THE NON-LINEAR THEORY OF THIN-WALLED MEMBER WITH OPEN CROSS SECTION

Dedicated to the 60 th Birthday of Professor Dr.Bruno Thurlimann, ETH-Zurich

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Introduction

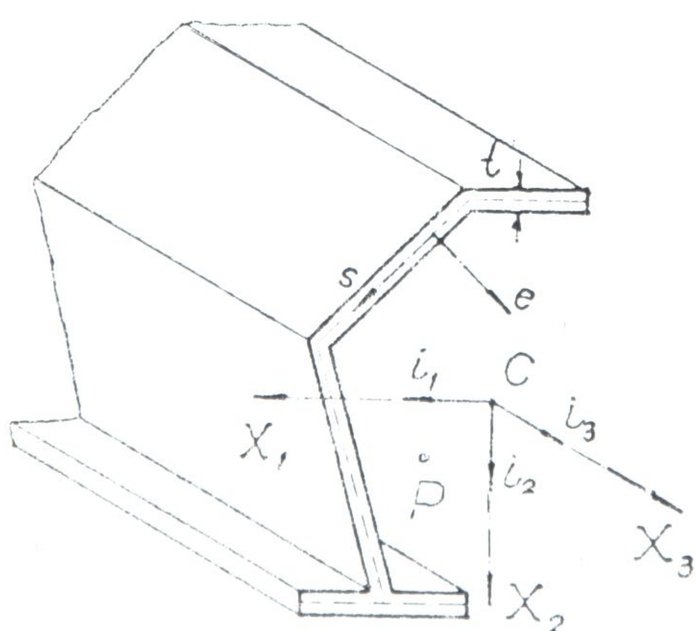
This paper represents an extension of authors contribution [1] to the second order theory for thin-walled member with the open cross section.

The presented theory differs form the conventional form by a unique treatment which includes the relations between the bimoment, St. Venant's torque, warping torque and other internal forces.

The system of six scalar equations and an additional equation relating the deplanation of cross section are derived applying the principle of virtual work to the deformed confiquration of the member.

Deformation of the member

Consider a thin-walled member with open cross section and an arbitrary polygonal centre line of the wall (Fig. 1).



The position vector of an arbitrary point of the underformed memebr can be written for a fixed rectangular cartesain system x_i $(i = 1,2,3)$ using the summation convention, in form

$$\mathbf{r}^* = x_i^* \mathbf{i}_i \qquad i = 1,2,3 \tag{1}$$

where x_i are coordinates of the observed point, i_1 and i_2 unit vectors along the principal axes of the cross section and i_3 the unit vector along the member axis.

The position vector can also be written using the coordinates s, x_3 of the middle surface of the member and the distance e of the observed point form the middle surface:

$$\mathbf{r}^* = \mathbf{r}(s, x_3) + e\mathbf{n} \tag{2}$$

where $\mathbf{n}$ is the unit vector of the normal to the middle surface, while s is the coordinate of the centre line of the cross section. The unit vector of the center line is denoted by $\mathbf{t}$.

The position vector in undeformed configuration can be, obviously, also represented in form:

$$\mathbf{r}^* = x_i \mathbf{i}_i + e\mathbf{n} \tag{3}$$

where x_i are the cartesian coordinates of the middle surface.

In deformed configuration the position of the same point is defined by

$$\mathbf{R}^* = \mathbf{r}^* + \mathbf{u}^* \tag{4}$$

where $\mathbf{u}^*$ is the displacement vector:

$$\mathbf{u}^* = u_i^* \mathbf{i}_i \tag{5}$$

The basic assumptions governing the kinematics [1] of deformation are:

i. The projection of the member cross section on the orginal plane behaves line a rigid plate (in its own plane) in course of deformation.

ii. The shear deformation ε_{s3} in the middle surface of the member can be neglected.

iii. A line element perpendicular to the middle surface before deformation remains perpendicular to the middle surface after the deformation.

On the basis of these assumptions the relations defining the components of the displacement vector $\mathbf{u}^*$ are derived in the following form:

$$\begin{aligned} u_1^* &= \xi_p - (x_2^* - x_{2p})\,\varphi_p \\ u_2^* &= \eta_p + (x_1^* - x_p)\,\varphi_{1p} \\ u_3^* &= W_0 - \xi_p\, x_1^* - \eta_p'\, x_2^* - \varphi_p'\, \omega_p \end{aligned} \tag{6}$$

where $(\ )' = \dfrac{d}{dx_3}$

The functions $\xi_p(x_3)$ and $\eta_p(x_3)$ are components of the displacement of an arbitrary centre of rotation (pole) $P(x_{1p} \cdot x_{2p})$ in the direction of axes X_1 and X_2, while φ_p is rotation of the cross section same centre. The sectorial coordinate is labeled by ω_p^*:

$$\omega_p^* = \int_0^s h_p \, ds + h_{np} \, e, \tag{7}$$

where

$$\begin{aligned} h_p &= (x_1 - x_{1p}) \, x_{2,s} - (x_2 - x_{2p}) \, x_{1,s} \\ h_{np} &= (x_1 - x_{1p}) \, x_{2,e} - (x_2 - x_{2p}) \, x_{1,e} \end{aligned} \tag{8}$$

are the distances of the tangent and normal to the centre line measured from the centre of rotation P, and

$$, \alpha = \frac{\partial}{\partial \alpha}, \qquad \alpha = e, \text{s}.$$

The position vector R^* after the deformation can be writen in form

$$\mathbf{R}^* = (x_i^* + u_i^*) \, \mathbf{i}_i \tag{9}$$

or

$$\mathbf{R}^* = \mathbf{R} + e\mathbf{n}', \tag{10}$$

where

$$\mathbf{R} = (x_i + u_i) \, \mathbf{i}_i \tag{11}$$

is the position vector of a point on the deformed middle surface, while $\mathbf{n}'$ is the unit vector of the normal on the deformed middle surface.

According to the assumption (iii.) the unit vector of the normal on the middle surface is given by

$$\mathbf{n}' = \frac{\mathbf{R}_{,e}^*}{(\mathbf{R}_{,e}^* \cdot \mathbf{R}_{,e}^*)^{1/2}} = \frac{(x_i^* + u_i^*)_{,e} \, \mathbf{i}_i}{(\mathbf{R}_{,e}^* \cdot \mathbf{R}_{,e}^*)^{1/2}}. \tag{12}$$

while the unit vectors of the coordinate curves s and x_i are:

$$\mathbf{t}' = \frac{\mathbf{R}_{,s}^*}{(\mathbf{R}_{,s}^* \cdot \mathbf{R}_{,s}^*)^{1/2}} = \frac{(x_i^* + u_i^*)_{,s} \, \mathbf{i}_i}{(\mathbf{R}_{,s}^* \cdot \mathbf{R}_{,s}^*)^{1/2}} \tag{13}$$

$$\mathbf{i}_i' = \frac{\mathbf{R}_{,i}^*}{(\mathbf{R}_{,i}^* \cdot \mathbf{R}_{,i}^*)^{1/2}} = \frac{(x_i^* + u_j^*)_{,i} \, \mathbf{i}_j}{(\mathbf{R}_{,i}^* \cdot \mathbf{R}_{,i}^*)^{1/2}} \tag{14}$$

where

$$, i = \frac{\partial}{\partial x_i}; \qquad i, j = 1, 2, 3.$$

Vectors $\mathbf{i}_i$ constitute an orthogonal triad according to assumptions on the kinematics of deformation.

Virtual Displacement and Deformation

The virtual displacement of points on the deformed member is

$$\mathbf{u}^* = \bar{u}_i^* \, \mathbf{i}_i, \tag{15}$$

so that the positon vector after additional displacement is

$$\bar{\mathbf{R}}^* = \mathbf{R}^* + \bar{\mathbf{u}}^*, \tag{16}$$

i.e.

$$\bar{\mathbf{R}}^* = (x_i^* + u_i^* + \bar{u}_i^*)\, \mathbf{i}_i, \tag{17}$$

or

$$\bar{\mathbf{R}}^* = \bar{\mathbf{R}} + e\mathbf{n}'', \tag{18}$$

where

$$\bar{\mathbf{R}} = (x_i + u_i + \bar{u}_i)\, \mathbf{i}_i \tag{19}$$

and

$$\mathbf{n}'' = \mathbf{R}_{,e}^* = \frac{(x_i^* + u_i^* + \bar{u}_i^*)_{,e}\, \mathbf{i}_i}{(\mathbf{R}_{,e}^* \cdot \mathbf{R}_{,e}^*)^{1/2}} \tag{20}$$

The projections of the virtual displacement $\mathbf{u}^*$ on axes and x_1, x_2 are determined by assumption (i.) about deformation:

$$\begin{aligned} \bar{u}_1^* &= \bar{\xi}_p - (x_2^* + u_2^* - x_{2p})\bar{\varphi}_p, \\ \bar{u}_2^* &= \bar{\eta}_p + (x_1^* + u_1^* + x_{1p})\bar{\varphi}_p, \end{aligned} \tag{21}$$

Functions $\bar{\xi}_p, \eta_p$ and $\bar{\varphi}_p$ represent virtual displacements of the cross section projection in the direction of axes x_1, x_2 and its rotation about the pole P.

Displacement component u_3^* is determined from conditions (ii.) and (iii.).

According to the assumption (ii.)

$$\bar{\varepsilon}_{s3} = \bar{\mathbf{R}}_{,s} \cdot \bar{\mathbf{R}}_{,3} = 0. \tag{22}$$

where

$$\bar{\mathbf{R}}_{,s} = (x_i + u_i + \bar{u}_i)_{,s}\, \mathbf{i}_i \tag{23}$$

and

$$\bar{\mathbf{R}}_{,3} = (x_i + u_i + \bar{u}_i)_{,3}\, \mathbf{i}_i \tag{24}$$

thus,

$$\bar{\varepsilon}_{s3} = \bar{u}_{i,is}\,(x_i + u_i)_{,3} + \bar{u}_{i,3}\,(x_i + u_i)_{,s} = 0 \tag{25}$$

Displacement component u_3^* i.e. $\bar{u}_3^*$ is significantly smaller then u_1^* and u_2^*, i.e. $\bar{u}_1^*$ and $\bar{u}_2^*$. Hence, products of the derivatives of these functions are neglected in the second-order theory:

$$u_{3,\alpha}^*\, u_{3,\beta}^* = 0 \quad \alpha,\beta = e,s,i \tag{26}$$

Substitution of (21) into equation (25), in conjuction with simplification (26) leads to

$$\begin{aligned} -\bar{u}_{3,s} &= \bar{\xi}_p\,(x_1 + u_1)_{,s} + \bar{\eta}_p\,(x_1 + u_2)_{,s} + \\ &+ \bar{\varphi}_p'\,[h_p + \xi_p\,(x_2 - x_{2p})_{,s} - \eta_p\,(x^{\mathrm{I}} - x_{1p})_{,s}] \end{aligned} \tag{27}$$

From condition (iii.) it follows that

$$\bar{\varepsilon}_{e3} = \bar{\mathbf{R}}_{,3}^{*} \cdot \bar{\mathbf{R}}_{,e}^{*} = 0 \tag{28}$$

leading to

$$\bar{\varepsilon}_{e3}^{*} = \bar{u}_{i,3}^{*} (x_i^{*} + u_i^{*})_{,e} + \bar{u}_{i,e}^{*} (x_i^{*} + u_i^{*})_{,3} = 0. \tag{29}$$

Substitution of (21) preceeding equation, in view of simplification (26) leads to

$$- \bar{u}_{3,e}^{*} = \bar{\xi}_p (x_1^{*} + u_1^{*})_{,e} + \bar{\eta}_p (x_2^{*} + u_2^{*})_{,e} +$$

$$+ \bar{\varphi}_p' [h_{np} + \xi_p (x_2^{*} - x_{2p}^{*})_{,e} - \eta_p (x_1^{*} - x_{1p}^{*})_{,e}]. \tag{30}$$

Intergrating $u_{3,e}^{*}$ and $u_{3,s}^{*}$

$$\bar{u}_3^{*} = \int_0^e \bar{u}_{3,e}^{*} \, de + \int_0^s \bar{u}_{3,s}^{*} ds + w_0 (x_3), \tag{31}$$

and making use of (27), (30) it finally follows that

$$\bar{u}_3^{*} = \bar{w}_0 - \bar{\xi}_p' (x_1^{*} + u_1^{*}) - \bar{\eta}_p' (x_2^{*} + u_2^{*}) -$$

$$- \bar{\varphi}_p' [\omega_p^{*} + \xi_p (x_2^{*} - x_{2p}) - \eta_p (x_1^{*} + x_{1p})] \tag{32}$$

The normal strain $\bar{\varepsilon}_{33}^{*}$ associated with virtual displacement field is defined by

$$\bar{\varepsilon}_{33}^{*} = (\bar{\mathbf{R}}_{,3}^{*} \cdot \bar{\mathbf{R}}_{,3}^{*})^{1/2} - (\mathbf{R}_{,3}^{*} \cdot \mathbf{R}_{,3}^{*})^{1/2}, \tag{33}$$

resulting in

$$\bar{\varepsilon}_{33}^{*} = (x_i^{*} + u_i^{*})_{,3} \cdot \bar{u}_{i,3}^{*}, \tag{34}$$

In scalar form, accounting for (26)

$$\bar{\varepsilon}_{33}^{*} = u_{1,3}^{*} \bar{u}_{1,3}^{*} + u_{2,3}^{*} \bar{u}_{2,3}^{*} + \bar{u}_{3,3}^{*} \tag{35}$$

The strain $\bar{\varepsilon}_{s3}^{*}$ becomes

$$\bar{\varepsilon}_{s3}^{*} = \bar{\mathbf{R}}_{,s}^{*} \cdot \bar{\mathbf{R}}_{,3}^{*} - \mathbf{R}_{,s}^{*} \cdot \mathbf{R}_{,3}^{*}, \tag{36}$$

i.e.

$$\bar{\varepsilon}_{s3}^{*} = e [\mathbf{R}_{,s}^{*} \cdot \bar{\mathbf{u}}_{,3}^{*} + \mathbf{R}_{,3}^{*} \cdot \bar{\mathbf{u}}_{,s}^{*}]_{,e} \tag{37}$$

Finally, making use of (9), (21) and (32).

$$\bar{\varepsilon}_{s3}^{*} = 2 \bar{\varphi}_p' e + e (u_{3,s}^{*} \cdot \bar{u}_{3,3}^{*} + u_{3,3}^{*} \cdot \bar{u}_{3,s}^{*}), \tag{38}$$

which according to (26) results in

$$\bar{\varepsilon}_{s3}^{*} = 2 \bar{\varphi}_p' e. \tag{39}$$

Equilibrium Conditions and the Governing Differential Equations of the Member

Consider the element of the member between cross sections $x_3 = x_{30}$ and $x_3 = x_{30} + dx_3$ before deformation. The stress vectors* acting as external forces on the bounding cross sections after the deformation are $\vec{\sigma}_3$ and $\vec{\sigma}_3 + \vec{\sigma}_{3,3}\, dx_3$. A surface load p is applied over the middle surface of the member.

Let $\bar{W}$ be the work of external forces and $\bar{U}$ the work of internal forces corresponding to the given virtual displacement vector $\mathbf{u}^*$. Then,

$$\bar{W} + \bar{U} = 0 \tag{40}$$

The work of external forces on the unit lenght of the member is

$$\bar{W} = \int_F (\vec{\sigma}_{3,3}\, \bar{\mathbf{u}} + \vec{\sigma}_3 \cdot \bar{\mathbf{u}}^*_{,3})\, dF + \int_S \bar{\mathbf{p}} \bar{\mathbf{u}}\, ds, \tag{41}$$

where the first integral is taken over the entire area F of the cross section while the latter one is taken over the entire length of the cross-sectional center line.

As a result of introduced assumptions four components of the virtual strain tensor vanish:

$$\bar{\varepsilon}^*_{ee} = \bar{\varepsilon}^*_{se} = \bar{\varepsilon}^*_{ze} = \varepsilon^*_{ss} = 0, \tag{42}$$

Therefore, the work of internal force per unit of lenght is

$$\bar{U} = -\int_F (\sigma_{33}\, \bar{\varepsilon}^*_{33} + \sigma_{s3}\, \bar{\varepsilon}^*_{s3})\, dF, \tag{43}$$

where

$$\sigma_{3i} = \vec{\sigma}_3 \cdot \mathbf{i}_i' \tag{44}$$

$$\sigma_{s3} = \vec{\sigma}_3 \cdot \mathbf{t}' \tag{45}$$

Vector $\bar{\sigma}_3$ can be defined in terms of its projections on the axes of the fixed coordinate system

$$\sigma_3 = \tilde{\sigma}_{3i} \cdot \mathbf{i}_i \tag{46}$$

From (44) and (46) it follows in view of (6) and (14):

$$\begin{aligned}
\tilde{\sigma}_{31} &= \sigma_{11} - \varphi_p\, \sigma_{32} + [\xi_p' - \varphi_p\, (x_2^* - x_{2p})]\, \sigma_{33} \\
\tilde{\sigma}_{32} &= \varphi_p\, \sigma_{31} + \sigma_{32} + [\eta_p - \varphi_p\, (x_1^* - x_{1p})]\sigma_{33} \\
\tilde{\sigma}_{33} &= [\xi_p' - \varphi_p'\, (x_2^* - x_{2p})]\, \sigma_{31} + [\eta_p + \varphi_p^*\, (x_1^* - x_{1p})]\, \sigma_{32} + \sigma_{33}
\end{aligned} \tag{47}$$

Using (15) and (46) the work of external forces is written as

$$\bar{W} = \int_F (\tilde{\sigma}_{3i,3}\, \bar{u}_i + \tilde{\sigma}_{3i}\, \bar{u}^*_{i,3})\, dF + \int_s \bar{p}_i\, \bar{u}_i\, ds \tag{48}$$

where

$$\bar{p}_i = \bar{\mathbf{p}} \cdot \mathbf{i}_i \tag{49}$$

The work of internal forces, after substitution of (35) and (39) becomes:

$$\bar{U} = -\int_F [\sigma_{33}\, (u^*_{1,3}\, \bar{u}^*_{1,3} + u^*_{2,3}\, \bar{u}^*_{2,3} + \bar{u}^*_{3,3}) + 2\, \sigma_{s3}\, \varphi_p' e]\, dF \tag{50}$$

Substituting derived relations for $\overline{W}$ and $\overline{U}$ into eg. (40) using formulas (6), (21) and 32), and collectiong the terms multiplying $\overline{w}_0, \overline{\xi}_p, \ldots, \overline{\varphi}_p$ obtained is the equation if form:

$$f_1 \overline{w}_0 + f_2 \overline{\xi}_p + f_3 \overline{\eta}_p + f_4 \overline{\varphi}_p - f_5 \overline{\xi}_p - f_6 \overline{\eta}_p' - f_7 \overline{\varphi}_p' = 0, \tag{51}$$

where

$$\begin{aligned}
f_1 &= \int_F \{\sigma_{33}' - [\sigma_{31} \xi_p' + \sigma_{23} \eta_p' - (\sigma_{31} (x_2^* - x_{2p}) - \sigma_{23} (x_1^* - \\
&\quad - x_{1p})) \varphi_p']'\} \, dF + \int_s \bar{p}_3 \, ds \\
f_2 &= \int_F \{\sigma_{31}' - [\sigma_{23} \varphi_p - (\xi_p' - \varphi_p' (x_2^* - x_{2p})) \sigma_{33}]'\} \, dF + \int_s \bar{p}_1 \, ds \\
f_3 &= \int_F \{\sigma_{23}' - [\sigma_{31} \varphi_p - (\eta_p' + \varphi_p' (x_1^* - x_{1p})) \sigma_{33}]'\} \, dF + \int_s \bar{p}_2 \, ds \\
f_4 &= \int_F \{\sigma_{23} (x_1^* - x_{1p}) - \sigma_{31} (x_2^* - x_{2p}) + [\xi_p \sigma_{23} - \eta_p \sigma_{31} - \\
&\quad - \sigma_{33} (\xi_p' (x_2^* - x_{2p}) - \eta_p' (x_1^* - x_{1p})) + \varphi_p' \sigma_{33} ((x_1^* - x_{1p})^2 + \\
&\quad + (x_2^* - x_{2p})^2)]'\} \, dF + \int_s [\bar{p}_2 (x_1 - x_{1p}) - \bar{p}_1 (x_2 - x_{2p}) + \xi_p \bar{p}_2 - \\
&\quad - \eta_p \bar{p}_1 - \varphi_p (\bar{p}_1 (x_1 - x_{1p}) + \bar{p}_2 (x_2 - x_{2p}))] \, ds \\
f_5 &= \int_F \{\sigma_{33}' x_1^* - \sigma_{31} - \varphi_p [\sigma_{33}' (x_2^* - x_{2p}) - \sigma_{23}] + \sigma_{33}' (\xi_p + x_{2p} \varphi_p) + \\
&\quad + \tau' x_1^*\} \, dF + \int_s \bar{p}_3 [x_1 - \varphi_p (x_2 - x_{2p}) + \xi_p] \, ds \\
f_6 &= \int_F \{\sigma_{33} x_1^* - \sigma_{23} + \varphi_p (\sigma_{33}' x_1^* - \sigma_{31}) + \sigma_{33}' (\eta_p - x_{1p} \varphi_p) + \\
&\quad + \tau' x_2^*\} \, dF + \int_s \bar{p}_3 [x_2 + \varphi_p (x_1 - x_{1p}) + \eta_p] \, ds \\
f_7 &= \int_F \{\sigma_{33} \omega_p^* - \sigma_{31} (x_2^* - x_{2p}) - \sigma_{23} (x_1^* - x_{1p}) + 2 \sigma_{s3} \varphi_p' e + \\
&\quad + \xi_p (\sigma_{33}' x_2^* - \sigma_{23}) - \eta_p (\sigma_{33}^* x_1^* - \sigma_{31}) - \\
&\quad - \sigma_{33}' (\xi_p x_{2p} - \eta_p x_{1p}) + \tau' \omega_p^*\} \, dF + \\
&\quad + \int_S \bar{p}_3 [\omega_p + \xi_p (x_2 - x_{1\mu}) - \eta_p (x_1 - x_{1p})] \, ds.
\end{aligned} \tag{52}$$

i

$$\tau = \sigma_{31} [\xi_p' - \varphi_p (x_2^* - x_{2p})] + \sigma_{23} [\eta_p' + \varphi_p' (x_1^* - x_{1p})]$$

and

$$\tau = \sigma_{31} [\xi_p' - \varphi_p (x_2^* - x_{2p})] + \sigma_{23} [\eta_p' + \varphi_p' (x_1^* - x_{1p})]$$

Since the equation (51) must be satisfied for arbitrary values of parameters $w_0. \ldots, \overline{\varphi}_p'$ the terms f_m $(m = 1,2, \ldots, 7)$ must be equal to zero

$$f_m = 0, \qquad m = 1, 2, \ldots, 7 \tag{53}$$

Integration of those terms over the surface leads to the following system of equations of equilibrium:

$$
\begin{aligned}
& N' - (Q_1 \xi_p' + Q_2 \eta_p' + T_p \varphi_p')' + p_3 = 0 \\
& Q_1' + [N(\xi_p' + x_{2p} \varphi_p') - M_2 \varphi_p' - Q_2 \varphi_p]' + p_1 = 0 \\
& Q_2' + [N(\eta_p' - x_{1p} \varphi_p') + M_1 \varphi_p' + Q_1 \varphi_p]' + p_2 = 0 \\
& T_p' + [Q_2 \xi_p - Q_1 \eta_p - \xi_p'(M_2 - N x_{2p}) + \eta_p'(M_1 - N x_{1p}) + \\
& + \varphi_p'(N i_p^2 + 2 M_1 \beta_1 + 2 M_2 \beta_2 + M_\omega \beta_\omega)]' + m_p + p_2 \xi_p - p_1 \eta_p - \\
& - \varphi_p \tilde{m}_p = 0 \\
& M_1' - Q_1 + m_1 - \varphi_p (M_2' - Q_2 + m_2) + (N' + p_3)(\xi_p - x_{2p} \varphi_p) + J_1 = 0 \\
& M_2' - Q_2 + m_2 + \varphi_p (M_1' - Q_1 + m_1) + (N' + p_3)(\eta_p - x_{1p} \varphi_p) + J_2 = 0 \\
& M_\omega' - T_\omega + m_\omega + \xi_p (M_2' - Q_2 + m_2) - \eta_p (M_1' - Q_1 + m_1) - \\
& - (N' + p_3)(\xi_p x_{2p} - \eta_p x_{1p}) + J_3 = 0
\end{aligned}
\tag{54}
$$

where

$$
\begin{aligned}
& N = \int_F \sigma_{33}\, dF \quad \text{— axial force,} \\
& Q_i = \int_F \sigma_{3i}\, dF, \ i = 1, 2 \quad \text{— shear forces,} \\
& M_i = \int_F \sigma_{33} x_i^*\, dF, \ i = 1, 2 \quad \text{— bending moments,} \\
& M_\omega = \int_F \sigma_{33} \omega_p^*\, dF \quad \text{— bimoment} \\
& T_p = \int_F [\sigma_{23}(x_1^* - x_{1p}) - \sigma_{31}(x_2'' - x_{2p})]\, dF \quad \text{— torque,} \\
& T_s = 2 \int_F \sigma_{33}\, e dF \quad \text{— st. Venant's torque,} \\
& T_\omega = T_p - Ts \quad \text{— warping torque,} \\
& J_1 = \int_F \tau' x_1^*\, dF, \quad J_2 = \int_F \tau' x_2^*\, dF, \quad J_3 = \int_F \tau' \omega_p^*\, dF
\end{aligned}
\tag{55}
$$

and

$$
\begin{aligned}
& \int_F \sigma_{33} [(x_1 - x_{1p})^2 + (x_2 - x_{2p})^2]\, dF = \\
& (N i_p^2 + 2 M_1 \beta_1^2 + 2 M_2 \beta_2 + M_\omega \beta_\omega) \\
& p_i = \int_S \bar{p}_i\, ds; \qquad m_i = \int_S p_3 x_i\, ds, \qquad i = 1, 2 \\
& m_p = \int_S [\bar{p}_2 (x_1 - x_{1p}) - \bar{p}_1 (x_2 - x_{2p})]\, ds \\
& \tilde{m}_p = \int_S [\bar{p}_1 (x_1 - x_{1p}) + \bar{p}_2 (x_2 - x_{2p})]\, ds \\
& m_\omega = \int_S \bar{p}_3 \omega_p\, ds
\end{aligned}
$$

Quantities $i_p^2, \ldots, \beta_\omega$ are defined as:

$$
i_p^2 = x_{1p}^2 + x_{2p}^2 + \frac{I_{11} + I_{22}}{F}
$$

$$\beta_1 = \frac{\int\limits_F x_1^* (x_2^{*2} + x_2^{*2})\, dF}{2\, I_{11}} - x_{1p} \tag{56}$$

$$\beta_2 = \frac{\int\limits_F x_2^* (x_1^{*2} + x_2^{*2})\, dF}{2\, I_{22}} - x_{2p}$$

$$\beta_3 = \frac{\int\limits_F \omega_p^* (x_1^{*2} + x_2^{*2}\, dF}{I_{\omega\omega}},$$

where

$$I_{11} = \int\limits_F x_1^{*2}\, dF,\ I_{22} = \int\limits_F x_2^{*2}\, dF,\ I_{\omega\omega} = \int\limits_F \omega_p^{*2}\, dF$$

The first six of equations (54), are alredy known equations of the second order theory for the member. The seventh equation of equilibrium represents a new relation characterizing the behaviour of the thin- walled member.

Linearisation of the system of equations (54), using the solutions of the linear theory for stress resultants in computing the products of the stress resultants and displacements, leads to the following system of equations

$$N' + p_3 - (\xi_p' Q_1 + \eta_p Q_2 + \varphi_p' T_p) = 0$$
$$Q_1' + p_1 + [N(\xi_p + x_{2p}\varphi_p') - \varphi_p' M_2 - \varphi_p' Q_2]' = 0$$
$$Q_2' + p_2 + [N(\eta_p - x_{1p}\varphi_p) + \varphi_p' M_1 + \varphi_p' Q_1]' = 0$$
$$T_p' + m_p + [\xi_p Q_2 - \eta_p Q_1 + \eta_p' (M_1 - x_1 N) -$$
$$- \xi_p' (M_2 - x_{2p} N) + \varphi_p (N i_p^2 + 2 M_1 \beta_1 + 2 M_2 \beta_2 +$$
$$+ M_\omega \beta_\omega)]' + p_2 \xi_p - p_1 \eta_p - \varphi_p \tilde{m}_p = 0$$
$$M_1' - Q_1 + m_1 = 0$$
$$M_2' - Q_2 + m_2 = 0$$
$$M_\omega' - T_\omega + m_\omega = 0$$

neglecting the integals J_1, J_2 and J_3, whose influence on the state of stress is insignificant.

The last equations have the same form as the corresponding equations of the linear theory.

Equations (57) can be further reduced to a system of four equations. Making use of well known relation of the linear theory [1] it follows:

$$E'F w_0'''' - (\xi_D' Q_1 + \eta_D' Q_2 + \varphi_D' T_p) = p_3$$
$$E'J_{11} \xi_D'''' - [N(\xi_D' + x_{2D}\varphi_D')]' + (\varphi_D M_2)'' + (\varphi_D m_2)' = p_1 + m_1'$$
$$E'J_{22} \eta_D'''' - [N(\eta_D' - x_{1D}\varphi_D')]' - (\varphi_D M_1)'' - (\varphi_D m_1)' = p_2 + m_2'$$
$$E'J_{\omega\omega} \varphi_D'''' - GK\varphi_D'' - [\varphi_D' (N i_\eta^2 + 2\beta_1 M_1 + 2\beta_2 M_2 + \beta_\omega M_\omega)]' - \tag{58}$$

$$- x_{2D} (N \xi'_D)' + x_{1D} (N \eta'_D)' + \xi''_D M_2 - \eta''_D M_1 -$$
$$- \xi'_D m_1 + \eta'_D m_2 + \varphi_D \tilde{m}_D = m_D + m'_\omega$$

where D, $(P \equiv D)$, is the shear centre of the cross section.

The last there equations of the system (58) coincide with Vlasov's equations [2] for

$$p_i = m_i = m_D = \tilde{m}_D = m_\omega$$

and

$$M_\omega = 0.$$

Neglecting the expression in the brackets, the first equation of the system (58) becomes the same as the corresponding equation of the linear theory.

REFERENCES

[1] Kollbrunner C. F. and Hajdin N., *Dünnwandige Stäbe*, Band 1, Springer--Verlag, Berlin—Heidelberg—New York, 1972.

[2[Власов В. З., *Изабранные шруди*, том 11 танкостенные упруиге стрежни, Москва 1963.

NICHT-LINEARE THEORIE DES DÜNNWANDIGEN STABES MIT OFFENEM QUERSCHNITT

Zusammenfassung

Die vorliegende Arbeit stellt eine Erweiterung des Beitrages [1] an die Theorie der zweiten Ordnung des dünwandigen Stabes mit offenem Querschnitt.

Durch die Anwendung des Prinzips der viruellen Verschiebugen bei der Aufstellung der Gleichgewichtsbedingungen an dem deformierten Element, werden alle charakteristischen Beziehungen für die Schnittkräfte auf eine einheitliche Art erhalten. Dies gilt sowohl fur die Schnittkräfte im klassischen Sinne als auch fur die dem dünnwandigen Stab zugehörigen Schnittkräfte, wie Bimoment, St. Venantsches Torsionsmoment und Wölbtorsionsmoment.

Die abgeitete Differentialgleichungen führen zu bekannten Gleichungen für eine bestimmte Belastung und Form des Querschitts.

NELINEARNA TEORIJA TANKOZDNOG ŠTAPA SA OTVORENIM POPREČNIM PRESEKOM

Izvod

U radu se daje teorija drugog reda tankozidnog štapa otvorenog poprečnog preseka zasnovana na teorijskom konceptu izloženom u knjizi Kollbrunnera i Hajdina [1].

Predložena teorija razlikuje se od konvencionalnog pristupa u jedinstvenosti razmatranja svih presečenih sila uključujući Bimoment, St. Vevant'ov moment i torzioni momenat deplanacije.

Sistem od sedam jednačina izvedenih primenom principa virtelnog rada uključuje i jednačinu koja povezuju veličine karakteristične za ograničenu torziju sa ostalim presečnim silama.

Iz izvedenih jednačina mogu se uz odgovarajuća uprošćenja i zanemarenja dobiti poznate veze za posebne vrste opterećenja ili oblika poprečnog preseka.

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