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STABILITY OF EQUILIBRIUM OF NONHOLONOMIC RHEONOMIC SYSTEMS

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Introduction

This paper is divided into two main parts. In the first part movement of the nonholonomic rheonomic system is discussed and the differential equations are given, while in the second part the theorem of stability of equilibrium is given and proved.

Motion of the representative particle is obtained in phase space V_{2n+2} with $q^{\circ} = t,\ q^1 \dots, q^n,\ \bar{p}^{\circ}, \dots, p_n$ as variables.

Stability theorem belongs to Liapunovs direct method, because we try to find some function W with certain characteristic. This function W depends on twice less variables than Liapunovs, so that, theoretically, it is easier to find it.

Idea to solve given problem on this way is not original; but now it is applied on both nonholonomic and rheonomic systems.

1. Moving of the system of particles

Let us consider the mechanical system of particles M_i $(i = 1, 2, \dots, N)$ whose masses are m_i. Motion of this system is constrained by K holonomic rheonomic ideal bilateral constraints:

$$f_\alpha (t, \vec{r}_1, \dots, \vec{r}_N) = 0 \quad (\alpha = 1, \dots, k) \tag{1.1}$$

i.e.

$$\sum_{i=1}^{N} \underset{\vec{r}i}{qrad}\, f\alpha \cdot \vec{V}_i + \frac{\partial f\alpha}{\partial t} = 0 \tag{1.2}$$

and by L nonholonomic rheonomic ideal bilateral constraints where

$$\varphi_\beta = \Sigma\, \vec{l}_{\beta i} \cdot \vec{V}_i + l_\beta = 0 \tag{1.3}$$

$$\vec{l}_{\beta i} = \vec{l}_{\beta i} (t, \vec{r}_1, \dots, \vec{r}_N), \quad l_\beta = l_\beta (t, \vec{r}_1 \dots, \vec{r}_N)$$

Instead of considering N praticles in threedimensional space we consider a single representative particle in $n + 1$ dimensional configuration space V_{n+1} with variables $q_0, \dots, q_n$ $(q_0 = t,\ n = 3N - K)$. In that case equations of holonomic constraints become the identities.

$$f\alpha \equiv 0 \qquad (\alpha = 1, \dots, K)$$

while equations of nonholonomic constraints will be:

$$\Phi_{\beta\bar{\mu}} \cdot \dot{q}^{\bar{\mu}} = 0 \quad (\beta = 1, \ldots, L;\ \bar{\mu} = 0, 1, \ldots, n) \tag{1.4}$$

and where

$$\Phi_{\beta\mu} = \sum_{i=1}^{N} \vec{l}_{si} \cdot \frac{\partial \vec{r}_i}{\partial q^{\mu}}, \quad \Phi_{\beta 0} = \sum_{i=1}^{N} \vec{l}_{\beta i} \cdot \frac{\partial \vec{r}_i}{\partial q_0} + l_s \quad (\mu = 1, \ldots, n)$$

Frequently we can write equations of nonholonomic constraints in a form of explicite dependance of L generalized velocities $\dot{q}^{\alpha'}$ on $m + 1 = n + 1 - L$ velocites $\dot{q}^{\bar{\alpha}}$

$$\dot{q}^{\alpha'} = \varphi^{\alpha'}_{\bar{\alpha}}\, \dot{q}^{\bar{\alpha}} \quad (\bar{\alpha} = 0, 1, \ldots, m; \quad \alpha' = m + 1 \ldots, n) \tag{1.5}$$

Because of (1.5) we can say that

$$\delta q^{\alpha'} = \varphi^{\alpha'}_{\bar{\alpha}}\, \delta q^{\alpha'} \tag{1.6}$$

where δ denotes the variation. According to (1.6) one can say that all $\delta q^{\bar{\alpha}}$ are independant, but if we take into consideration the nature of variable q^0, and if we know what the variation is, we can assert that independant virtual displacements are:

$$\delta q^1, \ldots, \delta q^m. \qquad m = n - L$$

As the number of independant virtual displacements is the number of degrees of freedom, the considered mechanical system will have m degrees of freedom.

In $Vn + 1$ space, in which we are considering the motion of a representative particle metric is introduced as follows:

$$ds^2 = a_{\bar{\mu}\bar{\nu}}\, dq^{\bar{\mu}} dq^{\bar{\nu}} \quad (\bar{\mu}, \bar{\nu} = 0, 1 \ldots, n)$$

where $a_{\bar{\mu}\bar{\nu}}$ is covariant coordinate of the metric tensor,

i.e.

$$a_{\bar{\mu}\bar{\nu}} = \sum_{i=1}^{N} m_i \frac{\partial \vec{r}_i}{\partial q^{\bar{\mu}}} \frac{\partial \vec{r}_i}{\partial q^{\bar{\nu}}} \quad (\bar{\mu}, \bar{\nu} = 0, 1, \ldots, n)$$

Let us say that now the representative particle is moving in the phase space $V2n + 2$, whose coordinates are: $q^0, q^1, \ldots, q^n, p_0, \ldots, p_n$.

Generalized momenta are defined in this way:

$$p_{\bar{\mu}} \overset{def}{=} \sum_{i=1}^{N} m_i \vec{V}_i \frac{\partial \vec{r}_i}{\partial q^{\bar{\mu}}} \tag{1.7}$$

so that

$$p_{\bar{\mu}} = \sum_{i=1}^{N} m_i \frac{\partial \vec{r}_i}{\partial q^{\bar{\mu}}} \frac{\partial \vec{r}_i}{\partial q^{\bar{\mu}}} \dot{q} = a_{\bar{\mu}\bar{\nu}}\, \dot{q}^{\bar{\nu}} = (\bar{\mu}, \bar{\nu} = 0, 1, \ldots, n) \tag{1.8}$$

If $det\ a_{\bar{\mu}\bar{\nu}} \neq 0$ is it possible to find contravariant coordinates of this tensor knowing that

$$a_{\bar{\mu}\bar{\nu}} \cdot a^{\bar{\mu}\bar{\theta}} = \delta^{\bar{\theta}}_{\bar{\nu}} = \begin{cases} 1, & \bar{\nu} = \bar{\theta} \\ 0, & \bar{\nu} \neq \bar{\theta} \end{cases} \tag{1.9}$$

From (18) and (1.9) it follows:

$$\dot{q}^{\bar{\mu}} = a^{\bar{\mu}\bar{\nu}} p_{\bar{\nu}} \tag{1.10}$$

Equations of nonholonomic constraints (1.5) can, according to (1.10), be expressed as follows:

$$(a^{\alpha'\bar{\nu}} - \varphi^{\alpha}_{\underline{\alpha}}{}' a^{\bar{\mu}\bar{\nu}}) p_{\bar{\nu}} = 0 \ (\bar{\alpha} = 0, 1, \ldots, m; \ \alpha' = m+1, \ldots n, \ \bar{\nu} = 0, 1 \ldots n) \tag{1.11}$$

From (1.11) we can say that among $n + 1$ generalized momenta there are L constraints, so that only $m + 1 = n + 1 - L$ of them are independant. This means that the particle will move only through that part of the phase space in which the conditions (1.11) are fulfilled in every point.

2. Differential equations of perturbed state of equilibrium

Let us assume that on the system, whose moving is limited by $K + L$ equations (1.1) and (1.2) forces F_1 are acting.

According to the Lagrange-E'Alambert principle:

$$\sum_{i=1}^{N} (m_i \vec{w}_i - \vec{F}_i - \vec{R}_i) \delta\vec{r}_i = 0 \tag{2.1}$$

knoving that

$$\delta\vec{r}_i = (\partial\vec{r}_i/\partial q\alpha + \varphi^{\alpha'}_{\alpha}/\alpha \, \partial\vec{r}_i/\partial q\alpha')\delta q\alpha \tag{2.2}$$

and because we said that the constraints are ideal:

$$\sum_{i=1}^{N} \vec{R}_i \cdot \delta\vec{r}_i = 0$$

we obtain first m equations:

$$\frac{Dp\,\alpha}{Dt} + \varphi^{\alpha'}_{\alpha} \frac{Dp\,\alpha}{Dt} = Q\alpha + \varphi^{\alpha'}_{\alpha} Q\alpha' \quad (\alpha = 1, \ldots, m; \ \alpha' = m+1, \ldots, n) \tag{2.3}$$

where:

$\frac{D}{Dt}$ — denotes the total derivative, and

$Q_\alpha = \sum_{i=1}^{N} \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_\alpha}$ — the generalized force.

Using the Lagrange equations of the first kind:

$$m_i \cdot \vec{w}_i = \vec{F}_i + \sum_{\alpha=1}^{K} \lambda_\alpha \operatorname{grad}_{\vec{r}i} f_\alpha + \sum_{\beta=1}^{L} \mu_\beta \vec{l}_{\beta i}$$

and m equations (2.3) we obtain:

$$\frac{Dpo}{Dt} + \varphi^{\alpha'}_{0} \frac{Dp\alpha'}{Dt} = Q_0 + Q_{\alpha}' \varphi^{\alpha'}_{0} - \sum_{\alpha=1}^{K} \lambda_\alpha \frac{\partial f\alpha}{\partial t} - \sum_{\beta=1}^{L} \mu_\beta l_\beta \tag{2.4}$$

Equations (2.3), (2.4), together with (1.10) and (1.11) form a system of $2n + 2$ equations of motion:

$$\frac{Dp\alpha}{Dt} + \varphi_{\alpha}^{\alpha'} \frac{Dp\alpha'}{Dt} = Q\alpha + \varphi_{\alpha}^{\alpha'} Q\alpha'$$

$$\frac{Dp_0}{Dt} + \varphi_3^{\alpha'} \frac{Dp\alpha'}{Dt} = Q_0 + \varphi_0^{\alpha'} Q\alpha' - \sum_{\alpha=1}^{K} \lambda\alpha \frac{\partial f\alpha}{\partial t} - \sum_{\beta=1}^{L} \mu_\beta l_\beta$$

$$\dot{q}^{\bar{\mu}} = a^{\bar{\mu}\bar{\nu}} p_{\bar{\nu}}$$

$$(a^{\alpha'\nu} - \varphi_{\underline{\alpha}}^{\alpha'} a^{\alpha\bar{\nu}} p_{\bar{\nu}} = 0 \tag{2.5}$$

$$\begin{pmatrix} \alpha = 1, \ldots, m; & \alpha' = m + 1, \ldots, n \\ \underline{\alpha} = 0, 1, \ldots, m; & \bar{\mu}, \bar{\nu} = 0, 1, \ldots, n \end{pmatrix}$$

3. State of Equilibrium

The equilibrium position is defined as a position in which $\vec{r}_i =$ const., so that $\vec{V}_i = 0 \quad (i = 1, 2, \ldots, N)$

Vector relations (3.1) are equivalent to the following scalar relations:

$$x_i = \text{const.}, \quad \dot{x}_i = 0 \quad (i = 1, 2, \ldots, 3N)$$

The state of equalibrium is defined with $2 \cdot 3N$ scalar relations. These relations also define the state of a equilibrium of a representative particle that is moving in $3N$ dimensional Euclidean space E_{3N}.

The state of equilibrium of a representative particle that is moving in $3N + 1$ dimensional space will be:

$$x_0 = t, \quad x_i = \text{const.}, \quad \dot{x}_0 = 1, \quad \dot{x}_i = 0 \quad i = 1, \ldots, 3N$$

Assuming the equations of holonomic and nonholonomic constraints the state of equilibrium will be defined by:

$$q^0 = t, \quad q^{\mu} = \text{const.}, \quad p_{\bar{\mu}} = 0$$

$$\frac{\partial \vec{r}_i}{\partial t} = 0, \frac{\partial f\alpha}{\partial t} = 0. \quad l_\beta = 0, \quad \varphi_0^{\alpha'} = 0$$

Substituing (3.3) into equations of motion they maintain the following system of equations:

$$0 = Q\alpha + \varphi_{\alpha}^{\alpha'} Q\alpha'$$

defining a nonholonomic manifold.

4. Stability of Equilibrium

Let us consider a certain state of equilibrium on $0 = Q\alpha + \varphi_\alpha^{\alpha'} Q\alpha' : q^1 =$ $=$ const., . . . , $q^n =$ const., $po = 0, \ldots, p_n = 0$.

With a conviniently chosen reference system the state of equilibrium can be writen as follows:

$$q^1 \ldots = q^n = p_0 = \ldots = p_n \tag{4.1}$$

Our aim is to derive the conditions of stability of the equilibrium state (4.1).

It can be proved that equations of perturbed equilibrium state are the same as equations of motion (2.6):

$$\frac{Dp\alpha}{Dt} + \varphi_\alpha^{\alpha'} \frac{Dp\alpha'}{Dt} = Q\alpha + Q\alpha' \varphi_\alpha^{\alpha'}$$

$$\frac{Dp_0}{Dt} + \varphi_0^{\alpha'} \frac{Dp\alpha'}{Dt} = Q_0 + Q\alpha' \varphi_0^{\alpha'} + \sum_{\alpha=1}^{K} \lambda\alpha \frac{\partial f\alpha}{\partial t} + \sum_{\beta=1}^{L} \mu_\beta l_\beta$$

$$\dot{q}^{\bar{\mu}} = a^{\bar{\mu}\bar{\nu}} p_{\bar{\nu}}$$

$$(a^{\alpha'\bar{\nu}} - a^{\alpha\bar{\nu}} \varphi_\alpha^{\alpha'}) p_{\bar{\nu}} = 0 \tag{4.2}$$

Here $q^\mu, p_{\bar{\mu}} (\bar{\mu} = 0, 1, \ldots, n;\ \bar{\mu} = 1, \ldots, n)$ are not coordinates but perturbations of these coordinates. We always identify q_0 as the coordinate $q^0 = t$ itself, and do not perturb it.

Statement of the stability. In the configuration space V_{n+1} for sufficiently small and sufficiently large q_0, there exists positive definite scalar function $W(q^0, q)$ such that

$$a^{\bar{\mu}\bar{\nu}} p_{\bar{\nu}} \left[Q\bar{\alpha} + \varphi_\alpha^{\alpha'} Q\alpha' + \frac{\partial w}{\partial q\bar{\alpha}} + \varphi_\alpha^{\alpha'} \frac{\partial w}{\partial q\alpha'} \right] - \sum_{\gamma=1}^{K} \lambda\gamma \frac{\partial f\gamma}{\partial t} - \sum_{\beta=1}^{L} \mu_\beta l_\beta \tag{4.3}$$

is less than or equal to zero in some interval $\Delta q^0 = \Delta t$ the state of equilibrium is stable in that interval.

Proof. We can represent Liapunov function fas a sum of two positive definite functions:

$V = T + W(q^0, q^\mu)$, where the kinetic energy T is a function of all $q^{\bar{\mu}}, p_{\bar{\mu}}$ $(\bar{\mu} = 0, 1, \ldots, n)$, but positive definite with respect to $p_{\bar{\mu}}$.

We assume W to be positive definite with respect to q^μ.

According to Liapunov theorem of stability, we know that if we can find some positive definite function V, such that it's derivative $\frac{dN}{dt}$ in the sence of equations of the perturbed state of equilibrium, is negative or identically equal to zero, the state of equilibrium is stable.

Hence let us find $\frac{dV}{dt}$ along the trajectories (4.2)

$$\frac{dV}{dt} = \frac{d}{dt}(T + w) = \frac{D\tau}{Dt} + \frac{dw}{Dt} \tag{4.4}$$

since for any scalar invariant $\frac{D}{Dt} = \frac{d}{dt}$.

$$\frac{DT}{Dt} = \frac{D}{Dt}\left(\frac{1}{2} a^{\bar{\mu}\bar{\nu}} p_{\bar{\mu}} p_{\bar{\nu}}\right) = a^{\bar{\mu}\bar{\nu}} \frac{Dp_{\bar{\mu}}}{Dt} p_{\bar{\nu}}, \tag{4.5}$$

because $\frac{Da^{\bar{\mu}\bar{\nu}}}{Dt} = 0$ if $a^{\bar{\mu}\bar{\nu}}$ is a metric tensor.

According to (4.2), (4.4) and (4.5) it follows that

$$\frac{dV}{dt} = a^{\bar{\mu}\bar{\nu}} p_{\bar{\nu}} \frac{Dp_{\bar{\mu}}}{Dt} + \frac{\partial w}{\partial q^{\bar{\mu}}} b^{\bar{\mu}}$$

$$\frac{dV}{dt} = a^{\bar{\mu}\bar{\nu}} p_{\bar{\nu}} \left(\frac{Dp_{\bar{\mu}}}{Dt} + \frac{\partial w}{\partial q^{\bar{\mu}}}\right)$$

$$\frac{dV}{dt} = a^{\bar{\alpha}\bar{\nu}} p_{\bar{\nu}} \left(\frac{Dp_{\bar{\alpha}}}{Dt} + \frac{Dw}{Dq_{\bar{\alpha}}}\right) + a'^{\bar{\alpha}\bar{\nu}} p_{\bar{\nu}} \left(\frac{Dp_{\alpha}'}{Dt} + \frac{\partial w}{\partial q^{\alpha'}}\right)$$

$$\frac{dV}{dt} = a^{\bar{\alpha}\bar{\nu}} p_{\bar{\nu}} \left[\frac{Dp_{\bar{\alpha}}}{Dt} + \varphi_{\bar{\alpha}}^{\alpha'} \frac{Dp_{\alpha}'}{Dt} + \frac{\partial w}{\partial q_{\bar{\alpha}}} + \varphi_{\bar{\alpha}}^{\alpha'} \frac{\partial w}{\partial q^{\alpha'}}\right]$$

$$\frac{dV}{dt} = a^{0\bar{\nu}} p_{\bar{\nu}} \left[\frac{Dp_0}{Dt} + \varphi_0^{\alpha'} \frac{Dp_{\alpha}'}{Dt} + \frac{\partial w}{\partial q^0} + \varphi_0^{\alpha'} \frac{\partial w}{\partial q^{\varkappa'}}\right] +$$
$$+ a^{\alpha\bar{\nu}} p_{\bar{\nu}} \left[\frac{Dp_{\alpha}}{Dt} + \varphi_{\alpha}^{\alpha'} \frac{Dp_{\alpha}'}{Dt} + \frac{\partial w}{\partial q^{\alpha}} + \varphi_{\alpha}^{\alpha'} \frac{\partial w}{\partial q^{\alpha'}}\right]$$

$$\frac{dV}{dt} = \left[Q_0 + \varphi_0^{\alpha'} Q_{\alpha}' - \sum_{\alpha=1}^{K} \lambda_{\alpha} \frac{\partial f_{\alpha}}{\partial t} - \sum_{\beta=1}^{L} \mu_{\beta} l_{\beta} + \frac{\partial W}{\partial q_0} + \varphi_0^{\alpha'} \frac{\partial W}{\partial q^{\alpha'}}\right] +$$
$$+ \left[Q_{\alpha} + \varphi_{\alpha}^{\alpha'} Q_{\alpha}' + \frac{\partial W}{\partial q^{\alpha}} + \frac{\partial W}{\partial q^{\alpha'}}\right] \cdot a^{\alpha\bar{\nu}} p_{\bar{\nu}}$$

$$\frac{dV}{dt} = a^{\bar{\alpha}\bar{\nu}} p_{\bar{\nu}} \left[Q_{\bar{\alpha}} + \varphi_{\bar{\alpha}}^{\alpha'} Q_{\alpha}' + \frac{\partial W}{\partial q_{\bar{\alpha}}} + \varphi_{\alpha}^{\alpha'} \frac{\partial W}{\partial q^{\alpha'}}\right] - \sum_{\alpha=1}^{K} \lambda_{\alpha} \frac{\partial f_{\alpha}}{\partial t} - \sum_{\beta=1}^{L} \mu_{\beta} l_{\beta} \tag{4.6}$$

As the right hadn of (4.6) is equal to (4.3) we can assert that theorem is proved.

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STABILITE D'EQUILIBRES DES SYSTEMES NONHOLONOMES RHEONOMES

R é s u m é

Dans ce travail, le mouvement de système de N points materiels soumis a K liaisons holonomes et L liaisons nonholonomes, qui sont toutes les deux rhéonomes, est remplacé par un problème équivalent du movement de points représentatifs dans l'espace de phase V_{2n+2} avec les coordonnées:

$$q^0 = t, \quad q^1, \ldots q^n, \quad p_0 = a_{0\bar{\mu}}\, \dot{q}^{\bar{\mu}}, \quad p_1, \ldots, p_n$$

$$(n = 3N - K - L, \quad \bar{\mu} = 0, 1, \ldots, n)$$

Dans cet espace nous dérivons les équations différentielles de mouvement et ensuite les équations différentielles de l'état perturbé d'équilibre.

La stabilité d'état d'équilibre est examinée par la méthode directe de Liapounoff. La fonction inconnue V est supposée être la somme de deux functions T et W (l'idée de travail /9/). La function RT est l'énergie kinétique qui est définite positive par rapport aux impulsions generalisées $p_0, p_1, \ldots, p_n$ et nous cherchons la fonction W comme une fonction de coordonnées $q_0, q_1, \ldots, q_n$ ayant le caractère de fonction de Liapounoff. De cette manière le probleme est, au moins en principe, simplifié (la fonction inconnue W depend de deux fois moins de variables que la fonction V).

STABILNOST RAVNOTEŽE NEHOLONOMNIH REONOMNIH SISTEMA

I z v o d

U ovom članku kretanje sistema od N materijalnih tačaka na koji deluje K holonomnih i L neholonomnih veza, gdje su i jedne i druge reonomne, zamjenjuje se ekvivalentnim problemom — kretanjem reprezentativne tačke u proširenom faznom prostoru V_{2n+2} sa koordinatama:

$$q^0 = t, q^1, \ldots, q^n; \quad p_0 = a_{0\mu}\dot{q}^\mu, \quad p_1, \ldots, p_n$$

U odnosu na tako dat prostor izvode se prvo dif. jednačine kretanja, a zatim i dif. jednačine poremećenog stanja ravnoteže.

Stabilnost ravnotežnog stanja se ispituje pomoću Ljapunovljevog direktnog metoda. No, Ljapunovljeva funkcija V se pretpostavlja u obliku zbira dve funkcije T i W (kao što je učinjeno u radovima koji slede u navedenoj literaturi).

T je kinetička energija za koju se pokazuje da je pozitivno definitna u odnosu na generalisane impulse $p_0, p_1, \ldots, p_n$; a funkciju W tražimo kao funkciju samo od generalisanih koordinata $q_0, q^1, \ldots, q^n$ i od nje zahtevamo da ima osobine Ljapunovljeve funkcije, Na taj način problem je, bar u principu, pojednostavljen — tražena funkcija W zavisi od dvostruko manje promenljivih nego funkcija V.

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