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NOTE ON VUJANOVIĆ' HAMILTON-JACOBI METHOD FOR HAMILTON EQUATIONS WITH NONCONSERVATIVE FORCES

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1. Introduction

In a recent paper [1], Vujanović has proposed a new integration procedure for Hamilton's equations in the case of additional nonconservative forces. It is based on the determination of a complete integral of a certian partial differential equation, and a subsequent algebraic procedure for determinig the motion. As such it may be termed an extension of the Hamilton-Jacobi method to nonconservative mechanics. It does not, however, reduce to the usual Hamilton-Jacobi method in case the nonconservative forces disappear, but rather to an alternative procedure, involving a quasi-linear partial differential equation in $2n$ independent variables, as compared to the $n+1$ independent variables of the usual Hamilton-Jacobi equation (n being the number of q's or p's). This enumeration of the number of independent variables already suggests that Vujanovic' fundamental equation may be related to some ordinary Hamilton-Jacobi equation, obtained through a doubling of the phase space variables.

It is the purpose ot this note to explore this relationship which in the first place will bring us to an extensive discussion of the connection between two partial differential equations. Secondly, we present a completion to the proof for the algebraic procedure proposed by Vujanović, and finally the above mentioned relation between two partial differential equations will show us alternative ways for obtaining the motion through algebraic manipulations on a complete integral. In the last section we discuss the possible relevance of this approach and put it in a wider perspective.

2. Vujanović' Method

Consider the following nonconservative dynamical system,

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} + f_i(t, q, p) \qquad i = 1, \ldots, n. \tag{1}$$

For any set of variables X (or functions of these variables) we will use a tilde to denote the same set of variables (or functions) minus one component (usually the first one). For example, if $x = (x_1, \ldots, x_m)$, then $\tilde{x} = (x_2, \ldots, x_m)$. In usig the summation convention over repeated indices, it will likewise be understood that a tilde on the variables indicates that the summation is over one less variable.

Now consider the following partial differential equation,

$$\frac{\partial \o}{\partial t}+\frac{\partial \o}{\partial q_1}\frac{\partial H}{\partial \o}+\frac{\partial \o}{\partial \tilde{q}_i}\frac{\partial H}{\partial \tilde{p}_i}+\frac{\partial \o}{\partial \tilde{p}_i}\left(-\frac{\partial H}{\partial \tilde{q}_i}+\tilde{f}_i\right)+\frac{\partial H}{\partial q_1}-f_1=0, \tag{2}$$

where H and f_i are considered as functions of t, q_i $\o$, $\tilde{p}_i$ (i. e. $\o$ replaces p_1). Let

$$\o = \Phi(t, q, \tilde{p}, c_1, \ldots, c_{2n}) \tag{3}$$

be a complete integral of this quasi-linear equation in $2n$ independent variables $(t, q, \tilde{p})$, depending on $2n$ arbitrary constants c_i. Let further the initial conditions of (1) be specified as,

$$q_i(t_0) = \alpha_i, \qquad p_i(t_0) = \beta_i, \tag{4}$$

and assume that the relation

$$\beta_i = \Phi(t_0, \alpha, \tilde{\beta}, c_1, \ldots, c_{2n}) \tag{5}$$

can be solved for c_1, yielding

$$c_1 = C_1(t_0, \alpha, \beta, \tilde{c}). \tag{6}$$

Finally, consider the relation

$$p_1 = \bar{\Phi}(t, q, \tilde{p}, t_0, \alpha, \beta, \tilde{c}), \tag{7}$$

where the function $\bar{\Phi}$ of the indicated variables is defined by,

$$\bar{\Phi} = \Phi(t, q, \tilde{p}, C_1(t_0\,\alpha, \beta, \tilde{c}), \tilde{c}). \tag{8}$$

Theorem (Vujanović)

Assuming $\det\left(\frac{\partial^2 \bar{\Phi}}{\partial \tilde{c}\, \partial \tilde{x}}\right) \neq 0$, where $\tilde{x} = (q, \tilde{p})$, the solution of equations (1) and (4) can be obtained from the $2n - 1$ algebraic relations

$$\frac{\partial \bar{\Phi}}{\partial \tilde{c}_i} = 0, \qquad i = 2, \ldots, 2n, \tag{9}$$

together with the relation (7) for p_1.

The algebraic manipulations following the determination of the complete solution Φ, as described above, may seem a bit far-fetched, especially because of the introduction of the initial values (α, β) in relations which already contain $2n$ arbitraty constants. In trying to explain what is behind these manipulations, we will attain to a completion of the proof of the above theorem. In addition, we will obtain an alternative procedure, which does not require introducing the constants α and β. There will remain, however, a number of redundant constants. This, together with the fact that (2) is a partial differential equation in $2n$ independent variables for a system with n degrees of freedom, is an indication that there should be some doubling of variables involved in the above procedure.

3. Canonization by Doubling of Coordinates and the Related Hamilton-Jacobi Equation

Every system of first-order differential equations can be turned into a set of Hamilton equations by doubling the number of variables, a procedure which is for instance well known in control theory. For our given system (1), we simply consider $x = (q, p)$ as coordinate-variables, introduce corresponding momentum-variables $X = (Q, P)$, and consider the Hamiltonian $\mathcal{H}(x, X, t)$ defined by

$$\mathcal{H} = Q_i \frac{\partial H}{\partial p_i} + P_i\left(-\frac{\partial H}{\partial q_i} + f_i\right). \tag{10}$$

The Hamilton equations derived from (10) then contain (1) as a sub-system, while the additional $2n$ equations are of no practical interest.

It is conceivable to write down the Hamilton-Jacobi equation for (10), which reads

$$\frac{\partial S}{\partial t} + \frac{\partial S}{\partial q_i}\frac{\partial H}{\partial p_i} + \frac{\partial S}{\partial p_i}\left(-\frac{\partial H}{\partial q_i} + f_i\right) = 0. \tag{11}$$

Assuming we would know a complete integral

$$S = S(t, q, p, \lambda_1, \ldots, \lambda_{2n})$$

for (11), we could apply the Jacobi theorem and conclude that the relations

$$\frac{\partial S}{\partial \lambda_i} = \mu_i \qquad i = 1, \ldots, 2n \tag{12}$$

would provide us with a complete set of first integrals for the given system, from which the solution could be determined. Of course, there are some redundant constants involved in (12), which is due to the fact that by joining to (12) the remaining transformation formulae $X_i = \partial S/\partial x_i$, we could at the same time solve the $2n$ artificially introduced supplementary equations. In other words, the $4n$ constants (λ, μ) could be interpreted as functions of the arbitrary initial values of both the x_i and X_j. A more serious drawback of such a procedure, however, is of practical nature. Indeed, (11) is nothing else but the partial differential equation for first integrals of (1). So it is hard to believe that some miraculous ad hoc procedure could deliver us first an S, from which the relations (12) then would lead to $2n$ first integrals. Instead, it is more realistic to think ... that we first should have to find $2n$ first integrals $F_i(t, q, p)$, after which an S could be constructed as $S = \lambda_i F_i$.

The conclusion therefore must be that, the way it stands, it does not make sense to proclaim this doubling of variables procedure as a Hamilton--Jacobi method for systems with nonconservative forces. Nevertheless, there may be ways to transform equation (11) in such a manner that determining a complete integral of the new equation is no longer directly and explicitly related to finding first integrals. One such way goes as follows : first interchange the role of coordinates and momenta in the doubled phase space, and then apply the usual Hamilton-Jacobi theory to this new situation, where the Hamiltonian now will be linear in the "coordinates".

The generating function which perfoms the first step for the Hamiltonian system (10) is the type 1 function $F_1 = - x_i y_i$, with transformation formulae,

$$X_i = \partial F_1 / \partial x_i = - y_i, \qquad Y_i = - \partial F_1 / \partial y_i = x_i,$$

where the $2n$ variables $y = (u, v)$ denote the new "coordinates", and the $2n$ $Y = (U, V)$ stand for the new "momenta". The new Hamiltonian becomes,

$$K(y, Y, t) = - u_i \frac{\partial H}{\partial p_i}(t, U, V) - v_i \left(- \frac{\partial H}{\partial q_i} + f_i \right)(t, U, V).$$

The momenta (U, V) now appear nonlinearly, so the Hamilton-Jacobi equation for K has a completely different structure. It reads,

$$\frac{\partial S}{\partial t} - u_i \frac{\partial H}{\partial p_i}\left(t, \frac{\partial S}{\partial u}, \frac{\partial S}{\partial v} \right) - v_i \left(- \frac{\partial H}{\partial q_i} + f_i \right)\left(t, \frac{\partial S}{\partial u}, \frac{\partial S}{\partial v} \right) = 0.$$

Once a complete integral of this equation is known, a straightforward application of the usual Jacobi theorem will (among other things) lead to the solutions $x(t)$ of the original equations. This procedure is exactly the one proposed by Arzhanikh [2].

It is our intention here to show that also Vujanovic' method can be understood as providing a different way of transforming (11) to a new situation, in which the complete integral no longer has to be a first integral itself.

To that end, let us first discuss the relationship between equations (2) and (11). Eq. (2) is a partial differential equation in which the coefficients depend on the unknown ø. The usual approach which is found in textbooks for determining a complete solution of such an equation (see e.g. [3] p. 106 or [4] p. 32), is to reduce it first to another equation, depending on one more independent variable, but no longer involving the unknown function. If this standard procedure is applied to (2), one precisely ends up with equation (11). But there is not really a one-to-one correspondence between the solutions of both equations, so let us explain the precise relationship in greater detail than it is usually done.

Let $S(t, x)$ be any solution of (11). Solving the relation $S(t, x) = 0$ for p_1, say $p_1 = \Phi(t, \tilde{x})$, it is straightforward to check, by using the identity $S(t, \Phi(t, \tilde{x}), \tilde{x}) \equiv 0$, that $\Phi(t, \tilde{x})$ will be a solution of (2). The converse, however, is slightly different. Let $ø \equiv \Phi(t, \tilde{x})$ be a solution of (2), and let

$$S(t, ø, \tilde{x}) = 0 \tag{13}$$

be any relation which defines this solution implicitly, i. e. we have

$$S(t, \Phi(t, \tilde{x}), \tilde{x}) \equiv 0, \tag{14}$$

then it is not exactly true that S as a function of its $2n + 1$ variables will satisfy (11) (p_1 being identified with ø). What is true, is that S will satisfy (11) along the constraint

$$p_1 = \Phi(t, \tilde{x}),$$

i. e. we will have

$$\left[\frac{\partial S}{\partial t}+\frac{\partial S}{\partial q_i}\frac{\partial H}{\partial p_i}+\frac{\partial S}{\partial p_i}\left(-\frac{\partial H}{\partial q_i}+f_i\right)\right]\Bigg|_{p_1=\Phi(t,\widetilde{x})}=0. \tag{15}$$

This subtle difference is of importance here, because it is seemingly the obstacle which prevents us to relate the algebraic procedure contained in Vujanovic, method directly to the formal Hamilton-Jacobi method (11), (12) in a doubled space.

4. Methods for the Determination of the Solution from a Complete Integral of (2)

4.1. *Vujanović' procedure*

Going back to section 2, we consider again a complete solution (3) of (2), and the related function $\Phi(t, q, \widetilde{p}, t_0, \alpha, \beta, \widetilde{c})$, as defined through (5), (6) and (8). As a result of this construction of Φ, it is obvious that we have the identity

$$\overline{\Phi}(t_0, \alpha, \widetilde{\beta}, t_0, \alpha, \beta, \widetilde{c}) \equiv \beta_1. \tag{16}$$

Now consider the equations

$$\frac{\partial\overline{\Phi}}{\partial\widetilde{c}}=0. \tag{17}$$

Solving them for $\widetilde{x}$,

$$\widetilde{x}=\widetilde{\chi}(t, t_0, \alpha, \beta, \widetilde{c}), \tag{18}$$

and substituting these time-functions into the relation $p_1=\overline{\Phi}$, we refer to [1] for a simple proof that the resulting expressions indeed satisfy the given system (1). Now in addition, from the identity (16) we know that,

$$\frac{\partial\overline{\Phi}}{\partial\widetilde{c}}\Bigg|_{\substack{t=t_0\\ q=\alpha,\ \widetilde{p}=\widetilde{\beta}}}\equiv 0. \tag{19}$$

These identities, through the use of the implicit function theorem, allow us to conclude that the obtained solutions for q and p actually verify the initial conditions (4). As a matter of fact, we have by definition of the relations (18) that

$$\frac{\partial\overline{\Phi}}{\partial \overline{c}}(t, \widetilde{\chi}(t, t_0, \alpha, \beta, \widetilde{c}), t_0, \alpha, \beta, \widetilde{c}) \equiv 0.$$

Expressing these identities for $t=t_0$, and comparing with (19), the result follows. Hence, Vujanovic' procedure seems specifically designed to produce the solutions immediately in the desired form, i. e. having the prescribed initial values (which is not the case in the algebraic procedure subsequent to solving the usual Hamilton-Jacobi equation).

4.2. *An alternative method*

We would like to come now to a procedure which more closely connects to the formal Hamilton-Jacobi method in doubled space (11) and (12). For that purpose we have to overcome the problem explained with (15) and thus try to arrive at an "unconstrained" solution of (11). This becomes possible if one considers families of solutions instead of single solutions (see [3] p. 31). In the present situation we can proceed as follows. We first solve the relation

$$\Phi(t, q, \tilde{p}, c_1, \ldots, c_{2n}) = p_1 \tag{20}$$

with respect to c_1 (not as in (6), however, i.e. without introducing first the initial values (4)),

$$c_1 = C_1(t, q, p, \tilde{c}). \tag{21}$$

Then we define S by

$$S(t, q, p, c) = c_1 - C_1(t, q, p, \tilde{c}). \tag{22}$$

Obviously this S fulfils the requirement that the relation $S = 0$ implicitly defines (20), and therefore (15) will be identically satisfied, and will reduce to,

$$\left[\frac{\partial C_1}{\partial t} + \frac{\partial C_1}{\partial q_i}\frac{\partial H}{\partial p_i} + \frac{\partial C_1}{\partial p_i}\left(-\frac{\partial H}{\partial q_i} + f_i\right)\right]\Bigg|_{p_1 = \Phi(t, q, \tilde{p}, c)} \equiv 0. \tag{23}$$

Since this is an identity for all $t, q, \tilde{p}, c$, it remains true if we replace c_1 by the function (21). But then

$$\Phi(t, q, \tilde{p}, C_1(t, q, p, \tilde{c}), \tilde{c}) \equiv p_1, \tag{24}$$

so that (23) simply becomes,

$$\frac{\partial C_1}{\partial t}(t, q, p, \tilde{c}) + \frac{\partial C_1}{\partial q_i}\frac{\partial H}{\partial p_i} + \frac{\partial C_1}{\partial p_i}\left(-\frac{\partial H}{\partial q_i} + f_i\right) \equiv 0, \tag{25}$$

expressing the fact that C_1 is an *incomplete* solution of the Hamilton-Jacobi equation (11); incomplete because it only depends on $2n - 1$ arbitrary constants (instead of $2n$). Nevertheless it is known that also for incomplete solutions of the Hamilton-Jacobi equation the relations (12) will yield first integrals (see e.g. Gelfand and Fomin [5]). So here we have,

$$\frac{\partial C_1}{\partial \tilde{c}_i}(t, q, p, \tilde{c}) = \tilde{\mu}_i \qquad i = 2, \ldots, 2n. \tag{26}$$

Actually, this observation in our case is quite trivial since the Hamilton-Jacobi equation (11) is just the partial differential equation for first integrals of (1). For the same reason, C_1 itself will constitute the remaining first integral, so that the relations (26), together with (21) or (20) will allow the determination of the motion by purely algebraic manipulations.

Now exactly as in the procedure of Section 4.1, we see that there are $4n-1$ constants involved in (26) and (21), so that $2n-1$ of them can be considered as redundant. Clearly, the most natural choice to make is to let the $\tilde{c}$ play the role of redundant constants. The $\tilde{\mu}$ together with c_1 then are considered as $2n$ independent constants, which makes it easy to identify which Jacobian determinant must be assumed to be non-zero, in order that (26) and (21) can be solved for q and p.

Note that the whole reasoning obviously could have been started with respect to any of the constants $c_1, \ldots, c_{2n}$ (instead of the specific c_1 we singled out), so that there is some freedom left in case the conditions for applying the implicit function theorem would not be met for a specifically chosen c_i.

Note also that the above analysis even exhibits another alternative procedure. Indeed, we have shown : if Φ is a complete solution of (2), then solving the relation (20) with respect to any of the constants c_i will immediately produce a first integral of the system (in which we then can even put the other constants equal to zero). It looks, however, like this way of determining $2n$ constants of the motion will provide less elegant relations for constructing the solution in practical applications, as will be seen already on the example below.

5. Illustrative Example

We content ourselves to illustrating the ideas developed before on one of the simplest possible examples, namely the harmonic oscillator,

$$\dot{q} = p, \quad \dot{p} = -\omega^2 q,$$

for which the partial differential equation (2) becomes,

$$\frac{\partial \varnothing}{\partial t} + \varnothing \frac{\partial \varnothing}{\partial q} + \omega^2 q = 0.$$

Vujanovic [1] obtained for this equation the complete solution

$$\Phi(t, q, c_1, c_2) = -\omega\, q tg(\omega t + c_1) + \frac{c_2}{\cos(\omega t + c_1)} \tag{27}$$

We refer to [1] for the determination of the solution by Vujanovic' method. Here we look at the alternative method of section 4.2. Solving $\Phi = p$ for c_2 we get,

$$c_2 = p\cos(\omega t + c_1) + \omega q \sin(\omega t + c_1), \tag{28}$$

and

$$\frac{\partial C_2}{\partial c_1} = -p\sin(\omega t + c_1) + \omega q \cos(\omega t + c_1) = \mu_1. \tag{29}$$

In accordance with (26) and (21), Eqs. (29) and (28) provide us with two first integrals, from which the solution easily can be obtained. The constant c_1 thereby is redundant and can as well be set equal to zero.

As an illustration of the remark at the end of the previous section, note that by solving $\Phi = p$ for c_1 we indeed find another constant of the motion, namely (choosing $c_2 = 0$),

$$c_1 = -\omega t + Arctg(-p/\omega q).$$

6. Discussion

It has been shown that the integration procedure for nonconservative systems as proposed by Vujanovic is related to a formal Hamilton-Jacobi equation in a doubled phase space. Vujanovic' fundamental equation appears to be obtained from that equation by taking a step which is exactly opposite to the one which is usually recommended in textbooks on partial differential equations. This could precisely be the relevance of Vujanovic' approach. To see this, let us recall the usual philosophy in the theory of partial differential equations, the way it is e.g. very well summarized in ref. [3] (pp. 106-107).

Generally speaking, it is considered to be a simplification if one can reduce solving a partial differential equation to solving a system of ordinary differential equations. For a general partial differential equation of the first order, such a reduction can e.g. be achieved by essentially two steps. If the equation explicitly contains the unknown function, the first step consists in passing to an equation with one more independent variable, but no longer containing the new unknown function (see before). Solving the resulting equation for the derivative with respect to one specific variable, we then get a partial differential equation of the Hamilton-Jacobi type. The second step then consists in reducing this last problem to the problem of solving the characteric equations, which are ordinary differential equations of Hamilton-type.

Now it is generally considered as Jacobi's merit that he recognized the fact that the second step may be reversed, i.e. that for given Hamilton equations, it may be easier to find first by some ad hoc procedure a complete integral of the related Hamilton-Jacobi equation. So it is not unthinkable that for certain applications, it may have merit to reverse the first step too. For conservative systems, this may seem far fetched, although it certainly can work, as illustrated by the harmonic oscillator example. For nonconservative systems, however, the idea gains importance. Indeed, if we first canonize the system by doubling the variables, "reversing only the second step" does not make sense for reasons explained in Section 3. This means that "reversing also the first step" may then be essential for arriving at a partial differential equation, for which there are reasonable chances to find a complete integral by some ad hoc procedure. Alternatively, one may of course use another device to transform the partial differential equation (11), such as the one inherent in Arzhanikh's method (see Section 3).

Let us finaly place the methods discussed in this note in a somewhat broader context, and show that we are in fact talking about a quite general procedure for generating first integrals of ordinary differential equations. Clearly, the peculiar form of the equations of motion (1) does not play any role. So consider a general system of first-order ordinary differential equations

$$\dot{x}_i = f_i(t,x) \qquad i = 1, \ldots, n. \tag{30}$$

First integrals of (30) have to satisfy

$$\frac{\partial F}{\partial t} + \frac{\partial F}{\partial x^i} f_i(t,x) = 0. \tag{31}$$

Now select one of the variables, say x_1, and replace the linear partial differential equation (31) by a quasi-linear as before:

$$F(t, x_1, \tilde{x}) = 0 \xleftrightarrow{\text{solve for } x_1} x_1 = \Phi(t, \tilde{x})$$

$$\Rightarrow \frac{\partial \Phi}{\partial t} + \frac{\partial \Phi}{\partial \tilde{x}_i} \tilde{f}_i - f_1(t, \Phi, \tilde{x}) = 0. \quad (32)$$

Then, as a general procedure, one can search for solutions of (32) which depend on some arbitrary constants c_i $(i = 1, \ldots, m \leqslant n)$. For each dependence on a constant, we find a first integral by solving the relation $x_1 = \Phi$ for that constant. If moreover the resulting expression still depends on other constants, we get further first integrals by computing partial derivatives with respect to these constants. The Hamilton-Jacobi approach discussed before then corresponds to the case that $m = n$, so that a complete set of first integrals becomes available, from which the solution can be determined.

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REMARQUE CONCERNANT LA METHODE HAMILTON- JACOBI DE VUJANOVIĆ POUR LES EQUATIONS HAMILTONIENNES AUX FORCES NON-CONSERVATIVES

Résumé

La généralisation de la théorie de Hamilton-Jacobi, comme présentée récemment par Vujanović est liée ici à une équation formelle du type de Hamilton-Jacobi, qui se situe dans un espace de phase à double dimension. On discute une manière alternative pour obtenir le mouvement de façon algébrique, après la détermination d'une intégrale complète. Finalement la théorie est posée dans un cadre plus large, en l'interprétant comme une procédure générale pour le calcul d'intégrales premières d'un système d'équations différentielles ordinaires.

O VUJANOVIĆEVOM METODU HAMILTON-JAKOBIJA ZA HAMILTONOVE JEDNAČINE SA NEKONZERVATIVNIM SILAMA

Izvod

U radu se razmatra Vujanovićeva generalizacija teorije Hamilton-Jakobija, za nekonzervativne sisteme, u udvojenom faznom prostoru. Diskutuje se jedan alternativan način dobijanja kretanja pomoću algebarskih operacija nad kompletnim integralom odgovarajuće parcijalne diferencijalne jednačine. Na kraju se ova teorija postavlja u nešto šire okvire. Diskutuje se potpuno opšti postupak formiranja prvih integrala sistema običnih diferencijalnih jednačina prvog reda.

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