TEORIJSKA I PRIMENJENA MEHANIKA 8, 95—102, 1982, UDK 531.131.2

ELASTIC-PLASTIC DEFORMATION WITH PLASTIC ANISOTROPY

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(Received December 4, 1980)

1. Introduction

By plastic anisotropy we mean the anisotropy of hardening. A general anisotropic elastic-plastic deformation would include both, elastic and plastic anisotropy. In this analysis oe restrict ourselves to materials which are elastically isotropic in its virgin configuration and which preserve that isotropy when plastically deformed. For many materials this is indeed so [5], because the basic crystal structure, which determines the elastic properties of the material, remains essentially intact after dislocation generation and migration which cause plastic flow. We, however, allow such materils to have the yield surface which expands during the course of deformation in the anisotropic manner (anisotropy of hardening). Elastic-plastic deformation of such materials is then a deformation with elastic isotropy and plastic anisotropy. This author shortly calls such deformation an elastic-plastic deformation with plastic anisotropy.

Most of the work related to anisotropic plastic deformation [6, 7, 8] is concerned with the handling of anisotropy of harding, i.e. with the mathematical representation of the way in which the yield surface expands during the course of plastic deformation. Althought that still presents perhaps the most important open problem in the theory of plasticity, this work is not orieneted toward that difficulty. It rather has as a goal to establish the global structure of the constitutive law for elastic-plastic deformation at arbitrary strains valid for any proposed anisotropic yield function. The result obtained here shows that the final structure of the constitutive law for elastic-plastic deformation with plastic anisotropy is globaly the same as the structure established in [2—4] for isotropic elastic-plastic deformation.

2. Kinematics

Consider the body in its initial (stress free) configuration $\mathcal{B}_0$. Let it be deformed under the action of some external agency into the configuration $\mathcal{B}_t$ such that the motion (deformation) from $\mathcal{B}_0$ to $\mathcal{B}_t$ is given by a single valued mapping

$$\underset{\sim}{x} = \underset{\sim}{\chi}(\underset{\sim}{X}, t) \tag{2.1}$$

which carries the material particle from its initial position $\underset{\sim}{X}$ into its current position $\underset{\sim}{x}$ at time t (Fig. 2.1).

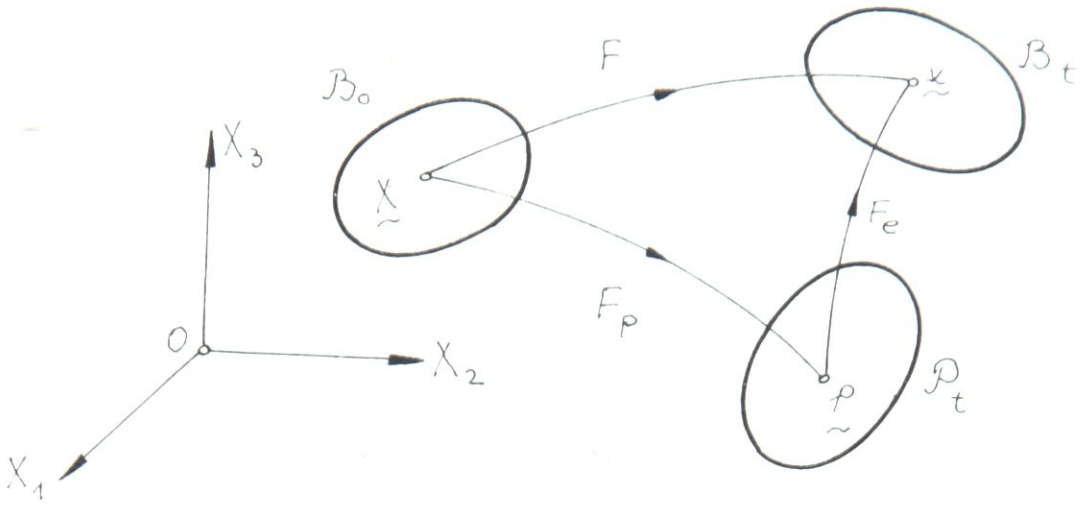

Fig. 2.1

Let the motion χ involve elastic-plastic deformation. Observations have shown such materials to be „simple" in the continuum sense, i.e. the deformation will be present in the constitutive equations only through the deformation gradient matrix

$$F = \frac{\partial x}{\partial X} \tag{2.2}$$

Following [1], we now introduce the intermidiate configuration $\mathcal{S}_t$ by destressing the whole body from its current configuration $\mathcal{B}_t$ and by reducing the temperature to the initial value. The configuration $\mathcal{S}_t$ then compises pure plastic deformation, for thermal expansion and elastic strain components are both zero. One can then establish at each point of the deformed body the decomposition

$$F = F_e F_p \tag{2.3}$$

of the deformation gradient F into the pure (thermo-) elastic part F_e which corresponds to mapping from $\mathcal{S}_t$ to $\mathcal{B}_t$ and pure plastic part F_p which corresponds to mapping from $\mathcal{B}_0$ to $\mathcal{B}_t$. The decomposition (2.3) is not unique, for we can always have

$$F = F_e F_p = (F_e Q)(Q^T F_p) = \bar{F}_e \bar{F}_p \tag{2.4}$$

for an arbitrary orthogonal Q. However, for elastically isotropic bodies, the above choice of F_e is immaterial and we can choose that F_e which is the most convinient for us. We therefore choose F_e corresponding to destressing without rotation, i.e. $R_e = 1$ and

$$F_e = V_e \tag{2.5}$$

V_e being the symmetric left strecth tensor and R_e the orthogonal rotation tensor from the polar decomposition theorem $F_e = V_e R_e$. The decomposition (2.3) consequently becomes

$$F = V_e F_p \tag{2.6}$$

Consider now the particle velocity in the state x

$$v = \frac{\partial x}{\partial t}\Big|_X \tag{2.7}$$

The velocity gradient of the total deformation is then

$$L = \frac{\partial \underset{\sim}{v}}{\partial \underset{\sim}{x}} = \frac{\partial \underset{\sim}{v}}{\partial \underset{\sim}{X}} \frac{\partial \underset{\sim}{X}}{\partial \underset{\sim}{x}} = \dot{F} F^{-1} \tag{2.8}$$

where the superposed dot denotes material derivative, time diferentiation at fixed $\underset{\sim}{X}$. Substitution from (2.6) then gives

$$L = L_e + V_e L_p V_e^{-1} \tag{2.9}$$

where $L_e = \dot{V}_e V_e^{-1}$ and $L_p = \dot{F}_p F_p^{-1}$ are the velocity gradients corresponding to the elastic and plastic part of deformation, respectively. Taking symmetric part of (2.9), we obtain

$$D = D_e + (V_e L_p V_e^{-1})_s \tag{2.10}$$

where D is the symmetric part of $L = D + W$ called velocity strain, D_e is the symmetric part of L_e and subscript s denotes the symmetric part. Further $L_p = D_p + W_p$, D_p and W_p being the symmetric and antisymetric parts of L_p, and (2.10) becomes

$$D = D_e + (V_e D_p V_e^{-1})_s + (V_e W_p V_e^{-1})_s \tag{2.11}$$

Now, the first and third term on right hand side of (2.11) can be combined to give

$$D_e + (V_e W_p V_e^{-1})_s = \frac{1}{2} V_e^{-1} \overset{\nabla}{C}_e V_e^{-1} \tag{2.12}$$

where $C_e = V_e^T V_e = V_e^2$ is the right Cauchy-Green deformation tensor, and $(\overset{\nabla}{})$ stands for the Jaumann derivative with respect to the plastic spin W_p, i.e.

$$(\overset{\nabla}{}) = (\dot{}) - W_p (\ ) + (\ ) W_p \tag{2.13}$$

Denoting (2.12) shortly by

$$\mathfrak{D}_e = \frac{1}{2} V_e^{-1} \overset{\nabla}{C}_e V_e^{-1} \tag{2.14}$$

we have from (2.11)

$$D = \mathfrak{D}_e + (V_e D_p V_e^{-1})_s \tag{2.15}$$

In the case of isotropic hardening [2—4], the principal directions of plastic stretching D_p coincide with the principal directions of stress, and since V_e has also the principal directions coincident with those of stress, matrices V_e and D_p in (2.15) are commutative and V_e and V_e^{-1} cancel each other. Hence, for the case of isotropic hardening (i.e. isotropic yield condition), (2.15) reduces to

$$D = \mathfrak{D}_e + D_p \tag{2.16}$$

which decomposes the stretching tensor D into the elastic part $\mathfrak{D}_e$ and plastic part D_p, and which presents the basis for the formulation of the rate-type elastic-plastic constitutive law [2—4].

However, when the yield conditionis not istoropic, i.e. when we allow the existance of the Bauschinger effect and the anisotropy of hardening, the principal directions of stress and plastic stretching are not coincident, hence matrices V_e

and D_p in (2.15) are not parallel. (commutative) and (2.15) doesn't reduce exactly to (2.16). Nevertheless, it turns wut that even in the case of anisotropy of hardening, the relation (2.16) is an appropraite kinematical relation to base the theory on. Indeed, for elastic-plastic deformation of metals, the finite elastic strains are predominantly dilatational, since an increase of elastic shear-strain components beyond the elastic limit ($\sim 10^{-3}$) produces plastic flow. Consequently, the stretch tensor V_e can be written as

$$V_e = v\,(I + \varepsilon) \tag{2.17}$$

where $v > 1$ is a scalar (volume compresion ratio), I is the identity matrix, and ε is a „small" matrix ($|\varepsilon_{ij}| \ll 1$) due to elastic shearing. It then follows that

$$V_e^{-1} \cong \frac{1}{v}\,(I - \varepsilon) \tag{2.18}$$

and

$$(V_e\, D_p\, V_e^{-1})_s \cong D_p - \varepsilon\, D_p\, \varepsilon \tag{2.19}$$

But the second term, on the right hand side of (2.19) is two order of magnitude smaller then the first term, and therefore

$$(V_e\, D_p\, V_e^{-1})_s \cong D_p \tag{2.20}$$

i.e.

$$D \cong \mathfrak{D}_e + D_p \tag{2.21}$$

In the case of infinitesimal elatic part of deformation, we can imidiatelly put $V_e = I$ in (2.15) to again get (2.21), without danger of loosing any important feature of the analysis, since all differentiation has been already done and rate measures earefully established. We also note that order of approximation involved in substituting $V_e = I$ in (2.15) for infinitesimal elastic deformation is of the same order as in the approximation $V_e = I$ normally made in the structure of the constitutive law for $\mathfrak{D}_e$ corresponding to infinitesimal elasticity.

Therefore, even with plastic anisotropy, an accurate theory can be based on the decomposition

$$D = \mathfrak{D}_e + D_p \tag{2.22}$$

of the total velocity strain D into the elastic part $\mathfrak{D}_e$ and plastic part D_p.[1)]

Our objective is now to use (2.22) and to establish the structure of the constitutive law for materials in elastic-plastic deformation with plastic anisotropy.

[1)] Alternatively, one can establish the exact relation

$$D_{ij} = \mathfrak{D}_{ij}^e + \mathfrak{L}_{ijke}^e\, D_{ke}^p$$

with

$$\mathfrak{L}_{ijke} = \frac{1}{2}\,(V_{ik}\, V_{ej}^{-1} + V_{ik}^{-1}\, V_{ej})$$

but there is no need in keeping the tensor $\mathfrak{L}_e$, since

$$\mathfrak{L}_{ijke} \cong \delta_{ik}\, \delta_{ej}$$

and hence

$$D \cong \mathfrak{D}_e + D_p.$$

3. Constitutive Laws

The elastic deformation F_e is governed by the classical finite elasticity law [9]

$$T = 2\frac{\rho}{\rho_0} F_e \frac{\partial \Psi_e}{\partial C_e} F^T$$

where T is the Cauchy stress tensor, $\Psi_e = \Psi_e(C_e\ \theta)$ is the Helmholtz free energy per unit initial volume, $C_e = F_e^T F_e$ is the right Cauchy — Green deformation tensor, θ is temperature, and ρ and ρ_0 are the densities in configurations $\mathcal{B}_t$ and $\mathcal{S}_t$, respectively. (We assume the incompressibility of plastic flow, hence the density in configuration $\mathcal{S}_t$ is the same as the initial density ρ_0 in the configuration $\mathcal{B}_0$. We also assume [8] that the elastic properties of material are not influenced by the previous plastic flow). But we have chosen $F_e = V_e$ and (3.1) becomes

$$\tau = 2\, C_e \frac{\partial \Psi_e}{\partial C_e} \tag{3.2}$$

where $\tau = \frac{\rho_0}{\rho} T$ is the Kirchhoff stress and Ψ_e is an isotropic function of $C_e = V_e^2$. The law (3.2) is the constitutive law for the elastic part of deformation. It is seen to be in the finite form as one-to-one relation between the deformation C_e and the stress τ.

The structure of the constitutive law for the plastic part of deformation is quite different. Plasticity is a fluid type phenomenom which is governed by a rate (incremental, flow) type relation which involves the strain rate rather then strain in its structure. Restricting ourselves to the case of time-independent plasticity (i.e. no rheological, viscous effects), the law governing the plastic part of (isothermal) deformation with arbitrary hardening, takes the form [10,11]

$$D_p = \frac{1}{\tilde{\mathcal{F}}} \left(\frac{\partial \tilde{f}}{\partial \tau} : \overset{\circ}{\tau} \right) \frac{\partial \tilde{f}}{\partial \tau} \tag{3.3}$$

where $\tilde{\mathcal{F}}$ is a scalar which contains information about the history of deformation, and

$$f = \tilde{f}\ (\mathcal{R}_p^T\, \tau\, \mathcal{R}_p,\ C_p,\ A) \tag{3.4}$$

is the yield function, A being the set of internal state variables [12,13], C_p the plastic deformation tensor and $\mathcal{R}_p$ the orthogonal rotation tensor, such that

$$\dot{\mathcal{R}}_p\, \mathcal{R}_p^{-1} = W \tag{3.5}$$

where $W = \operatorname{asym} L$ is the spin tensor.

The law (3.3) can be rewritten as

$$D_p = \Lambda_p\, [\overset{\circ}{\tau}] \tag{3.6}$$

i.e.

$$D_{ij}^p = \Lambda_{ijmn}^p\, \overset{\circ}{\tau}_{mn} \tag{3.7}$$

with

$$\Lambda_{ijmn}^p = \frac{1}{\mathcal{F}} \frac{\partial \tilde{f}}{\partial \tau_{ij}} \frac{\partial \tilde{f}}{\partial \tau_{mn}} \tag{3.8}$$

4. Rate-Type Constitutive Law for Elastic-Plastic Deformation with Plastic Anisotropy

The established kinematic relation (2.22) for the rate measures of the deformation will be the basis for the assembling of elastic and plastic constitutive laws, (3.2) and (3.6), respectively, into a single law. By taking the Jaumann derivative [14] of (3.2) with respect to the total spin W

$$(^{o}) = (\dot{\ }) - W(\ ) + (\ )W \tag{4.1}$$

we obtain (for details, see [2—4])

$$\overset{\circ}{\tau} = \Pi_e [\mathcal{D}_e] \tag{4.2}$$

where (droping the index „e" for the moment)

$$\Pi_{ijmn} = \varphi_{ij\,\alpha\beta}\, \Sigma_{\alpha\beta\,mn} \tag{4.3}$$

with

$$\varphi_{ij\alpha\beta} = 2\left[\delta_{i\alpha}\frac{\partial \Psi}{\partial C_{\beta j}} + C_{ik}\left(\frac{\partial^2 \Psi}{\partial C^2}\right)_{kj\alpha\beta}\right] \tag{4.4}$$

and

$$\Sigma_{\alpha\beta mn} = \delta_{\alpha m}\, C_{n\beta} + \delta_{\beta n}\, C_{\alpha m} \tag{4.5}$$

Inverting (4.2) for $\mathcal{D}_e$, we obtain

$$\mathcal{D}_e = \Lambda_e [\overset{\circ}{\tau}] \tag{4.6}$$

which is the desired form of the rate-type constitutive law for the elastic part of deformation.

Now, we substitute the laws (3.6) and (4.6) into (2.22) to get

$$D = (\Lambda_e + \Lambda_p)[\overset{\circ}{\tau}] \tag{4.7}$$

i.e.

$$D = \Lambda [\overset{\circ}{\tau}] \tag{4.8}$$

This is the rate-type law for the materials under the conditions of elastic-plastic deformation with plastic anisotropy. It gives the velocity strain D as a function of the stress rate $\overset{\circ}{\tau}$ and the tensor (operator) Λ which is a function f the current state (i.e. stress and other quantities which define the state). Inverting (4.8) for $\overset{\circ}{\tau}$, we obtain the equivalent form of the rate-type constitutive law

$$\overset{\circ}{\tau} = \mathfrak{L}[D] \tag{4.9}$$

We observe that the laws (4.8) and (4.9) have the same global structure as the laws established in [2—4] for the elastic-plastic deformation with plastic isotropy. The difference is, of cource, in the different nature of the yield function corredponding to plastic anisotropy and plastic isotropy, as seen by comparing the equation (3.4) here, with the corresponding yield function of isotropic hardening utilised in [2—4].

The constitutive laws (4.8) and (4.9) for the elastic-plastic deformation with plastic anisotropy are in this form explicitely established for the first time in this paper. They are not restricted to small strain or small rotations, as is usually the case with the other theories of plasticity, and the only major restrictions are the elastic isotropy and time independence. Extensions of these (isothermal) laws to

nonisothermal case is a straight forward procedure analogous to one which was performed in [15] for the isotropic elastic-plastic deformation.

In the final part oa this paper we consider an important special case of anistropic hardening and show that the corresponding constitutive law has nature of here established general law (4.8) or (4.9).

5. Kinematic Hardening Case

The simplest case of the anistropic hardening is the case of kinematic hardening [6,8] in which the yield surface does not change its size or shape but merely translates in stress space τ_{ij} in the direction of its normal. If the initial yield surface is defined by

$$f(\tau_{ij}) - C = 0 \tag{5.1}$$

where f is an isotropic function of τ, and c is the radius of the yield surface, then the kinematic hardening assumption means that the new yield surface is given by

$$f(\tau_{ij} - \alpha_{ij}) - C = 0 \tag{5.2}$$

where α_{ij} are the coordinates of the new centre of the yield sufrace, and f in (5.2) has the same dependence on $(\tau_{ij} - \alpha_{ij})$ as f in (5.1) on τ_{ij}. In particular, f in (5.2) is therefore an isotropicfunction of $(\tau_{ij} - \alpha_{ij})$. To specify the position of the center of the yield surface in the cource of deformation we consider α_{ij} as the internal state variables and propose for them the evolution law [10]

$$\overset{\circ}{\alpha}_{ij} = \mathcal{A}^{\alpha}_{ijmn} \, D^{p}_{mn} \tag{5.3}$$

where the tensor $\mathcal{A}^{p}_{ijmn}$ depends on the current state of material. (Almost all application of (5.3) have been based on taking $\mathcal{A}^{\alpha}_{ijmn} = a\delta_{im}\,\delta_{jn}$, where „$a$" is a constant).

Now, the plastic stretching D_p obeys the law

$$D_p = \frac{1}{\mathcal{J}} \left(\frac{\partial f}{\partial \tau} : \dot{\tau} \right) \frac{\partial f}{\partial \tau} \tag{5.4}$$

where

$$\mathcal{J} = \mathcal{A}^{\alpha}_{ijkc} \, \frac{\partial f}{\partial \tau_{ij}} \, \frac{\partial f}{\partial \tau_{ke}} \tag{5.5}$$

as can be easily shown similarly to the procedure in [10]. But, due to the isotropic nature of function f, the material derivative of τ in (5.4) can be replaced by the Jaumann derivative, and (5.4) becomes

$$D_p = \frac{1}{\mathcal{J}} \left(\frac{\partial f}{\partial \tau} : \overset{\circ}{\tau} \right) \frac{\partial f}{\partial \tau} \tag{5.6}$$

or

$$D_p = \Lambda_p \, [\overset{\circ}{\tau}]$$

with

$$\Lambda^{p}_{ijmn} = \frac{1}{\mathcal{J}} \, \frac{\partial f}{\partial \tau_{ij}} \, \frac{\partial f}{\partial \tau_{mn}}$$

Substitution of (4.6) and (5.7) into (2.22) leads to the final form of the constitutive law for elastic-plastic deformation with kinematic hardening:

$$D = \Lambda\,[\overset{\circ}{\tau}] \tag{5.9}$$

$$\overset{\circ}{\tau} = \mathfrak{L}\,[D] \tag{5.10}$$

which is seen to be the same structure as already established by the equations (4.8) and (4.9) for the general anisotropic hardening.

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LA DEFORMATION ELASTO-PLASTIQUE AVEC L'ANISOTROPIE PLASTIQUE

Résumé

Nous analysons ici les matériaux conditions de la déformation élasto-plastique avec l'isotropie élastique et anisotropie plastique. L'anisotropie plastique signifie l'anisotropie de la fonction du coulage. Le but de ce travail est l'établissement de la structure globale la loi constitutive pour les matériaux élasto-plastiques aux conditions de la anisotropie plastique. Le résultat obtenu indique que cette structure ne diffère pas formalement de la structure qui correspond à la déformation élasto-plastique avec l'isotropie plastique [2—4]. Cette conclusion est obtenue ici de manière explicite pour la première fois.

ELASTO-PLASTIČNA DEFORMACIJA SA PLASTIČNOM ANIZOTROPIJOM

Izvod

Pod plastičnom anizotropijom mi podrazumijevamo anizotropiju očvršćavanja. Potpuno anizotropna elasto-plastična deformacija bi bila deformacija i sa elastičnom i sa plastičnom anizotropijom. U ovoj analizi mi se ograničavamo na materijale koji su elastično izotropni u svojoj početnoj konfiguraciji i koji očuvaju tu izotropiju i nakon plastične deformacije. Mi, međutim, dopuštamo da takvi materijali imaju površinu tečenja koja ekspandira u toku deformacije na proizvoljan anizotropan način (anizotropnost očvršćavanja). Elasto-plastična deformacija takvih materijala je onda deformacija sa elastičnom izotropijom i plastičnom anizotropijom. Autor ovog rada naziva kratko takvu deformaciju elasto-plastična deformacija sa plastičnom anizotropijom.

Cilj ovog rada je da se formira globalna struktura konstitutivnog zakona za elasto-plastičan materijal pri konačnim deformacijama u uslovima proizvoljne plastične anizotropije. Dobijeni rezultat pokazuje da se ta struktura formalno ne razlikuje od korespodentne strukture za elasto-plastičnu deformaciju sa plastičnom izotropijom, formirane u prethodnom autorovom radu [2—4]. Ovo se pojavljuje ekspicitno dokazano po prvi put u ovom radu.

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