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# A CONTRIBUTION TO THE CYLINDER THEOREMS FOR INCOMPRESSIBLE VISCOUS FLUIDS

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## 1. Introduction

Let $\vec{V}$ bi the velocity field of a two-dimensional incompressible flow of a slow viscous fluid. In this case the Navier-Stokes equation is simplified and can be written as

$$\mu \Delta u - \frac{\partial p}{\partial x} = 0 \tag{1.1}$$

$$\mu \Delta v - \frac{\partial p}{\partial y} = 0 \tag{1.2}$$

and

$$div \, \vec{V} = 0 \tag{1.3}$$

where $\vec{V} = (u(x,y), \, v(x,y), 0)$, $p = p(x,y)$ and $\mu$ is a constant. Introducing the complex variable $z = x + iy$, we obtain two expressions which are eequivalent to the above equations [1]:

$$i(u + iv) = 2[\varphi(z) + z\overline{\varphi'(z)} + \overline{\psi(z)}] \tag{1.4}$$

$$p = -8\mu \, Im[\varphi'(z)] + \text{const.} \tag{1.5}$$

where $\varphi(z)$ and $\psi(z)$ are two analytical functions which have to be defined for each particular problem. Further, considering still the expression for force $\vec{F}$ on the arc $\widehat{z_1 z_2}$, we get:

$$\int_{z_1}^{z_2} (X + iY) \, ds = 4\mu [\varphi(z) - z\overline{\varphi'(z)} - \overline{\psi(z)}]_{z_1}^{z_2} \tag{1.6}$$

where $\vec{F} = (X(x,y), \, Y(x,y), 0)$ and s is the arc lenght $\widehat{z_1 z_2}$.

Now let $\varphi_0(z)$ and $\psi_0(z)$ be two arbitrary analytical functions having in domain $E$ only point singularities of which noonc should lie on the given curve $C \subset D$.

Designating the domain inside the curve $C$ by $D_C$, we can write the functions $\varphi_0(z)$ and $\psi_0(z)$ as follows:

$$\varphi_0(z) = \varphi_{01}(z) + \varphi_{02}(z) \tag{1.7}$$

$$\psi_0(z) = \psi_{01}(z) + \psi_{02}(z) \tag{1.8}$$

where $\varphi_{01}(z)$ and $\psi_{01}(z)$ are regulator in the domain $D_1 = D - (D_C U C)$, and $\varphi_{02}(z)$ and $\psi_{02}(z)$ in the domacin $D_C$. If domain $D$ represents the entire plane $z$, then for the given functions $\varphi_0(z)$ and $\psi_0(z)$ the summands on the right sides till the additive constant are precisely defined. Now introducing curve $C$ in the initial domain $D$ as an additional boundary, we obtain a changed velocity field. In what way this new velocity field depends on the initial one is explained by the cylinder theorem. Let the new velocity field be defined by the functions $\varphi(z)$ and $\psi(z)$ and let it be

$$\varphi(z) = \varphi_0(z) + \varphi_1(z) \tag{1.9}$$

$$\psi(z) = \psi_0(z) + \psi_1(z) \tag{1.10}$$

where the functions $\varphi_1(z)$ and $\psi_1(z)$ comprise only the local singular behaviour of the fluid flow in the neighbourhood of the new buwndary. If $C$ is a closed curve and the fluid flow autside curve $C$ is wanted, i.e. in Domain $D_1$, then the functions $\varphi_1(z)$ and $\psi_1(z)$ in $D_1$ are regular. We shall restain to the case when domain $D$ is represented by the entire plane $z$. The determination of the new velocity field in domain $D_1$ then corresponds to the following boundary value problem:

$$[\varphi(z) + z\,\overline{\varphi'(z)} + \overline{\psi(z)}]^+ = 0, \qquad z \in C \tag{1.11}$$

while — designates the limit from domain $D_C$ to the boundary point on curve $C$. As a boundary condition the most frequent pequirement about the zero value of the velocity on the boundary is taken.

Considering the equations (1—7), (1—8), (1—9) and (1—10) we then obtain

$$\begin{aligned}\{[\varphi_{01}(z) + \varphi_{02}(z) + \varphi_1(z))] + z\,[\overline{\varphi_{01}'(z)} + \overline{\varphi_{02}'(z)} + \overline{\varphi_1'(z)}] + \\ + [\overline{\psi_{01}(z)} + \overline{\psi_{02}(z)} + \overline{\psi_1(z)}]\}^+ = 0, z \in C\end{aligned} \tag{1.12}$$

With regard to the agreed properties of the present functions, the above equation can be written still as

$$\begin{aligned}[\varphi_1(z) + z\,\overline{\varphi_1'(z)} + \overline{\psi_1(z)}]^+ = \{-[\varphi_{01}(z) + z\,\overline{\varphi_{01}'(z)} + \overline{\psi_{01}(z)}]^+ - \\ - [\varphi_{02}(z) + z\,\overline{\varphi_{02}'(z)} + \overline{\psi_{02}(z)}]^-\} = \varphi(z), \quad z \in C\end{aligned} \tag{1.13}$$

For the particular case the function $f(z)$ is given. From the theory of the boundary value problems of the form (1—13) it is also known [2] that in the case of the outer boundary value problem the solution always exists. The proof of the existence can be done by Šerman's method [2]. Under the following headings we shall consider some special cases by transforming the problem to the Riemann--Hilbert's boundary value problem of analytical functions.

## 2. Cylinder Theorem for Circular Boundary

Let circular boundary $C$ have the equation $z\bar{z} = 1$. The functions $\varphi_{01}(z)$, $\psi_{01}(z)$, $\varphi_1(z)$ and $\psi_1(z)$ are regular in domain $D_1 = \{z,|z| > 1\}$, and the functions $\varphi_{02}(z)$ and $\psi_{02}(z)$ in domain $D_C = \{z,|z| > 1\}$. With this in view we obtain for large $|z|$.

$$\varphi_{01}(z) = a_0 + \frac{a_1}{z} + \dots \tag{2.1}$$

$$\varphi_{01}{}'(z) = -\frac{a_1}{z^2} + \dots \tag{2.2}$$

$$\psi_{01}(z) = b_0 + \frac{b_1}{z} + \dots \tag{2.3}$$

$$\varphi_1(z) = A_0 + \frac{A_1}{z} + \dots \tag{2.4}$$

$$\varphi_1{}'(z) = -\frac{A_1}{z^2} + \dots \tag{2.5}$$

$$\psi_1(z) = B_0 + \frac{B_1}{z} + \dots \tag{2.6}$$

and for small $z$

$$\varphi_{02}(z) = \tilde{a}_0 + \tilde{a}_1 z + \tilde{a}_2 z^2 + \dots \tag{2.7}$$

$$\varphi_{02}'(z) = \tilde{a}_1 + 2\,\tilde{a}_2 + z \dots \tag{2.8}$$

$$\psi_{02}(z) = \tilde{b}_0 + \tilde{b}_1 z + \dots \tag{2.9}$$

Considering the equations

$$\overline{\varphi_1'(z)}^{+} = \bar{\varphi}_1'\left(\frac{1}{2}\right)^{-} \tag{2.10}$$

$$\overline{\psi_1(z)}^{+} = \bar{\psi}_1\left(\frac{1}{2}\right)^{-} \tag{2.11}$$

$$\overline{\varphi_{01}'(z)}^{+} = \bar{\varphi}_{01}'\left(\frac{1}{z}\right)^{-} \tag{2.12}$$

$$\overline{\psi_{01}(z)}^{+} = \bar{\psi}_{01}\left(\frac{1}{z}\right)^{-} \tag{2.13}$$

$$\overline{\varphi_{02}'(z)}^{-} = \bar{\varphi}_{02}'\left(\frac{1}{z}\right)^{+} \tag{2.14}$$

$$\overline{\psi_{02}(z)}^{-}\,\bar{\psi}_{02} = \left(\frac{1}{z}\right)^{+} \tag{2.15}$$

equation (1—13) can be written as follows

$$\left[\varphi_1(z) + \varphi_{01}(z) + z\,\overline{\varphi}_{02}'\left(\frac{1}{z}\right) + \overline{\psi}_{02}\left(\frac{1}{z}\right)\right]^+ - \left[-z\,\overline{\varphi}_1'\left(\frac{1}{z}\right) - \right.$$

$$\left. - \overline{\psi}_1\left(\frac{1}{z}\right) - z\,\overline{\varphi}_{01}'\left(\frac{1}{z}\right) - \overline{\psi}_{01}\left(\frac{1}{z}\right) - \varphi_{02}(z)\right]^- = 0,\ z \in C \tag{2.16}$$

From the above enumerated properties of the presenmt functions follows that the function $\Phi(z)$, defined by

$$\Phi(z) = \varphi_1(z) + \varphi_{01}(z) + z\,\overline{\varphi}_{02}'\left(\frac{1}{z}\right) + \overline{\psi}_{02}\left(\frac{1}{z}\right),\ |z| > 1 \tag{2.17}$$

$$\Phi(z) = -z\,\overline{\varphi}_1'\left(\frac{1}{z}\right) - z\,\overline{\psi}_1\left(\frac{1}{z}\right) - \varphi_{01}'\left(\frac{1}{z}\right) - \overline{\psi}_{01}\left(\frac{1}{z}\right) - \varphi_{02}(z),\ |z| > 1 \tag{2.18}$$

is a polynominal of the first order

$$\Phi(z) = Az + B \tag{2.19}$$

$$A = \overline{\tilde{a}}_1 \tag{12.9}$$

wherefrom follows

$$\varphi_1(z) = -\varphi_{01}(z) + \overline{\tilde{a}}_1 z - z\,\overline{\varphi}_{02}'\left(\frac{1}{z}\right) - \overline{\psi}_{02}\left(\frac{1}{z}\right) + B \tag{2.20}$$

$$\psi_1(z) = -\psi_{01}(z) - \overline{\varphi}_{02}\left(\frac{1}{z}\right) - \frac{\tilde{a}_1 + \overline{a}_1}{z} + \frac{1}{z}\frac{d}{dz}\left[z\,\overline{\varphi}_{02}'\left(\frac{1}{z}\right) + \right.$$

$$\left. + \overline{\varphi}_{02}\left(\frac{1}{z}\right)\right] - \overline{B} \tag{2.21}$$

The solution of the problem is now

$$\varphi(z) = \varphi_{02}(z) + \overline{\tilde{a}}_1 z - z\,\overline{\varphi}_{02}'\left(\frac{1}{z}\right) - \overline{\psi}_{02}\left(\frac{1}{z}\right) + B \tag{2.22}$$

$$\psi(z) = \psi_{02}(z) - \frac{\tilde{a}_1 + \tilde{a}_1}{z} - \overline{\varphi}_{02}\left(\frac{1}{z}\right) + \frac{1}{z}\frac{d}{dz}\left[z\,\overline{\varphi}_{02}'\left(\frac{1}{z}\right) + \right.$$

$$\left. + \overline{\psi}_{02}\left(\frac{1}{z}\right)\right] - B \tag{2.23}$$

Constant $B$ still remains arbitrary.

The comparison of the cylinder theorem contained in the above two equations to the analogue theorem in [1] shows that the theorem in [1] is carried out for the conditions $\varphi_{01}(z) = \psi_{01}(z) = 0$ and $\varphi_0'(z) = 0$ and thus still $\varphi_{02}(z) =$ constant or $\varphi_0(z) =$ constant respectively. So the results (2—22) and (2—23) represent an essential generalization.

### 3. Cylinder Theorem for a Straight Infinite Boundary

Let boundary $C$ be axis $x$, domain $D_1$ the upper and domain $D$ the lower semi-plane. If to equations (2—1) to (2—15) the corresponding equations are formed for the here discussed conditions, then from (1—13) follows:

$$[\varphi_1(z) + \varphi_{01}(z) + z\,\bar{\varphi}'_{02}(z) + \bar{\psi}_{02}(z)]^+ - [- z\,\bar{\varphi}'_1(z) - \psi_1(z) - z\,\varphi'_{01}(z) - \\ - \bar{\psi}_{01}(z) - \varphi_{02}(z)]^- = 0, \qquad z \in C \tag{3.1}$$

From the behaviour of the present functions follows that the function $\Phi(z)$ defined by

$$\Phi(z) = \varphi_1(z) + \varphi_{01}(z) + (z)\,\bar{\varphi}'_{02}(z) + \bar{\psi}_{02}(z), \quad Im[z] > 0, \tag{3.2}$$

$$\Phi(z) = - z\bar{\varphi}'_1(z) - \bar{\psi}_1(z) - z\,\bar{\varphi}'_{01}(z) - \bar{\psi}_{01}(z) - \varphi_{02}(z), \quad Im[z] > 0 \tag{3.3}$$

is simply a constant:

$$\Phi(z) = B \tag{3.4}$$

wherefrom follows that

$$\varphi_1(z) = - \varphi_{01}(z) - z\,\varphi'_{02}(z) - \bar{\psi}_{02}(z) + B \tag{3.5}$$

$$\psi_1(z) = - \psi_{01}(z) - \bar{\varphi}_{02}(z) + z\,\frac{d}{dz}[z\,\varphi'_{02}(z) + \bar{\psi}_{02}(z)] - \bar{B} \tag{3.6}$$

Thus the solution of our problem is

$$\varphi(z) = \varphi_{02}(z) - z\,\bar{\varphi}'_{02}(z) - \bar{\psi}_{02}(z) + B \tag{3.6}$$

$$\psi(z) = \psi_{02}(z) - \bar{\varphi}_{02}(z) + z\,\frac{d}{dz}[z\,\bar{\varphi}'_{02}(z) + \bar{\psi}_{02}(z)] - B \tag{3.8}$$

The remark from the previous heading is valid also for this solution.

From equations (2—22), (2—23) as well as from (3—7) and (3—8) it is evident that the summands $\varphi_{01}(z)$ and $\psi_{01}(z)$ exert no influence on functions $\varphi(z)$ and $\psi(z)$, therefore thay can be omitted in practical calculations. This is also generally valid. From equations (1—7), (1—8) and (1—9), (1—10) and from the properties of the present functions follows that the generality is in no way inflicted if we take the sum of $\varphi_{01}(z) + \varphi_1(z)$ for a new func$\Phi$ion $\varphi_1(z)$ and $\psi_{01}(z) + \psi_1(z)$ for $\psi_1(z)$.

All the properties of the newly created functions remain unchanged when compared to the original two.

Example:

Let $\quad \varphi_0(z) = 0, \; \psi_0(z) = \dfrac{pi}{z - z_0} + \dfrac{qi}{z - \bar{z}_0}, \; Im[z_0] > 0.$

Here it is

$$\varphi_{01}(z) = \varphi_{02}(z) = 0, \quad \psi_{01}(z) = \frac{qi}{z - \bar{z}_0}, \quad \psi_{02}(z) = \frac{pi}{z - z_0}$$

and

$$\varphi(z) = \frac{pi}{z - \bar{z}_0}$$

$$\psi(z) = \frac{pi}{z - z_0} + \frac{pi\,z}{(z - \bar{z}_0)^2}$$

## 4. Slip Cylinder Theorems

A cylinder theorem is called a slip-cflinder theorem when instead of the zero value of velocity on boundary $C$ the zero value of the normal velocity component and of the tangential force component per unit of lenght of boundary $C$ [3] is taken as a boundary condition.

Let curve $C$ be given by equation $z = z(s)$ where $s$ is the arc lenght of the curve. We use the designation $\frac{dz}{ds} = \dot{z}$. Then the normal velocity component is given by the expression

$$V_n = \vec{V}\vec{n} = Re\,[i\,(u + i\,v)\dot{\bar{z}}] \tag{4.1}$$

and the tangential force component per unit of lenght by

$$\frac{d\vec{F}}{ds}\vec{t} = Re\,[(X + iY)\,\dot{\bar{z}}] \tag{4.2}$$

Taking into account equations (1—4) and (1—6) the above two equations take the form:

$$\vec{V}\vec{n} = 2\,Re\,\{\dot{\bar{z}}\,[\varphi(z) + z\,\overline{\varphi'(z)} + \overline{\psi(z)}]\} \tag{4.3}$$

$$\dot{\vec{F}}\vec{t} = 4\,\mu\,Re\,\{\dot{\bar{z}}\,[\varphi(z) - z\,\overline{\varphi(z)} - \overline{\psi(z)}]^{\cdot}\} =$$

$$= 4\,\mu\,Re\,\{2\,\varphi'(z) - \dot{\bar{z}}\,[\varphi(z) + z\,\overline{\varphi'(z)} + \overline{\psi(z)}]^{\cdot}\} \tag{4.4}$$

Multiplying equation (4—3) by $2\mu$ deriving it to $s$ and adding it to the other equation, we obtain

$$2\,\mu\,(\vec{V}\vec{n})^{\cdot} + \dot{\vec{F}}\dot{\vec{t}} = 4\,\mu\,Re\,\{2\,\varphi'(z) + \ddot{\bar{z}}\,[\varphi(z) + z\,\overline{\varphi'(z)} + \overline{\psi(z)}]\} \tag{4.5}$$

Equation (4—3) and (4—5) cab be unified into equation [4]

$$[\varphi'(z) + \overline{\varphi'(z)}] + \ddot{\bar{z}}\,[\varphi(z) + z\,\overline{\varphi'(z)} + \overline{\psi(z)}] =$$

$$= \frac{1}{2}\left\{\left[(\vec{V}\vec{n})^{\cdot} + \frac{1}{2\,\mu}\,(\dot{\vec{F}}\vec{t})\right] - i\,\frac{1}{\rho}\,(\vec{V}\vec{n})\right\} \tag{4.6}$$

where $\rho$ is the radius of the osculating circle of curve $C$ in point $z$.

The boundary value problem can be stated by equation (4—6) where the right side is equalized to 0, in cases when curve $C$ has no straight sections. In the case when $C$ is the axis $x$, we simply remain at equations (4—3) and (4—5). Since in this case $s = x$, $\dot{z} = 1$, $\ddot{z} = 0$, we obtain

$$Re\,[\varphi(z) + z\,\overline{\varphi'(z)} + \overline{\psi(z)}]^{+} = 0 \tag{4.7}$$

$$Re\,[\varphi'(z)]^{+} = 0, \quad z = x \tag{4.8}$$

Due to (4—7') in (4—7) still the middle term in square brackets can be omitted. Considering the remark it the end of the previous heading, (4—7') yields

$$[\varphi_1'(z) + \bar{\varphi}_{02}'(z)]^{+} - [-\bar{\varphi}_1'(z) - \varphi_{02}'(z)]^{-} = 0 \tag{4.8}$$

wherefrom, taking into account the properties of the present aunctions follows

$$\varphi_1'(z) + \bar{\varphi}_{92}'(z) = 0 \tag{4.9}$$

and

$$\varphi_1(z) = -\,\bar{\varphi}_{02}(z) + B \tag{4.10}$$

where $B$ is an arbitrary constant.

From equation (4—7) then follows

$$Re\,[\varphi_1(z) + \varphi_{02}(z) + \overline{\psi_1(z)} + \overline{\psi_{02}(z)}]^{+} = 0$$

wherefrom considering (4—10)

$$[\psi_1(z) + \bar{\psi}_{02}(z)]^{+} - [-\bar{\psi}_1(z) - \psi_{02}(z)]^{-} = -(B + \bar{B}), \quad z = x \tag{4.11}$$

and taking into account all that bas been solved before

$$\psi_1(z) + \bar{\psi}_{02}(z) = iK - \bar{B} \tag{4.12}$$

where $K$ is an arbitrary real constant.

Out of this the solution of the problem is obtained

$$\varphi(z) = \varphi_{02}(z) - \bar{\varphi}_{02}(z) + B \tag{4.13}$$

$$\psi(z) = \psi_{02}(z) - \bar{\psi}_{02}(z) - \bar{B} + iK \tag{4.14}$$

Opposed to the solution in [3], our solution contains still the arbitrary real constant $K$ which can e.g. define rthe velocity of the fluid on the boundary in $z = \infty$.

If boundary $C$ is circular having the equation $z \cdot \bar{z} = 1$, then formulation (4—6) will be used in order to solve the problem where $z = e^{-is}$, $\dot{z} = -\,iz$, $\ddot{z} = -\,z$ will be taken into account. Thus from (4—6) follows

$$[\varphi'(z) + \overline{\varphi'(z)}]^{+} - \bar{z}[\varphi(z) + z\overline{\varphi'(z)} + \overline{\psi(z)}]^{+} = 0 \qquad z \in C \tag{4.15}$$

abd wherefrom

$$\left[\varphi_1'(z) - \frac{1}{z}\,\varphi_1(z) - \frac{1}{z}\,\bar{\psi}_{02}\left(\frac{1}{z}\right)\right]^+ - \left[-\varphi_{02}'(z) + \frac{1}{z}\,\varphi_{02}(z) + \right.$$

$$\left. + \frac{1}{z}\,\bar{\psi}_1\left(\frac{1}{z}\right)\right]^- = 0 \quad z \in C \tag{4.16}$$

The function $\Phi(z)$ defined by

$$\Phi(z) = \varphi_1'(z) - \frac{1}{z}\,\varphi_1(z) - \frac{1}{z}\,\bar{\psi}_{02}\left(\frac{1}{z}\right), \quad |z| > 1 \tag{4.17}$$

$$\Phi(z) = -\,\varphi_{02}'(z) + \frac{1}{z}\,\varphi_{02}(z) + \frac{1}{z}\,\bar{\psi}_1\left(\frac{1}{z}\right), \quad |z| < 1 \tag{4.18}$$

is, except in $z = 0$ where it has the pole of order one, everywhere regular (also in $z = \infty$) thus it is

$$\Phi(z) = \frac{A}{z} + B \tag{4.19}$$

From (4—17) follows that $B = 0$. (4—18) yields

$$\psi_1(z) = -\,\bar{\varphi}_{02}\left(\frac{1}{z}\right) + \frac{1}{z}\,\bar{\varphi}_{02}\left(\frac{1}{z}\right) + \bar{A} \tag{4.20}$$

where $A$ is an arbitrary constant.

From (4—17) we obtain still the equation

$$\varphi_1'(z) - \frac{1}{z}\,\varphi_1(z) = \frac{A}{z} + \frac{1}{z}\,\bar{\psi}_{02}\left(\frac{1}{z}\right) \tag{4.21}$$

The solution in the frame of the required propderties is

$$\varphi_1(z) = -\,z \cdot \bar{\chi}_{02}\left(\frac{1}{z}\right) - A \tag{4.22}$$

In this equation it is

$$\bar{\chi}_{02}(u) = \int\limits_0^u \bar{\psi}_{02}(u)\,du, \quad u = \frac{1}{z}$$

Finally the solution can be written as

$$\varphi(z) = \varphi_{02}(z) - z\,\bar{\chi}_{02}\left(\frac{1}{z}\right) - A \tag{4.23}$$

$$\psi(z) = \psi_{12}(z) - \bar{\varphi}_{02}\left(\frac{1}{z}\right) + \frac{1}{z}\,\bar{\varphi}_{01}'\left(\frac{1}{z}\right) + \bar{A} \tag{4.24}$$

The theorem in [3] is just a special case of the above solved.

## 5. Cylinder Theorems for the Boundary Mapped on a Circle or Line by a Polynomial

Let the polynominal $z = \omega(\zeta) = P_n(\zeta)$ map the exterior of the domain bounded by curve $C$ into the exterior of a circle with the equation of circumference $\zeta\bar{\zeta} = 1$. Using the indices 0, 01 and 02 in plane $\zeta$ analogously to the previous use, the functions $\varphi(z) = \tilde{\varphi}(\zeta)$ and $\psi(z) = \tilde{\psi}(\zeta)$ at zero boundary velocity can be written as

$$\tilde{\varphi}(\zeta) = \tilde{\varphi}_{02}(\zeta) + \sum_{k=0}^{k=n-1} \beta_k \zeta^{n-k} - \omega(\zeta)\bar{\tilde{\Phi}}_{02}\left(\frac{1}{\zeta}\right) - \bar{\tilde{\psi}}_{02}\left(\frac{1}{\zeta}\right) + B \tag{5.1}$$

$$\tilde{\psi}(\zeta) = \tilde{\psi}_{02}(\zeta) - \sum_{k=0}^{k=n-1} \bar{\beta}_k \frac{1}{\zeta^{n-k}} - \bar{\tilde{\varphi}}_{02}\left(\frac{1}{\zeta}\right) -$$

$$- \frac{\bar{\omega}\left(\frac{1}{\zeta}\right)}{\omega'(\zeta)} \left\{ \sum_{k=0}^{k=n-1} \beta_x (n-k)\, \zeta^{n-k-1} - \frac{d}{d\zeta}\left[\omega(\zeta)\,\bar{\tilde{\Phi}}_{02}\left(\frac{1}{\zeta}\right) + \right.\right.$$

$$\left.\left. + \bar{\tilde{\Phi}}_{02}\left(\frac{1}{\zeta}\right)\right]\right\} - \bar{B} \tag{5.2}$$

where

$$\tilde{\Phi}(\zeta) = \Phi(z) = \frac{d\varphi}{dz} = \frac{\tilde{\varphi}'(\zeta)}{\omega'(\zeta)}$$

and

$$\omega(\zeta)\bar{\tilde{\Phi}}_{02}\left(\frac{1}{\zeta}\right) = \sum_{k=0}^{k=n} \beta_x \zeta^{n-k} + o(1)$$

It is also required that the polfnom $\omega'(\zeta)$ has all the zeros outside domain $\tilde{D}_1$.

If the polynomial $z = \omega(\zeta) = P_n(\zeta)$ maps the exterier of curve $C$ into the upper semi-plane of plane $\zeta$, then at the previous boundary condition follows

$$\tilde{\varphi}(\zeta) = \tilde{\varphi}_{02}(\zeta) - \omega(\zeta)\,\bar{\tilde{\Phi}}_{02}(\zeta) - \bar{\tilde{\psi}}_{02}(\zeta) + B \tag{5.3}$$

$$\tilde{\psi}(\zeta) = \tilde{\psi}_{02}(\zeta) - \bar{\tilde{\varphi}}_{02}(\zeta) + \frac{\bar{\omega}(\zeta)}{\omega'(\zeta)}\frac{d}{d\zeta}\left[\omega(\zeta)\,\bar{\tilde{\Phi}}_{02}(\zeta) + \bar{\tilde{\psi}}_{02}(\zeta)\right] - \bar{B} \tag{5.4}$$

Both theorems can be proved by the same methodology as the previous theorems, therefore the proofs have been omitted.

Example:

Into the fluid flow a boundary is inserted along the positive part of axis $x$ The thus created domain is mapped by $z = \omega(\zeta) = \zeta^2$ into the upper semi-plane of plane $\zeta$. In this way, at the original boundary condition of zero boundary velocity, we get

$$\tilde{\varphi}(\zeta) = \tilde{\varphi}_{02}(\zeta) - \frac{1}{2}\zeta\,\overline{\tilde{\varphi}'_{02}}(\zeta) - \overline{\tilde{\psi}_{02}}(\zeta) + B$$

$$\tilde{\psi}(\zeta) = \tilde{\psi}_{02}(\zeta) - \overline{\tilde{\varphi}_{02}}(\zeta) - \frac{1}{2}\zeta\frac{d}{d\zeta}\left[\frac{1}{2}\overline{\tilde{\varphi}'_{02}}(\zeta) + \overline{\tilde{\psi}_{02}}(\zeta)\right] - \overline{B}$$

And in the case of the boundary conditions of the Slip-Cylinder Theorem we get

$$\tilde{\varphi}(\zeta) = \tilde{\varphi}_{02}(\zeta) - \overline{\tilde{\varphi}_{02}}(\zeta) + B$$

$$\tilde{\psi}(\zeta) = \tilde{\psi}_{02}(\zeta) - \overline{\tilde{\psi}_{02}}(\zeta) - \overline{B} + iK$$

where $K$ is an arbitrary real constant.

## REFERENCES


[1] Ionesco D. G.: *Circle and cylinder theorems for slow viscous flow*; Journa de Mechanique, v. 10, No 3, 345—355, (1971)

[2] Krušić B.: *An application of complex analysis at threedimensional flow of a fluid*; TAM 5, 73—84, (1979)

[3] Atanacković T. M. and Leigh D., C.: *S'lip plane and slip cylinder theorems for slow viscous flow*; Acta Mechanica 27, 121—126, (1977)

[4] Krušić B. : *Bending of a simply supported plate*; TAM 5, 63—72, (1979)


## BEITRAG ZU DEN ZYLINDER THEOREMEN BEI DEN ZÄHIGEN FLÜSSIGKEITEN

### Zusammenfassung


In diesem Beitrag werden die Zylinder Theoreme für die obene Strömung der langsamen zähigen Flüssigkeit bei verschiedenen Randbedingungen behandelt. Mehr verallgemeinerte Abstimmungen als diejenige die bisher bekannt waren, werden gefunden. Alle theoreme werden durch die selbe Methodologie geprüft, d.h. durch die Umformung des Problems auf Riemann-Hilbert Randwertproblem der analytischen Funktionen.


## DOPRINOS K CILINDRSKIM IZREKOM PRI VISKOZNIH TE TEKOČINAH

### Povzetek


V tem sestavku so obravnavani cilindrski izreki za dvodimenzijski tok počasne viskozne tekočine pri raznih robnih pogojih. Najdene so splošnejše formulacije od doslej znanih. Vsi izreki so dokazani po enoti metodologiji — s prevedbo problema na Riemann-Hilbertov robni problem pri analitičnih funkcijah.



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