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PROPAGATION OF WEAK MHD DISCONTINUITIES ALONG BICHARACTERISTICS IN AN OPTICALLY THICK MEDIUM OF MAGNETO-FLUIDS

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1. Introduction

Thomas [1] introduced the singular surface theory to study the propagation of week discontinuities in an ideal gas flow. Several workers [2, 3, 4, 5, 6] further generalised and developed the singular surface theory to cover up some complicated cases of real gases. Here we shall apply the singular surface theory to investigate the phenomena associated with weak discontinuities in unsteady flows of magneto fluids in an optically thick inhomogeneous medium. Nariboli and Secrest [7] extended the analysis of Thomas to MHD flows with finite electrical conductivity. Nariboli and Ranga Rao [8] combined the singular surface theory and the ray theory to study the propagation of weak discontinuities in non-linear anisotropic media. Using the ray theory Ramashankar [9] obtained the growth equation for weak discontinuities propagating through conducting fluids, but he did not study the behaviour of the wave amplitude. Recently, Elcrat [10] studied the propagation of sonic discontinuities in an unsteday flow of a perfect gas.

The object of the present work is to obtain the growth equation which will govern the growth and decay of weak discontinuities propagating in unsteady flows of thermally and electrically and electrically conducting fluids. The inhomogeneity effects are accounted for and the growth equation for the wave amplitude has been solved analytically. An explicit criteria for decay or "blow up" of weak discontinuities is presented.

At very high temperature, the radiation energy density and the radiation pressure cannot be neglected. The thick gas approximation for the radiation energy density, the radiation pressure and the radiative heat flux has been used in the present analysis. Under the approximation of local thermodynamic equilibrium the radiative heat flux term is similar to the heat conduction term [1]. In this case, the effective thermal conductivity is given by [11],

$$K_{eff} = K + 4\, D_R\, a_R\, T^3,$$

where K, D_R, a_R and T are respectively the coefficient of thermal condustivity, Rosseland diffusion constant, Stefan — Boltzman constant and temperature.

The set of non-linear differential equations governing the two dimensional unsteady flow of *MHD* with infinite electrical conductivity are:

$$\frac{\partial \rho}{\partial t} + \rho_{,k} u_k + \rho u_{k,k} = 0, \tag{1.1}$$

$$\rho \frac{\partial u_i}{\partial t} + \rho u_k u_{i,k} + (1 + 4 R_p) p_{,i} - 4 R_p \frac{p}{\rho} \rho_{,i} + \mu H H_{,i} = 0, \tag{1.2}$$

$$\frac{\partial H}{\partial t} + u_i H_{,i} + 0, \tag{1.3}$$

$$\{1 + 12(\nu - 1) R_p\} \frac{\partial p}{\partial t} - \{\nu + 16(\nu - 1) R_p\} \frac{p}{\rho} \frac{\partial \rho}{\partial t} +$$

$$+ \{1 + 12(\nu - 1) R_p\} u_i p_{,i} - \{\nu + 16(\nu - 1) R_p\} \frac{p}{\rho} u_i \rho_{,i} +$$

$$+ (K_{eff} T_{i})_{,i} = 0, \tag{1.4}$$

$$p = \rho R T, \tag{1.5}$$

where t, p, u_i, ρ and H respectively represent the time, the pressure, the velocity components, the density and the transverse magnetic field vector. μ is the magnetic permeability. R and ν are the universal gas constant and heat exponent. The radiation pressure number R_p defined by

$$R_p = \frac{\text{Radiation pressure}}{\text{gas pressure}} = \frac{P}{p}.$$

A comma followed by a Latin index denotes partial differentiation with respect to a space variable and $(\partial / \partial t)$ denotes the partial differentiation with respect to time t.

We assume the existence of a moving singular surface $S(t)$ across which the temperature field and its normal derivative are continuous with possible discontinutities in the higer derivatives, while other flow quantities are continuous but their first and higher derivatives are discontinuous. Suppose that $S(t)$ is given by $\Phi(x_i, t) = 0$ and we denote by n_i the compobents of unit normal vector $\Phi_{i} / |\text{grad}\, \Phi|$, and by G the normal speed of advance of this surface, that is $G = - \left(\frac{\partial \Phi}{\partial t}\right) / (|\text{grad}\Phi|)$. We assume that the surface $S(t)$ has two sides denoted by 1 and 2 and the normal vector n_i points into 2. The relative speed of advance of the surface $S(t)$ in the fluid is denoted by $U = G - u_i n_i$.

Let $[f] = (f)_2 - (f)_1$ denote the jump in a flow quantity f across $S(t)$. Then the geometrical and kinematical compatibility conditions [1] can be written in the form,

$$[f_{,i}] = B n_i, \left[\frac{\partial f}{\partial t}\right] = - BG,$$

where $[f]$, a square bracket, denotes jump in the quantity enclosed and B is a scalar function defined over $S(t)$ by $B = [f_{,i}]\, n_i$.

2. Law of Propagation

From the law of conservation of energy across the surface $S(t)$, we have [12].

$$[T_{,i}]\ n_i = 0 \tag{2.1}$$

Now taking jumps in equations (1.1), (1.2), and (1.3) and making use of the equation (1.6) we obtain,

$$U\zeta = \rho\lambda, \tag{2.2}$$

$$-PU\lambda + (1 + 4R_p)\xi - 4\frac{p}{\rho}R_p\zeta + \mu H\eta = 0, \tag{2.3}$$

$$U\eta = H\lambda. \tag{2.4}$$

Differentiating the equation of state (1.5) with respcet to x_i and taking jumd across $S(t)$ and making use of (2.1) and (1.6) we get,

$$\xi = a^2\zeta, \tag{2.5}$$

where a is the isothermal velocity of sound. The functions λ, ζ, ξ and η are defined over the wave front by the relations:

$$\lambda_i = [u_{i,j}]\ n_j,\ \zeta = [\rho_{,i}]\ n_i,\ \xi = [p_{,i}]\ n_i,\ \eta = [H_{,i}]\ n_i.$$

Now putting the values of η, ζ and ξ from equations (2.4), (2.9), (2.5) into the equation (2.3) we get,

$$\lambda\{-U^2\rho + (11 + 4R_p)a^2\rho + \mu H^2 - 4R_p a^2\rho\} = 0 \tag{2.6}$$

In view of the assumption that $S\ t)$ is a regular singular surface, we have $\lambda \neq 0$ and hence we have

$$U^2 = a\varphi + b^2, \tag{2.7}$$

where b is the Alfven speed.

3. Growth Equation

Differentiating equation (1.2) with respect to x_j and taking jump across $S(t)$ and making use of second order compatibility conditions [1] we get:

$$\rho\left[\frac{\partial^2 u_i}{\partial x_j\,\partial t}\right] + \left.\frac{\partial\rho}{\partial x_j}\frac{\partial u_i}{\partial t}\right] + \rho u_k\ [u_{i\ kj}] + \rho\,[u_{k,j}\,u_{i,k}] + u_k\ [\rho_j\,u_{i,k}] +$$

$$+ (1 + 4R_p)\,[p_{,ij}] + 4\,[R_{p,j}\,p_{,i}] - 4\,R_p\frac{p}{\rho}\,[\rho_{,ij}] -$$

$$-4\frac{p}{\rho}\,[R_{p,j}\,\rho_{,i}] - 4\,\frac{R_p}{\rho}\,[\rho_{,i}\ p_{,j}] + \frac{4}{\rho^2}\,R_p\,p\,[\rho_{,j}\,\rho_{,i}] +$$

$$+ \mu\,H\,[H_{,ij}] + \mu\,[H_{,j}\,H_{,i}] = 0 \tag{3.1}$$

In the present work it will be shown that it is natural to assume the transport of discontinuities along bicharacteristic curves in the characteristic manifold $\Sigma = \underset{t}{=} US(t)$ of governing differential equations. These bicharacteristics coincide with orthogonal trajectories in the case of uniform medium at rest [13].

Using the genarilised geometrical and kinematical compatibility conditions to evaluate (3.1) across a singular surface $S(t)$ and simplifying we get,

$$\rho \frac{\delta\lambda}{\delta t} + (1 + 4 R_p)\, \bar{\xi} - \rho U \bar{\lambda}_i n_i + u_k \rho^{\alpha\beta} \lambda_{,\alpha} x_{k,\beta} + \zeta n_i \left(\frac{\partial u_i}{\partial t} + u_j u_{i,j}\right)_2 +$$
$$+ \lambda \left(\frac{\partial \rho}{\partial n}\right)_2 \left\{- U + \frac{8p\, R_p}{\rho U}\right\} + 2 U \zeta (u_{i,j}\, n_i\, n_j)_2 -$$
$$- \frac{8 R_p}{p} \xi \left(\frac{\partial p}{\partial n}\right)_2 - \frac{4p\, R_p}{\rho} - \mu \eta^2 + \mu H \eta + 2\mu \left(\frac{\partial H}{\partial n}\right)_2 = 0 \qquad (3.2)$$

Similfrly from (1.1) and (1.2) we obtain

$$U \frac{\delta \zeta}{\delta t} - U^2 \bar{\zeta} + \rho U \bar{\lambda}_i n_i + 2U\lambda \left(\frac{\partial \rho}{\partial n}\right)_2 + U g^{\alpha\beta} \tau_{,\alpha} u_k x_{k,\beta} - 2\lambda U \zeta +$$
$$+ 2 U \zeta (u_{i,j}\, n\, n_j)_2 - 2 U \dot{\lambda} \rho \Omega = 0, \qquad (3.3)$$

$$\frac{\delta \eta}{\delta t} - U \bar{\eta} + g^{\alpha\beta} \eta_{,\alpha} u_i x_{i,\beta} + \eta (u_{i,j}\, n_i\, n_j)_2 + \lambda \left(\frac{\partial H}{\partial n}\right)_2 - 2\lambda\eta + H \bar{\lambda}_i n_i +$$
$$+ g^{\alpha\beta} H (n_i)_{,\alpha} x_{i,\beta} + (u_{i,j}\, n_i\, n_j)_2\, \eta + (H_{,j})\, 2\lambda n_j = 0, \qquad (3.4)$$

where

$$\bar{\lambda}_i = [u_{i,jk}]\, n_j n_k, \quad \bar{\zeta} = [\rho_{,jk}]\, n_j n_k, \quad \bar{\xi} = [p_{,ij}]\, n_i n_j, \quad \bar{\eta} = [H_{,ij}]\, n_i n_j,$$

and $(\delta/\delta t)$ denotes differentiation along an orthogonal trajectory of the surface $S(t)$, and Ω is the mean wean curvature of $S(t)$ defined by $2\Omega = g^{\alpha\beta} b_{\alpha\beta}$ where $g^{\alpha\beta}$ nad $b_{\alpha\beta}$ are the first and second fundeamental forms of $S(t)$ respectively.

Differentiating the state equation (1.5) twice with respect to x_i and evaluating across $S(t)$ we get,

$$\bar{\theta} = \frac{1}{R\rho} \left\{\bar{\xi} - \frac{p}{\rho} \bar{\zeta} - 2 R \zeta \left(\frac{\partial T}{\partial n}\right)_2\right\}, \qquad (3.5)$$

where $\bar{\theta} = [T_{,ii}]$.

Evaluating (1.4) across $S(t)$ and using (3.5) we get

$$- \{1 + 12(\nu - 1) R_p\}\, U \bar{\xi} + \{\nu + 16(\nu - 1) R_p\} \frac{p}{\rho} U \bar{\zeta}$$
$$+ \frac{K_{eff}}{R\rho} \left\{\bar{\xi} - \frac{p}{\rho} \bar{\zeta} - 2 R \zeta \left(\frac{\partial T}{\partial n}\right)_2\right\} = 0 \qquad (3.6)$$

Eliminating $\bar{\xi}$, $\bar{\zeta}$, $\bar{\lambda}_i$ and $\bar{\eta}$ from equations (3.6), (3.3), (3.2) and (3.4) we obtain

$$\rho \frac{\delta\lambda}{\delta t} + \rho g^{\alpha\beta} \lambda_{,\alpha} u_k x_{k,\beta} + \frac{p}{\rho U}\left\{\frac{\delta\zeta}{\delta t} + g^{\alpha\beta} \zeta_{,\alpha} u_k x_{k,\beta}\right\} +$$

$$+ \frac{\mu H}{U}\left\{\frac{\delta\eta}{\delta t} + g^{\alpha\beta} \eta_{,\alpha} u_k x_{k,\beta}\right\} + \frac{\rho\lambda}{U}\left\{n_i \left(\frac{\partial u_i}{\partial t} + u_j u_{i,j}\right)_2 +\right.$$

$$+ \frac{U}{\rho}\left(\frac{\partial\rho}{\partial n}\right)_2 \left(- U + \frac{8 R_p p}{\rho U} + \frac{2 p}{U\rho}\right) + 4 U (u_{i,j}\, n_i\, n_j)_2 +$$

$$+ \frac{RU(1 + 4R_p)}{K_{eff}}\left(- (\nu - 1)\, (1 + 4 R_p)\, p + \frac{2 K_{eff}}{U}\left(\frac{\partial T}{\partial n}\right)_2\right) -$$

$$\left. - \frac{8 R_p}{\rho}\left(\frac{\partial p}{\partial n}\right)_2 + \frac{4 \mu H}{\rho}\left(\frac{\partial H}{\partial n}\right)_2 - 2 U^2 \Omega\right\} - \lambda^2 \rho \left\{2 + \frac{b^2}{U^2}\right\} = 0 \qquad (3.7)$$

The equation (3.7) is a differential equation for ($U\zeta = \rho\lambda$) and, therefore, one for λ, one for ξ and one for η, along the orthogonal trajectories of $S(t)$. However, the „inhomogeneous terms" arising from the surface derivatives cause some difficulty in interpretation and if we transform (3.7) into a differential equation along bicharacteristics, this difficulty disappears. It is remarkable that the results for two dimensioal systems in [14] and the results for three dimensional potential flow in [15] utilize the bicharacteristic directions in order to avoid such complexities.

$$\left.\begin{aligned} \frac{d_r \lambda}{dt} &= \frac{\delta\lambda}{\delta t} + g^{\alpha\beta} \lambda_{,\alpha} u_i x_{i,\beta} \\ \frac{d_r \zeta}{dt} &= \frac{\delta\zeta}{\delta t} + g^{\alpha\beta} \zeta_{,\alpha} u_i x_{i,\beta} \\ \frac{d_r \eta}{dt} &= \frac{\delta\eta}{\delta t} + g^{\alpha\beta} \eta_{,\alpha} u_i x_{i,\beta} \end{aligned}\right\} \qquad (3.8)$$

where $\left(\frac{d}{dt}\right)$ has the same meaning as in [10] and represents a material derivative along bicharacteristic direction.

Now using equations (2.2), (2.4) and (3.8) in (3.7) we get

$$\frac{d}{dt} + \zeta\left\{\frac{1}{2}\frac{d}{dt}\log\left(\frac{U}{\rho}\right) + \frac{1}{2} Q \frac{d}{dt}\log\left(\frac{H}{\rho}\right) + \frac{1}{2U} E\right\} = \frac{U}{2} M \zeta^2 \quad (3.9)$$

where

$$E = n_i \left(\frac{\partial u_i}{\partial t} + u_j u_{i,j}\right)_2 + \frac{U}{\rho}\left(\frac{\partial\rho}{\partial n}\right)_2 \left\{- U + \frac{8 R_p p}{\rho U} + \frac{2 p}{\rho U}\right\} +$$

$$+ 4 U (u_{i,j} n_i n_j)_2 + \frac{RU(1 + 4R_p)}{K_{eff}}\left\{-(\nu - 1)(1 + 4 R_p) + \frac{2 K_{eff}}{U}\left(\frac{\partial T}{\partial n}\right)_2\right\} -$$

$$- \frac{8 R_p}{\rho}\left(\frac{\partial p}{\partial n}\right)_2 + 4 \frac{\mu H}{\rho}\left(\frac{\partial H}{\partial n}\right)_2 - 2 U^2 \Omega.$$

$$Q = \frac{\mu H^2}{p U^2} = \text{Alfven Number}$$

$$M = \left\{2 + \frac{b^2}{U^2}\right\}.$$

4. Global behaviour of wave amplitude

The equation (3.9) is the fundamental differential equation for the growth and decay of weak magnetohydroynamic discontinuities associated with the wave surface $S(t)$.

The solution of the equation (3.9) is given by

$$\zeta = \frac{\zeta_0 \left(\frac{U}{U_0}\right)^{-\frac{1}{2}} \left(\frac{H}{H_0}\right)^{-Q/2} \left(\frac{\rho}{\rho_0}\right)^{\left(\frac{Q+1}{2}\right)} \exp\left\{-\frac{1}{2}\int_0^t \frac{E}{U}\,dt\right\}}{1 - \frac{1}{2} M \zeta_0 \left(\frac{U}{\rho}\right) U_0^{\frac{1}{2}} H_0^{Q/2} \rho_0^{-\left(\frac{Q+1}{2}\right)} \int_0^t \left\{\left(\frac{U}{\rho}\right)^{-\frac{1}{2}} \left(\frac{H}{\rho}\right)^{Q-2} \exp\left(-\frac{1}{2}\int_0^t \frac{E}{U}\,d\zeta\right)\right\} dt} \tag{4.10}$$

The solution (4.10) shows that, whan $\zeta_0 > 0$, the wave amplitude ζ decreases continuously with time and vanishes ultimately. When $\zeta_0 - 0$, the wave amplitude grows without limit and after finite critical time t_c a weak discontinuity breaks down and a shock type discontinuity appears at time t_c given by

$$\int_0^t \left\{\left(\frac{U}{\rho}\right)^{-\frac{1}{2}} \left(\frac{H}{\rho}\right)^{-Q/2} \exp\left(-\frac{1}{2}\int_{t}^{\zeta} \frac{E}{U}\,d\tau\right)\right\} dt =$$

$$= \frac{2}{MU} U_0^{-\frac{1}{2}} H_0^{-Q/2} \rho_0^{\left(\frac{Q+1}{2}\right)} \zeta_0^{-1} \tag{4.9}$$

Such that,

$$\lim_{t \to t_c} \zeta(t) = \infty.$$

The solution (4.1) of the growth equation shows the global behaviourof the wave amplitude $\zeta(t)$ on time t. The initial wave amplitude ζ_0 also plays an important role in the formation of a shock wave. When the initial amplitude of a compressive weak wave ($\zeta - 0$) exceeds a critical value ζ_c given by

$$\zeta_c = \frac{2}{M}\left(\frac{U}{\rho}\right)^{-1} U_0^{-\frac{1}{2}} H_0^{-Q/2} \rho_0^{\left(\frac{Q+1}{2}\right)} \left\{\int_0 \left(\frac{U}{\rho}\right)^{-\frac{1}{2}} \left(\frac{H}{\rho}\right)^{-Q/2}\right.$$

$$\left.\exp\left(-\frac{1}{2}\int_0^{\zeta} \frac{E}{U}\,d\tau\right) dt\right\}^{-1} \tag{4.3}$$

then a shock wave will be formed at finite time t_c given by (4.2). On the other hand when $\zeta_0 > \zeta_c$, a strage for the shock formation will never appear and the wave amplitude will decay with time.

REFERENCES

[1] T. Y. Thomas, *Jour. Math. Mech.*, vol (61957), p. 453—469.
[2] C. N. Caul, *Jour. Math. Mech.*, vol (10 (1961), p. 393.
[3] C. A. Narboli, *Jour. Math. Mech.*, vol (12 (1963), p. 141.
[4] S. Srinivasan and Rishi Ram, *Z. Angev. Math. Phys.*, vol 26 (1975), p. 307.
[5] D. Y. Ludwig, A.E.C. Report (1961), NYO — 9351, New York.
[6] G. F. D. Duff, *Comm. Pure. Appl. Math.*, vol 1 (1964), p. 189.
[7] G. A. Nariboli and B. G. Secrest, *Tensor (N. S.)*, vol. 18 (1967), p. 22.
[8] G. A. Nariboli and M. P. Ranga Rao, *J. Math. Phys. Sci.*, vol. 1 (1967), p. 302.
]9] Ramshankar, *International J. Engg. Sci.*, vol. 17, No. 1 (1979), p. 17.
[10] A. R. Elcrat. *Int. J. Engg. Sci.*, vol 15 (1977), p. 29.
[11] S. I. Pai, *Radiation Gas Dynamics*, Springer-Verlag, 1966.
[12] J. C. Pant and R. S. Mshra, *Rendiconti Circolo Mathematisc*, Tomo XI (1963), p. 1.
[13] T. Y. Thomas, *J. Math. Mech.*, vol. 6 (1957), pp. 311.
[14] M. Schiffer, *Handbuch Phys.*, vol .8 (3), (1969), p. 1.
[15] R. Courant and D. Hilbert, *Methods of Mathematical Physics*, II. Interscience — Wiley, New York (1962).

PROPAGATION DES DISCONTINUITES MHD FAIBLES LE LONG DES BICARACTERISTIQUES DANS UN MILIEU OPTIQUE EPAIS DES MAGNETO-FLUIDES

Résumé

Dans ce travuil on considère les phenomènes associés aux discontinuités d'un cours non-stationnaire d'un fluide conducteur de chaleur et d'une conduction infinie d'électricité. On a obtenu et résolu analytiquement les équations différentielles fondamentales qui décrivent la croissance et la décroissance fondamentales qui décrivent la croissance et la décroissance des discontinutés faibles le long des bicaractéristiques. Dans la discussion on présente le critère explicite de la décroissance des discontinuités faibles et aussi le comportement global de l'amplitude de l'onde.

PROSTIRANJE SLABIH MHD DISKONTINUITETA DUŽ BIKARAKTERISTIKA U OPTIČKI DEBELOJ SREDINI MAGNETOFLUIDA

Izvod

Rad se odnosi na aktuelnu oblast magnetohidrodinamike, a sadrži rezultate u vezi sa rasprostiranjem slabih diskontinuiteta u slučaju nestacionarnog strujanja termički provodljivog fluida sa beskonačnom električnom provodljivošću. U radu su izvedene i rešene analitički osnovne jednačine koje opisuju porast ili opadanje slabih diskontinuiteta duž bikarakteristika. U diskusiji ovog rešenja izveden je kriterijum za formiranje udarnog talasa i procenjeno je vreme koje je za to potrebno.

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