TEORISJKA I PRIMENJENA **MEHANIKA** 8, pp. 143—148, 1982, UDK 531.391

ON OPTIMAL CONTROL OF MOTION OF A MECHANICAL SYSTEM OF A RESTRICTED KINETIC ENERGY

J. Vuković

(Received, July 10, 1981)

The present paper deals with the problem of optimal control of motion of a mechanical system with a restricted kinetic energy when such a system is subject to the action of control which has the nature of a generalised force of a restricted intensity. The motion is considered in a phase space V_{2n} of q^α, p_α variables ($\alpha = 1,2,\ldots,n$), so that the condition of restriction of the kinetic energy defines a closed set B in that space. Here, the optimal trajectory can be, partially, located within the open interior of the region B, and partially, on the boundary of the region B. By applying the principle of the maximum and other necessary conditions [2, 3], the structure of the optimal control is defined, and a sufficient number of equations and conditions are given for the solution of the problem in its final form.

We shall consider a holonomic, sceleronomic mechanical system which, in addition to potential and non-potential forces, is also acted upon by controls having the nature of a generalised force. Differential equations of motion of such a system in a phase space V_{2n} have the form of

$$\begin{cases} \dfrac{dq^\alpha}{dt} = \dfrac{\partial H}{\partial p_\alpha} \\ \dfrac{dp_\alpha}{dt} = -\dfrac{\partial H}{\partial q^\alpha} + Q_\alpha^N + u_\alpha, \quad \alpha = 1,2,\ldots,n. \end{cases} \tag{1}$$

where H is Hamilton's function of a scleronomic mechanical system, Q_α^N — is a non-potential force, while u_α — is the force of control.

Let the controls of a restricted intensity be permitted, i.e., let the set U be determined by the inequality

$$a^{\alpha\beta} u_\alpha u_\beta - K^2 \leqslant 0, \quad K = \text{const}, \tag{2}$$

where $a^{\alpha\beta}$ is a contravariant metric tensor of the configuration space R_n.

Let the kinetic energy of the system be restricted, i.e.

$$T - C^2 \leqslant 0, \qquad C = \text{const.}, \tag{3}$$

thus, the region B of permissible phase positions is defined.

Let the criterion of optimality be given in the form of a requirement that the functional

$$\mathcal{J} = \int_{t_0}^{t_1} \omega\,(q^1, q^2, \ldots, q^n; p_1, p_2, \ldots, p_n)\, \mathrm{dt} \tag{4}$$

has the minimum value, where, in a general case, the limit t_1 is not determined.

Let the initial and final phase conditions of the system belong to the open interior of the region B, and let s be the number of sections of the optimal trajectory lying on the boundary of the region B. Then,

$$T - C^2 < 0 \;\forall\; t \in [t_0, \tau_1)\,(\tau_2, \tau_3), \ldots, (\tau_{2s-2}, \tau_{2s-1})\,(\tau_{2s}, t_1] \tag{5}$$

$$T - C^2 = 0 \;\forall\; t \in [\tau_{2k-1}, \tau_{2k}], \tag{6}$$

where τ_{2k-1} are the times of arrival of the phase point to the boundary of the region B, while τ_{2k} $(k = 1, 2, \ldots, s)$ are the times of departure of the phase point from the boundary.

The principle of the maximum provides the sufficient number of necessary conditions which are satisfied by each part of the optimal trajectory within the open interior (5) of the region B. In that case, considering (1) and (4), Pontryagin's function has the form of

$$\mathcal{H} = \lambda_\alpha \frac{\partial H}{\partial p_\alpha} + \nu^\alpha \left(- \frac{\partial H}{\partial q^\alpha} + Q_\alpha^N + u_\alpha \right) - \omega, \tag{7}$$

and hence, on the grounds of the principle of the maximum, we have on the optimal trajectory

$$(\mathcal{H})_{\substack{0\\u_\alpha}} = \sup_{u_\alpha \in U} (\mathcal{H}), \tag{8}$$

where the vector λ_α, ν^α satisfies the system of equations

$$\begin{cases} \dot{\lambda}_\alpha = - \dfrac{\partial \mathcal{H}}{\partial q^\alpha} = - \lambda_\beta \dfrac{\partial^2 H}{\partial q^\alpha\, \partial p_\beta} - \nu^\beta \left(- \dfrac{\partial^2 H}{\partial q^\alpha\, \partial p^\beta} + \dfrac{\partial Q_\beta^N}{\partial q^\alpha} \right) + \dfrac{\partial \omega}{\partial q^\alpha} \\[2ex] \dot{\nu}^\alpha = - \dfrac{\partial \mathcal{H}}{\partial p_\alpha} = - \lambda_\beta \dfrac{\partial^2 H}{\partial p_\alpha\, p_\beta} - \nu^\beta \left(- \dfrac{\partial^2 H}{\partial p_\alpha\, \partial q^\beta} + \dfrac{\partial Q_\beta^N}{\partial p^\alpha} \right) - \dfrac{\partial \omega}{\partial p_\alpha}. \end{cases} \tag{9}$$

Since, in our case, the function $\mathcal{H}$ is linearly dependent on the control u_α, then, by excluiding the case $\nu^\alpha = 0$, the condition (8) will be fulfilled on the boundary of the region U

$$a^{\alpha\beta}\, u_\alpha\, u_\beta - K^2 = 0. \tag{10}$$

In that case, we can introduce a new function

$$\mathcal{H}_1 = \mathcal{H} - \eta\,(t)\,(a^{\alpha\beta}\, u_\alpha\, u_\beta - K^2). \tag{11}$$

thus, since the boundary (10) is smooth along its entire length, the condition (8) obtains the form

$$\left(\frac{\partial \mathcal{H}_1}{\partial u_\alpha} \right)_{\overset{0}{u_\alpha}} = 0, \; \left(\frac{\partial^2 \mathcal{H}_1}{\partial u_\alpha\, \partial u_\beta} \right)_{\overset{0}{u_\alpha}} u^\alpha\, u^\beta < 0, \tag{12}$$

whence

$$u_\alpha^0 = \frac{1}{2\eta} a_{\alpha\beta} \nu^\beta, \qquad \eta(t) > 0 \tag{13}$$

while the vector λ_α, ν^α satisfies the system of equations

$$\begin{cases} \dot{\lambda}_\alpha = -\dfrac{\partial \mathcal{H}_1}{\partial q} = -\dfrac{\partial \mathcal{H}}{\partial q^\alpha} + \eta \dfrac{\partial a^{\beta\gamma}}{\partial q^\alpha} u_\beta u_\gamma, \\ \dot{\nu}^\alpha = -\dfrac{\partial \mathcal{H}_1}{\partial p_\alpha} = -\dfrac{\partial \mathcal{H}}{\partial p_\alpha}. \end{cases} \tag{14}$$

Let us observe that here we have excluded the case where the function is stationary over the region U, i.e. when

$$\frac{\partial \mathcal{H}}{\partial u_\alpha} = 0,$$

or $\nu^\alpha = 0$. This case belongs to the class of problems of the so-called singular controls [1], and will not be dealt with in this paper.

When the optimal trajectory lies on the boundary (6) of the region B, then following condition is satisfied

$$\frac{dT}{dt} = \frac{\partial T}{\partial q^\alpha}\frac{\partial H}{\partial p_\alpha} + \frac{\partial T}{\partial p_\alpha}\left(-\frac{\partial H}{\partial q^\alpha} + Q_\alpha^N + u_\alpha\right) = 0,$$

where, for the holonomic, scleronomic system,

$$H = T + \Pi, \quad \frac{\partial H}{\partial p_\alpha} = \frac{\partial T}{\partial p_\alpha}$$

and hence, we have

$$\frac{\partial T}{\partial p_\alpha}\left(-\frac{\partial \Pi}{\partial q^\alpha} + Q_\alpha^N + u_\alpha\right) = 0 \tag{15}$$

On the grounds of this inequality, which represents a new relationship between the phase coordinates and optimal controls, we propose to form the following function

$$\mathcal{H}_2 = \mathcal{H} - \mu(t)\frac{\partial T}{\partial p_\alpha}\left(-\frac{\partial \Pi}{\partial q_\alpha} + Q_\alpha^N + u_\alpha\right)$$

or

$$\mathcal{H}_2 = \lambda_\alpha \frac{\partial T}{\partial p_\alpha} + \nu^\alpha\left(-\frac{\partial H}{\partial q^\alpha} + Q_\alpha^N + u_\alpha\right) - \omega - \eta\,(a^{\alpha\beta} u_\alpha u_\beta - K^2) - - \mu \frac{\partial T}{\partial p_\alpha}\left(-\frac{\partial \Pi}{\partial q^\alpha} + Q_\alpha^N + u_\alpha\right). \tag{16}$$

Analogously to the preceding considerations, the optimality conditions have the following form

$$\left(\frac{\partial \mathcal{H}_2}{\partial u_\alpha}\right)_{u_\alpha^0} = 0, \quad \left(\frac{\partial^2 \mathcal{H}_2}{\partial u_\alpha\, \partial u_\beta}\right)_{u_\alpha^0} u_\alpha u_\beta < 0 \tag{17}$$

whence

$$u_\alpha^0 = \frac{1}{2\eta}\,(a_{\alpha\beta}\, \nu^\beta - \mu\, p_\alpha), \quad \eta > 0 \tag{18}$$

while the vector λ_α, ν^α satisfies the system of equations

$$\begin{cases} \dot\lambda = -\dfrac{\partial \mathcal{H}_2}{\partial q^\alpha} = -\dfrac{\partial \mathcal{H}}{\partial q^\alpha} + \eta\,\dfrac{\partial a^{\beta\gamma}}{\partial q^\alpha}\, u_\beta\, u_\gamma + \mu\,\dfrac{\partial^2 T}{\partial q^\alpha \partial p_\beta}\left(-\dfrac{\partial \Pi}{\partial q^\beta} + Q_\beta^N + u_\beta\right) + \\ \qquad + \mu\,\dfrac{\partial T}{\partial p_\beta}\left(-\dfrac{\partial^2 \Pi}{\partial q^\alpha\, \partial q^\beta} + \dfrac{\partial Q_\beta^N}{\partial q^\alpha}\right), \\ \dot\nu = -\dfrac{\partial \mathcal{H}_2}{\partial p_\alpha} = -\dfrac{\partial \mathcal{H}}{\partial p_\alpha} + \mu\,\dfrac{\partial^2 T}{\partial p_\alpha\, p_\beta}\left(-\dfrac{\partial \Pi}{\partial q^\beta} + Q_\beta^N + u_\beta\right) + \mu\,\dfrac{\partial T}{\partial p^\beta}\dfrac{\partial Q_\beta^N}{\partial p_\alpha}. \end{cases} \tag{19}$$

From the preceding we can conclude that, provided $q^\alpha(t), p_\beta(t)\ \forall\ t \in [\tau_{2k-1}, \tau_{2k}]$ is an optimal trajectory which lies entirely on the boundary of the region B, there exists a continuous vector function $\lambda_\alpha(t)$, $\nu^\alpha(t)$, as well as continuous function $\mu(t)$ and $\eta(t)$ so that the equations (1) and (19), and the condition (8) are satisfied.

In addition, $\dfrac{d\mu}{dt} \leqslant 0$ which appears as a consequence of the fact that the variated function also lies within the boundary B.

The complete solution of the problem of optimal congrol of motion of a holonomic, scleronomic mechanical system with restrictions (2) and (3) consists in determining the functions $q^\alpha(t)$, $p_\alpha(t)$, $\lambda_\alpha(t)$, $\nu^\alpha(t)$, $u_\alpha(t)$, $\mu(t)$, $\eta(t)$ in all sub-intervals of the interval $[t_0, t_1]$, of which there are $2s + 1$. Since in each sub-interval we have $5n$ functions q^α, p_α, λ_α, ν^α, u_α, and one function η, and since the function μ appears is a sub-intervals only, in which the trajectory lies on the boundary of the region B, the total number of functions looked for is $(5n + 1)(2s + 1) + s$. For their determination we have $4n(2s + 1)$ equations (1), (14) and (19), $n(2s + 1)$ equations (13) and (18), $2s + 1$ equations (10) and s equations (15), thus all these equations are equal to the mumber of functions that are looked for.

By integrating the differential equations we obtain $4n(2s + 1)$ constants of integration. In addition, we also have the terms $\tau_1, \tau_2, \ldots, \tau_{2s}, t_1$ as unknows so that the total number of unknown constants is $(4n + 1)(2s + 1)$. To find them all we have $4n$ initial and final conditions, and conditions of transversality. We shall find the remaining conditions on the grounds of behaviour of functions in points $\tau_1, \tau_2, \ldots, \tau_{2s}, t_1$. Namely, the optimal trajectory is a continuous one over the enire interval $]t_0, t_1]$, and we have

$$\begin{cases} q^\alpha(\tau_{2k-1} - 0) = q^\alpha(\tau_{2k-1} + 0), \quad p_\alpha(\tau_{2k-1} - 0) = p_\alpha(\tau_{2k-1} + 0), \\ q^\alpha(\tau_{2k} - 0) = q^\alpha(\tau_{2k} + 0), \quad p_\alpha(\tau_{2k} - 0) = (\tau_{2k} + 0), \quad k = 1, 2, \ldots, s \end{cases} \tag{20}$$

which yields $4ns$ conditions. At the same time, in the same points, the vector λ_α ν^α has discontinuities]3], which in our case have the form

$$\begin{cases} \lambda_\alpha(\tau_{2k-1} - 0) = \lambda_\alpha(\tau_{2k-1} + 0), \quad \nu^\alpha(\tau_{2k-1} - 0) = \nu^\alpha(\tau_{2k-1} + 0), \\ \lambda_\alpha(\tau_{2k} - 0) = \lambda_\alpha(\tau_{2k} + 0) + \alpha \dfrac{\partial T}{\partial q^\alpha}, \quad \nu^\alpha(\tau_{2k} - 0) = \nu^\alpha(\tau_{2k} + 0) + \alpha \dfrac{\partial T}{\partial p_\alpha}. \end{cases} \tag{21}$$

Of these conditions there are, also, $4ns$. The functions $\mathcal{H}$ satisfies $2s + 1$ conditions of the form

$$(\mathcal{H})_{\tau_{2k-1}-0} = (\mathcal{H})_{\tau_{2k-1}+0}, \ (\mathcal{H})_{\tau_{2k}-0} = (\mathcal{H})_{\tau_{2k}+0}, \ (\mathcal{H})_{t_1} = 0 \tag{22}$$

so that, now, we have $(4n + 1)(2s + 1)$ conditions. But, in conditions (21) there appear s unknown constants α. In view of the fact that at the moment of arrival of the phase point at the boundary of the region B, the following condition is fulfilled

$$(T)_{\tau_{2k-1-0}} = C^2, \tag{23}$$

we have obtained the total number of conditions required for the determination of unknown constants.

REFERENCES

[1] Габасов Р., Кириллова Ф.М., *Особые оптимальные управления*, Наука, Москва, 1973.

[2] Leitmann, G., *An Introduction to Optimal Control*, McGraw-Hill Book Company, New York, 1966.

[3] Понтрягин Л.С., *Математическая теория оптимальных процессов*, Наука, Москва, 1976.

ОБ ОПТИМАЛЬНОМ УПРАВЛЕНИИ ДВИЖЕНИЕМ МЕХАНИЧЕСКОЙ СИСТЕМЫ ОГРАНИЧЕНОЙ КИНЕТИЧЕСКОЙ ЭНЕРГИИ

Р е з ю м е

Анализированно оптимальное движение голономной, склерономной механической системы ограниченной кинетической энергии под влиянием управления вида ограниченной обобщенной силы. Применением принципа максимума Понтрягина определена структура управления и поставлены уравнения и условия, число которых достаточно для конечного решения.

O OPTIMALNOM UPRAVLJANJU KRETANJEM MEHANIČKOG SISTEMA OGRANIČENE KINETIČKE ENERGIJE

I z v o d

Razmatran je problem optimalnog upravljanja kretanjem holonomnog, skleronomnog mehaničkog sistema ograničene kinetičke energije, pod dejstvom upravljanja koje ima prirodu generalisane sile, ograničenog inteziteta.

Kretanje je posmatrano u faznom prostoru, tako da, uslov ograničenosti kinetičke energije, određuje jedan zatvoren skup u tom prostoru. Korišćenjem Pontrjaginovog principa maksimuma i drugih neophodnih uslova, određena je struktura optimalnog upravljanja i postavljen dovoljan broj jednačina i uslova za rešenje pooblema u konačnom obliku.

Josif Vuković, Mašinski fakultet
27. marta 80, 11000 Beograd