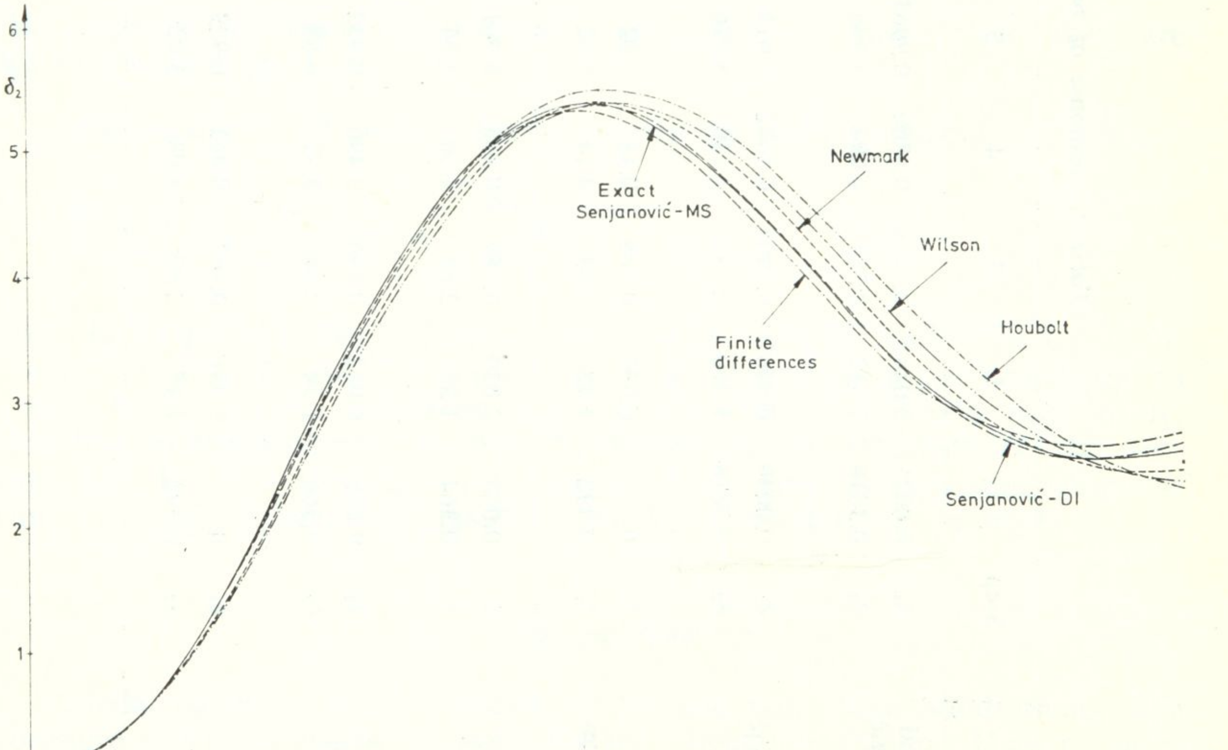

Fig. 4. Response δ_2 of two d. o f. system

6.6. *Correlation analysis*

Response of the two d. o f. system determined by the direct integration and the mode superposition methods assuming the harmonic acceleration is compared to the exact solution in Table 1 as well as in Figs. 3 and 4. The mode superposition method gives the exact solution since in the considered example the constant load and no damping are specified. In the case of direct integration some discrepancies between the approximate and exact solution exist.

In Table 1 and Figs. 3 and 4 the results obtained by the finite difference, the Newmark, the Wilson θ and the Houbolt method are also included, [5]. Each of these methods applied for direct integration and modal integration gives the same results, against the harmonic acceleration method.

In order to evaluate the above methods, the error of the solution may be defined as the mean value of the relative discrepancies between the approximate and exact solution. Thus, for the both d. o f. we can write

$$\varepsilon = \frac{1}{20} \sum_{i=1}^{2} \sum_{j=3}^{12} \left| \frac{\delta_{ij} - \bar{\delta}_{ij}}{\bar{\delta}_{ij}} \right| .$$

The first two time steps are excluded from the analysis since the small response at these steps causes large relative error. The values of ε, given in Table 1, point out different degree of accuracy of the analysed integration methods.

The resulting error of the direct integration method assuming harmonic acceleration, $\varepsilon = 1.6\%$, is somewhat higher than the predicted value $\varepsilon = 1.4\%$, because the error of the load integration is not taken into account in the latter case.

In order to verify the stability of the solutions obtained by the considered methods, which are all unconditionally stable besides the finite differences, the same example is solved taking the large value of time step into account, $\Delta t = 10\, T_2 = 28$. The results, given in Table 2, are discussed as follows.

The mode superposition method gives the exact solution. The results of the direct integration method are inaccurate. However, they are bounded up and down by the maximum and minimum value of the response respectively. The limits are determined according to the analytical solution (j), $-\frac{4}{3} \leqslant \delta_1 \leqslant \frac{10}{3}$ and $0 \leqslant \delta_2 \leqslant 6$. Thus the solution indicates order of the response magnitude.

In the case of the finite differences the results approach to infinity as a consequence of the conditional stability. Namely, the method is unstable if $\Delta t > \frac{T_{\min}}{\Pi}$.

The results of the Newmark method exceed the bounds of the exact solution to some extent and seems to converge to the static solution $\delta_1 = 1$ and $\delta_2 = 3$, which is, of course, within the bounds of the exact dynamic response.

Table 2. Response of two d. o f. system, $\Delta t = 28$.

Method	Step.	1	2	3	4	5	6	7	8	9	10	11	12
Exact and Senjanović MS	δ_1	2.187	2.923	0.136	0.995	2.959	0.152	—0.235	2.313	1.037	—1.035	1.241	1.575
	δ_2	2.236	3.116	0.536	1.732	4.069	2.039	2.406	4.720	3.865	2.155	4.770	5.351
Senjanović DI	δ_1	0.3222	0.5426	1.779	1.958	1.532	0.860	0.276	0.159	0.582	1.126	1.589	1.674
	δ_2	0.9824	1.6440	5.3924	5.919	4.557	2.451	0.703	0.460	1.883	3.623	4.955	5.006
Finite differences	δ_1	0	$3.03\times10^6 \to \pm\infty$										
	δ_2	$3.83\ 10^3$	$—1.21\times10^7 \to \pm\infty$										
Newmark	δ_1	1.99	0.028	1.94	0.112	1.82	0.248	1.67	0.429	1.47	0.648	1.23	0.894
	δ_2	5.99	0.045	5.99	0.177	5.72	0.393	5.47	0.685	5.14	1.04	4.76	1.45
Wilson θ	δ_1	1.09	2.82	—2.61	5.85	—4.47	6.59	—4.38	5.97	—3.46	4.92	—2.39	3.89
	δ_2	1123.	—834.	674.	—519.	406.	—038.	24.2	—181,	144.	—105.	86.1	—60.8
Houbolt	δ_1	2.23	2.98	1.00	1	1	1	1	1	1	1	1	1
	δ_2	2.28	3.17	3.00	3	3	3	3	3	3	3	3	3

The results obtained by the Wilson θ method exceed the bounds of the exact solution considerably.

In the case of the Houbolt method the results approach the static solution extremly rapid.

7. Conclusion

Since the dynamic response of a structure may be expanded into the harmonic series, it seems that the best approximation of the response may be obtained also by the harmonic interpolation function. Concerning the numerical methods for solution of this problem, the assumption of harmonic acceleration in the case of direct integration of the equilibrium equation and its transformation into the set of modal equations, results in two different numerical methods. The both methods are unconditionally stable and very accurate comparing to some other methods, especially the mode superposition version which gives the exact solution in the case of undamped system with a constant load.

Physical meaning of the methods is maintained through all the derivations. That makes possible a proper choice of the integration parameters and checking of the results. This is an advantage of the methods for successful engineering application.

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REFERENCES

[1] J. C. Houbolt, *A reccurence matrix solution for the dynamic response of elastic aircraft,* J. Aeronaut. Sci., Vol. 17, pp 540-550, 1950

[2] N. M. Newmark, *A method of computation for structural dynamics,* Proc. Am. Soc. Civ. Engs., Vol. 85, EM3, pp 67-94, 1959

[3] E. L. Wilson, *A computer program for the dynamic stress analysis of underground structures,* SESM Report 68-1, Department of Civil Engineering, University of California, Berkeley, 1968

[4] K. J. Bathe and E. L. Wilson, *Stability and accuracy analysis of direct integration methods,* Earthquake Eng. and Struct. Dyn., Vol. 1, pp 283-291, 1973

[5] K. J. Bathe, and E. L. Wilson, *Numerical Methods in Finite Element Analysis,* Prentice — Hall, 1976

[6] G. L. Goudrean and R. L. Taylor, *Evaluation of numerical integration methods in elastodynamics,* "Computer methods in applied mechanics and engineering", no 2, pp 69-97, 1972

[7] E. L. Wilson, E. Farhoomand and K. J. Bathe, *Nonlinear dynamic analisis of complex structures,* Earthquake Eng. and Struct. Dyn., Vol. 1, pp 241-252, 1973

[8] T. J. R. Hughes, *Stability, convergence, and decay of energy of the average acceleration method in nonlinear structural dynamics,* Computers & Structures, Vol. 6, pp 313-322, 1976

[9] C. A. Fellipa and K. C. Park, *Computation aspects of time integration procedures in structural dynamics,* ASME Journal of applied mechanics, Vol. 45, pp 595-611, 1978

[10] P. S. Jensen, *Transient analysis of structures by stiffly stable methods,* Computers & Structures, Vol. 4, pp 615-626, 1974

[11] P. S. Jensen, *Stiffly stable methods for undamped second-order equations of motion,* SIAM Journal of Numer. Analysis, Vol. 13, pp 549-563, 1976

[12] I. Senjanović, *Numerical methods for ship vibration analysis by finite elnments,* Det norske Veritas, Report no. 74-55-S, Oslo, 1974

[13] O. C. Zienkiewicz, *The finite element method in engineering science,* McGraw-Hill, London, 1971

[14] E. L. Wilson and J. Penzien, *Evaluation of orthogonal damping matrices,* Int. J. Num. Meth. Eng., Vol. 4, pp 5-10, 1972

[15] R. S. Dunham, R. E. Nickell and D. C. Stickler, *Integration operators for transient structural response,* Computers & Structures, Vol. 2, pp 1-15, 1972

[16] E. L. Wilson and R. W. Clough, *Dynamic rasponse by step-by-step matrix analysis,* Symposium "Computers in Civil Engineering", Lisbon, October, 1962

METHODE DER HARMONISCHEN BESCHLEUNIGUNG ZUR DINAMISCHEN ANALYSE DER KONSTRUKTIONEN

Zusammenfassung

Die harmonische Beschleunigung ist für jeden Schritt beim Integration von Differentialgleichungen des Gleichgewichts, und ihrer Modaltransformation, angenommen. Als Ergebnis bekam man zwei Methoden der nummerischen Integration; die direkte Methode und die Methode der Normalfunktionen. Beide Methoden sind stabil und in Vergleichen mit heutzutage am meisten verwendeten Methoden sehr genau.

METODA HARMONIJSKOG UBRZANJA ZA DINAMIČKU ANALIZU KONSTRUKCIJA

Izvod

Harmonijsko ubrzanje je pretpostavljeno u svakom koraku integracije diferencijalne jednadžbe dinamičke ravnoteže konstrukcije, odnosno njene modalne transformacije. Kao rezultat toga dobivene su dvije metode numeričke integracije, tj. direktna metoda i metoda superpozicije normalnih funkcija. Obe metode su bezuvjetno stabilne i vrlo točne u usporedbi s metodama koje se danas najčešće primjenjuju.

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