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AN EXTREMAL VARIATIONAL PRINCIPLE FOR A CLASS OF BOUNDARY VALUE PROBLEMS

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1. Introduction

Variational principles play an important role in mechanics, for theoretical as well as pragmatic reasons. They provide, among the other things, a basis for many methods for finding approximate solutions of linear and non-linear problems. When the variational principle is used sa a direct method, the problem of estimating the erorr involved in the approximation scheme may be connected with the values that the functional takes on the approximate solution. This is an important property of variational principles. However, to achieve this, the variational principle must satisfy certain restrictions, that usually amount to the requirement that the principle is extremal either maximal or minimal. The problem of using variational principles to estimate the error of an approximate solution is treated, for example, in [1], [2], [3], [4].

The aim of this paper is to construct an extremal variational principle for boundary value problems governed by a second-order difefrentional equation of the form

$$y''(1+y'^2)^{-2/3} - F(y, x) = 0, \qquad a > x > b \tag{1.1}$$

where $F(x, y)$ is an arbitrary function having the continuous derivatives with respect to both variables. The following boundary conditions will be considered.

$$a_3 y'(a) + a_1 y(a) = a_2, \tag{1.2}$$

$$b_3 y'(b) + b_1 y(b) = b_2; \tag{1.3}$$

here a, b, a_3, a_1, a_2, b_3, b_1 and b_2 are known constants. Equation (1.1) together with the boundary conditions (1.2) and (1.3) arises in various problems of post-buckling behaviour of elastic columns. In this context the function $F(y,x)$ takes the form $F(y,x) = -\mathrm{P}y/\mathrm{EI}(x)$, where P is the applied load and EI(x) is the stiffness.

After constructing a variational principle for the boundary value problem (1.1)—(1.3) we will use it to obtain an error estimate or approximate solutions. The methods of error estimation are basically the same as those developed in [4].

Finaly a concrete example will be treated for which we will determine the approximate solution by applying the Ritz method to the variational principle. Also an error estimate for this approximate solution will be given.

2. Extremum variational principle

It can bi easily verified that the boundary value problem (1.1)—(1.3) is equivalent to the condition that the following functional is stationary at y:

$$E(y) = \gamma [y(a), y(b)] + \int_a^b L(y, y', x)\, dx; \tag{2.1}$$

where,

$$L = (1 + y'^2)^{1/2} + f(y,x); f(y,x) = \int_0^y F(\xi, x)\, d\xi \tag{2.2}$$

$$\gamma [y(a), y(b)] = \frac{1}{b_1}\left\{1 + \left[\frac{1}{b_3}(b_2 - b_1 y(b))\right]^2\right\}^{1/2} - \frac{1}{a_1}\left\{1 + \left[\frac{1}{a_3}(a_2 - a_1 y(a))\right]^2\right\}^{1/2}, \tag{2.3}$$

under the condition that the variations of y at the endpoints are different from zero.

From (2.2) we find that the generalized momentum is given by

$$p = \frac{\partial L}{\partial y'} = y'(1 + y'^2)^{-1/2}. \tag{2.4}$$

Defining the Hamiltonian H by

$$H = p y' - L, \tag{2.5}$$

and using (2.4), the differential equation (1.1) can be written as

$$y' = p(1 - p^2)^{-1/2} \tag{2.6}$$

$$p' = F(y, x). \tag{2.7}$$

Assuming that (2.7) can be solved for y for every x in the interval $[a,b]$ so that y can be written as

$$y = F^{-1}(p', x), \tag{2.8}$$

and transforming the second term in $E(y)$ as follows

$$\int_a^b L dx = \int_a^b \{py' - H\}\, dx = [py]_a^b - \int_a^b \{p'y + H\}\, dx = [py]_a^b - \int_a^b \Lambda(p, p', x)\, dx, \tag{2.9}$$

where

$$\Lambda(p, p', x) = p' F^{-1}(p', x) + p^2(1 - p^2)^{-1/2} - [(1 - p^2)^{-1/2} + f(F^{-1}(p', x), x)]. \tag{2.10}$$

we get the dual variational principle in the form

$$G(p, y) = \gamma [y(a), y(b)] + [py]_a^b - \int_a^b \Lambda(p, p', x)\, dx. \tag{2.11}$$

At this point we note that the theory of complementary variational principles, as used by Arthurs [3] for example, requires that the functionals (2.1) and (2.11) be extremal, minimum and maximum, respectively (or vice versa). This restricts the class of functions $F(y,x)$ in the eequation (1.1) more than the procedure that we will follow. Namely, we define a new functional I by

$$I(y, p) \equiv E - G = \int_a^b [L(y, y', x) + \Lambda(p, p', x)]\, dx - [py]_a^b. \qquad (2.12)$$

The functional (2.12) is the basic functional to be used in this paper. Its main property is that it may be in minimum (maximum) on the solution of the problem (1.1)—(1 3) even in the case when E and G are not extremal. We intend to obtain approximate solutions to the problem (1.1)—(1.3) by minimizing I given by (2.12). This is a novel approach for obtaining approximate solutions.

It is obvious, from the procedure of constructing I, that on the exact solution of the problem (1.1)—(1.3) the value of I is equal to zero. To show that I is also stationary on the solution of (1.1)—(1.3) we calculate the first variation (treating y and p as independent) of (2.12).

$$\delta I = \int_a^b \left[\frac{\partial L}{\partial y}\delta y + \frac{\partial L}{\partial y'}\delta y' + \frac{\partial \Lambda}{\partial p}\delta p + \frac{\partial \Lambda}{\partial p'}\delta p'\right] dx - [\delta p y + p\delta y]_a^b. \qquad (2.13)$$

Using (2.2) and (2.10) in (2.19) and performing integration by parts, we get

$$\begin{aligned}\delta I = \int_a^b \{[&-y''(1+y'^2)^{-3/2} + F(y,x)]\,\delta y + \\ &+\left[-\frac{d}{dx}\left\{F^{-1} + \frac{1}{\dfrac{\partial F}{\partial p'}}(p' - F)\right\} + p(1-p^2)^{-1/2)}\right]\delta p\, dx - \\ &- [\delta p y + p\delta y]_a^b + y'(1-y'^2)^{-1/2}\,\delta y|_a^b - \\ &- \left[F^{-1} - \frac{1}{\dfrac{\partial F}{\partial p'}}(p' - F)\right]\delta p\,|_a^b. \end{aligned} \qquad (2.14)$$

From (2.14) ,after the use of (2.6), (2.7) and (2.8), we conclude that $\delta I = 0$, that is I is stationary on the solution of the problem (1.1)—(1.3). To examine the nature of the stationarity, we calculate the second variation of I

$$\begin{aligned}\delta^2 I = \frac{1}{2}\int_a^b \Bigg[&\frac{\delta^2 L}{\partial y^2}(\delta y)^2 + 2\frac{\partial^2 L}{\partial y\,\partial y'}(\delta y)(\delta y') + \frac{\partial^2 L}{\partial y'^2}(\delta y')^2 + \frac{\partial^2 \Lambda}{\partial p^2}(\delta p)^2 + \\ &+ 2\frac{\partial^2 \Lambda}{\partial p\,\partial p'}(\delta p)(\delta p') + \frac{\partial^2 \Lambda}{\partial p'^2}(\delta p')^2\Bigg] dx - \delta p\,\delta y|_a^b \end{aligned} \qquad (2.15)$$

Using (2.2) and (2.10) in (2.15) and calculating the coefficients with variations of y and p on the curves where the first variation vanishes i.e., by using (1.1), (2.4), (2.6) and (2.7), the equation (2.15) becomes

$$\delta^2 I = -\, \delta p\, \delta y\,|_a^b + \frac{1}{2}\int_a^b \left\{ (1 + y'^2)^{-3/2} (\delta y')^2 + \frac{\partial F}{\partial y} (\delta y)^2 + \frac{1}{\frac{\partial F}{\partial y}} (\delta p')^2 + \right.$$

$$\left. + (1 - p^2)^{-3/2} (\delta p)^2 \right\} dx. \tag{2.16}$$

Now, from the fact that on the curves where the first variation vanishes the fol- owing relation holds*

$$\delta p = \delta y' \,(1 + y'^2)^{-3/2}, \tag{2.17}$$

lso that on integrating by parts in (2.16) we get

$$\delta^2 I = \frac{1}{2}\int_a^b \frac{1}{\frac{\partial F}{\partial y}} \left[\frac{d}{dx} \{\delta y' \,(1 + y'^2)^{-3/2}\} - \frac{\partial F}{\partial y}\, \delta y \right]^2 dx. \tag{2.18}$$

From the equation (2.18) we conclude that:

(*i*) The functional I has a local minimum on the solution of the equations (1.1)—(1.3) if

$$\frac{\partial F}{\partial y} > 0 \text{ for } x \in (a, b). \tag{2.19}$$

(*ii*) The functional I has a local maximum on the solution of the problem (1.1)—(1.3) if

$$\frac{\partial F}{\partial y} < 0 \text{ for } x \in (a,b). \tag{2.20}$$

It should be noted that in the case when $\partial F/\partial y$ changes sing in the interval $[a,b]$ we can not say anything about the nature of the stationarity of I. Also it is easy to see that either one of (2.19) or (2.20) does not imply that both E and G, given by (2.1) and (2.11) respectively, are extremal variational principles. However, if E and G are extremal principles then I is also extremal principle. Examples illustrating both situations are given in [4].

3. Error estimate procedure

Suppose we have an approxmate solution Y to the equation (1.1) which is satisfying the boundary conditions (1.2) and (1.3). Our goal is to estimate the L_2 norm of the difference, $\delta y = Y - y$, between the approximate and the exact solution of the problem (1.1)—(1.3). The method of the error estimation to be given here is presented in more details in [4].

To connect the value of the functional I on the approximate solution (Y,P) where

$$P = Y' \,(1 + Y'^2)^{-1/2}, \tag{3.1}$$

* The equation (2.17) follows from the method that we will use in choosing neighbouring curves, namely, we will calculate the values of I on the neighboring curves (Y, P) where $Y = y + \delta y$ and $P = Y' \,(1 + Y'^2)^{-1/2} = p + \delta p$, y being the exact solution of (1.1) — — (1.3) and $p = y' \,(1 + y'^2)^{-1/2}$.

with the second variation given by (2.18) we first expand the terms $(\partial F/\partial y)_{y=Y}$ and $(1+Y'^2)^{-3/2}$ in power series with respect to δy, about $\partial F/\partial y$ and $(1+y'^2)^{-3/2}$ respectively, calculated on the exact solution y. The result is

$$\left(\frac{\partial F}{\partial y}\right)_Y = \frac{\partial F}{\partial y} + \frac{\partial^2 F}{\partial y^2}\,\delta y + 0\,((\delta y)^2, \dots) \tag{3.2}$$

$$(1+Y'^2)^{-3/2} = (1+y'^2)^{-3/2} - 3y'(1+y'^2)^{-5/2}\delta y' + 0((\delta y')^2, \dots). \tag{3.3}$$

Using (3.2) and (3.3) in (2.18) we get

$$\delta^2 I = \frac{1}{2}\int_a^b \frac{1}{\left(\frac{\partial F}{\partial y}\right)_Y}\left[\frac{d}{dx}\{\delta y'(1+Y'^2)^{-3/2}\} - \left(\frac{\partial F}{\partial y}\right)_Y \delta y\right]^2 dx + 0((\delta y)^3, \dots). \tag{3.4}$$

Now, since the value and the first variation of I on the exact solution of the boundary value problem (1.1)–(1.3) is zero, the expansion

$$I(Y,P) = I(y,p) + \delta I(y,p,\delta y,\delta p) + \frac{1}{2}\,\delta^2 I(y,p,\delta y,\delta p) + {} + 0\,((\delta y)^2, (\delta p)^2, \dots), \tag{3.5}$$

becomes,

$$I(Y,P) = \frac{1}{2}\int_a^b \frac{1}{\left(\frac{\partial F}{\partial y}\right)_Y}\left[\frac{d}{dx}\{\delta y'(1+Y'^2)^{-3/2}\} - \left(\frac{\partial F}{\partial y}\right)_Y \delta y\right]^2 dx, \tag{3.6}$$

where we used (3.1).

Supposing that*

$$-\left(\frac{\partial F}{\partial y}\right)_Y = R\,(x) > 0, \tag{3.7}$$

in what follows, we will distinguish the following two methods for error estimate:

Method 1. First we introduce a new independent variable τ by the relation

$$\tau = \int_a^x (1+Y'^2)^{3/2}\, d\xi, \tag{3.8}$$

so that from (3.6) we get

$$\frac{1}{[R(1+Y'^2)^{-3/2}]_{\max}}\int_0^{\tau(b)}\left(\frac{d^2\,\delta y}{d\tau^2}\right)^2 d\tau + 2\int_0^{\tau(b)} \delta y\,\frac{d^2\,\delta y}{d\tau^2}\,d\tau + {} + [R(1+Y'^2)^{-3/2}]_{\min}\int_0^{\tau(b)} (\delta y)^2\, d\tau \leqslant -2I. \tag{3.9}$$

In (3.9) $\tau(b)$ is given by

$$\tau\,(b) = \int_a^b (1+Y'^2)^{3/2}\, d\xi, \tag{3.10}$$

* We treat here only the case where (3.7) holds. The case $(\partial F/\partial y)_Y > 0$ could be treated similarly (see [4]).

and we used $[\,\cdot\,]_{m}$ and $[\,\cdot\,]_{min}$ to denote the maximal and minimal value in the interval $[a,b]$ of the function in brackets. Further, we consider a generalized Fourier series corresponding to the error δy

$$\delta y = \sum_{n=1}^{\infty} C_n \Phi_n, \tag{3.11}$$

where C_n are the Fourier constants, and Φ_n are the eigenfunctions of the fololwing spectral problem

$$\begin{aligned} &\frac{d^2 \Phi_n}{d\tau^2} + \lambda_n^2 \Phi_n = 0 \\ &a_3 \frac{d\Phi_n(0)}{d\tau} + a_1 \Phi_n(0) = 0 \\ &b_3 \frac{d\Phi_n(\tau(b))}{d\tau} + b_1 \Phi_n(\tau(b)) = 0. \end{aligned} \tag{3.12}$$

Substituting (3.11) and $(3.12)_1$ in (3.9) and using Parseval's equation we obtain the following error estimate

$$||\delta y||_{L_2} \leqslant \left\{ \frac{-2I[(1+Y'^2)^{-3/2}]_{max}}{[G(\lambda_n^2)]_{min}} \right\}^{1/2} \tag{3.13}$$

where

$$||\delta y||_{L_2} = \left(\int_0^{\tau(b)} \delta y^2(\tau)\, d\tau\right)^{1/2}. \tag{3.14}$$

In (3.13) the function $G(\lambda_n^2)$ is given by

$$G(\lambda_n^2) = \frac{\lambda_n^4}{[R(1+Y'^2)^{-3/2}]_{max}} - 2\lambda_n^2 + [R(1+Y'^2)^{-3/2}]_{min} \tag{3.15}$$

and we assumed that $G(\lambda_n^2) > 0$ for every eigenvalue λ_n of the spectral problem (3.12). If the condition $G(\lambda_n^2) > 0$ is not satisfied, the error estimation procedure could be performed by:

Method 2. We expand now the error δy in the series (3.11), where the eigenfunctions Φ_n are the solutions of the following problem

$$\begin{aligned} &\frac{d}{dx}\{(1+Y'^2)^{-3/2}\Phi'_n\} + \lambda_n R \Phi_n = 0 \\ &a_3 \Phi'_n(a) + a_1 \Phi_n(a) = 0 \\ &b_3 \Phi'_n(b) + b_1 \Phi_n(b) = 0. \end{aligned} \tag{3.16}$$

Substituting (3.11) into (3.6) and using the orthogonality property of the eigenfunctions of the problem (3.16) we get the following error estimate

$$||\delta y||_{L_2} \leqslant \left[\frac{-2I}{[(1-\lambda_n)^2]_{min}[R]_{min}} \right]^{1/2}, \tag{3.17}$$

under the condition that no eigenvalue of the problem (3.16) is equal to one, i.e.,

$$\lambda_n \neq 1, \qquad n = 1, 2, 3, \ldots \tag{3.18}$$

4. Example

Let us consider the classical problem of buckling of an elastic column. The function $F(y,x)$ in this case becomes

$$F(x,y) = -k^2 y. \tag{4.1}$$

We take the boundary conditions (1.2) and (1.3) with the following values for the constants

$$a_3 = 0, \quad a_1 = 1, \quad a_2 = 0, \quad a = 0 \tag{4.2}$$

$$b_3 = 0, \quad b_1 = 1, \quad b_2 = 0, \quad b = \pi.$$

The boundary conditions (1.2), (1.3) and (4.2) correspond to pin ended column. We assume the approximate solution for the above problem in the form

$$Y = Ax\,(\pi - x) + Bx^2\,(\pi - x)^2 \tag{4.3}$$

where the unknown constants A and B are to be determined by minimizing the functional (2.12), which in the present case becomes

$$I = \int_0^\pi \left\{(1 + Y'^2)^{1/2} - \frac{k^2}{2} Y^2 - \sqrt{1 - P^2} - \frac{P'^2}{2\,k^2}\right\} dx \tag{4.4}$$

In (4.4) P is related to Y' by equation (2.4), i.e.,

$$P = Y'\,(1 + Y'^2)^{-1/2}. \tag{4.5}$$

Substituting (4.3) into (4.4), the procedure of minimization od (4.4) with respect to arbitrary constants A and B have to be performed numerically for every given k^2, due to the eliptic nature of the integrals involved.

We performed calculations for $k^2 = \mathrm{P/EI} = 0.6325$ and obtained $A = 0.566$, $B = -0.0325$, which gives $I = -0.008859$. It is easy to conclude that in this example the Method 2 for error estimation must be used. Since the eigenvalues of the problem (3.16) in this case are all positive and increasing and since $\lambda_1 < 0.459$ and $\lambda_2 > 1.344$*, we get the following error estimate

$$\|\,\delta y\,\|^2_{L_2} < 0.236 \tag{4.6}$$

In conclusion we note that the example presented corresponds to the buckling of an elastic column when the buckling load P is equal 1.15 P_{cr}, where P_{cr} is the critical load. The solution obtained here

$$Y = 0.566\,x\,(\pi - x) - 0.0325\,x^2\,(\pi - x)^2, \tag{4.7}$$

when compared with the exact solution (see for example [6]) agrees well in both maximal deflection and the slope at the ends of the column.

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* An upper bound for λ_1 and a lower bound for λ_2 is found by a method given in [5].

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ÜBER EIN EXTREMALES VARIATIONSPRINZIP FÜR EINE KLASSE VON RANDWERTPROBLEMEN

Zusammenfassung

Das Referat bringt der Formulierung des Variationprinzips fur eine Klasse von Randwertproblemen, die mit den gewöhnlichen Differentialgleichungen zweiter Ordnung darstellend sind. Das gebrauchte Variationsprinzip gibt die appr oximativen Lösungen mit der Ritz'sche Methode. Es wird das Verfahren für Fehlerabschätzung, fur mit dieser Methode gegebene approximativen Lösungen, ergeben. Für ein konkretes Beispiel der Knickung mit der Teilnahme die geometrischen Nichtlinearitäten wird einer approximativen Lösung und die Fehlerabschätzung ergeben.

O JEDNOM EKSTREMALNOM VARIJACIONOM PRINCIPU ZA KLASU GRANIČNIH PROBLEMA

Izvod

U radu je formulisan varijacioni princip za jednu klasu graničnih problema opisanih običnim diferencijalnim jednačinama drugog reda. Varijacioni princip je korišćen za dobijanje približnih rešenja Ritzovim postupkom. Dat je postupak za ocenu greške ovako dobijenih približnih rešenja. Za konkretan primer izvijanja, sa učešćem geometrijske nelinearnosti, određeno je približno rešenje i ocenjena njegova greška.

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