TEORIJSKA I PRIMENJENA MEHANIKA 8, pp. 119—123, 1982, UDK 539.

EXPANSION TENSOR IN A MODIFIED DE SITTER METRIC

D. Radojević

(Received December 1981)

The cosmological model of de Sitter had been published in 1917. In that model the proper density of matter is zero so that it represents an empty universe model. In connection with the de Sitter model, Einstein suggested the possibility that matter could be distributed on a sphere. That idea was realised by Mal'cev and Markov, sixty years later [1]. In this paper we investigate the expansion tensor in such a modified metric.

The classical de Sitter metric is [3]:

$$ds^2 = \frac{dr^2}{1 - \dfrac{r^2}{R^2}} + r^2 (d\theta^2 + \sin^2\theta \, d\varphi^2) - \left(1 - \frac{r^2}{R^2}\right) dt^2$$

The modified de Sitter metric with „displaced" horizon can be represented in the form

$$ds^2 = \frac{dr^2}{1 - \dfrac{r^2}{R^2}} + r^2 (d\,\theta^2 + \sin^2\theta \, d\,\varphi^2) - \left(\frac{a + R\sqrt{1 - \dfrac{r^2}{R^2}}}{a + R}\right)^2 dt^2$$

where a and R are constants.

We investigate the expansion tensor which is defined as

$$\vartheta_{\alpha\beta} = \mathfrak{L}_\xi (g_{\alpha\beta} + \xi_\alpha \xi_\beta) \tag{1}$$

where ξ_α is the unit time-like vector pointed into the future, tangent on the world lines, $g_{\alpha\beta}$ is the metric tensor of the considered space, and $\mathfrak{L}_\xi$ denotes Lie derivative with respect to ξ^α. This tensor satisfies

$$\vartheta_{\alpha\beta} \xi^\beta = 0 \tag{2}$$

Since ξ_α is time-like, we can choose a coordinate system in which ξ_α is the unit vector of proper time. In this paper we try to find out the field of expansion relative to a system of radially orientied four velocities. The vector ξ_α has the form

$$\xi_\alpha = \{\xi_1, 0, 0, \xi_4\}$$

We shall suppose that coordinate transformations $x^\alpha \to \bar{x}^\alpha$ are of form

$$\begin{aligned}
\bar{x}^1 &= \bar{r} = \psi(r, t) = \psi(x^1, x^4)\\
\bar{x}^2 &= \bar{\theta} = \theta = x^2\\
\bar{x}^3 &= \bar{\varphi} = \varphi = x^3\\
\bar{x}^4 &= \bar{\tau} = \tau + \mu(r) = x^4 + \mu(x^1)
\end{aligned} \tag{3}$$

The components of the vector ξ_α could be expressed as

$$\begin{aligned}
\xi_1 &= \frac{\dfrac{\partial \bar{\tau}}{\partial r}}{\sqrt{-g^{11}\left(\dfrac{\partial \bar{\tau}}{\partial r}\right)^2 - g^{44}\left(\dfrac{\partial \bar{\tau}}{\partial \tau}\right)^2}} = \frac{\dot{\mu}}{\sqrt{-g^{11}\,\dot{\mu}^2 - g^{44}}}\\
\xi_4 &= \frac{\dfrac{\partial \bar{\tau}}{\partial \tau}}{\sqrt{-g^{11}\left(\dfrac{\partial \bar{\tau}}{\partial r}\right)^2 - g^{44}\left(\dfrac{\partial \bar{\tau}}{\partial \tau}\right)^2}} = \frac{1}{\sqrt{-g^{11}\,\dot{\mu}^2 - g^{44}}}
\end{aligned} \tag{4}$$

The system of coordinates $\bar{x}^\alpha$ is ortogonal also, and thus

$$g^{\alpha\beta}\,\xi_\alpha\,\frac{\partial \psi}{\partial x^\beta} = 0 \tag{5}$$

In order to find out the function μ relevant for transformations (3) we shall use the relations having $\dfrac{\partial \bar{x}^\alpha}{\partial x^\beta}$.

We find the components of tensor $\vartheta_{\alpha\beta}$ from definition (1).

$$\begin{aligned}
\vartheta_{11} &= 2\,(1 + g^{11}\,\xi_1\,\xi_1)\,\frac{\partial \xi_1}{\partial r} + 2\,g^{44}\,\xi_1\,\xi_4\,\frac{\partial \xi_4}{\partial r} - g^{11}\,\xi_1\,(1 + g^{11}\,\xi_1\,\xi_1)\,\frac{\partial g_{11}}{\partial r} -\\
&\quad - 2\,g^{44}\,g^{44}\,\xi_1\,\xi_4\,\xi_4\,\frac{\partial g_{44}}{\partial r}\\
\vartheta_{14} &= 2\,g^{11}\,\xi_1\,\xi_1\,\frac{\partial \xi_1}{\partial r} + \xi_1\,\xi_1\,\xi_4\,\frac{\partial g^{11}}{\partial r} - g^{44}\,\xi_4\,(1 + g^{44}\,\xi_4\,\xi_4)\,\frac{\partial g_{44}}{\partial r}\\
\vartheta_{22} &= g^{11}\,\xi_1\,\frac{\partial g_{22}}{\partial r}\\
\vartheta_{33} &= g^{11}\,\xi_1\,\frac{\partial g_{33}}{\partial r}\\
\vartheta_{44} &= 2\,g^{11}\,\xi_1\left(\xi_4\,\frac{\partial \xi_4}{\partial r} + \frac{\partial g_{44}}{\partial r}\right)
\end{aligned} \tag{6}$$

Tensor $\vartheta_{\alpha\beta}$ is symmetric. Other components are zero.

The contravariant components of the expansion tensor could be obtained by

$$\vartheta^{\alpha\beta} = g^{\alpha\rho} g^{\beta\varphi} \vartheta_{\rho\varphi} \tag{7}$$

We shall use the contravariant components of the expansion tensor in the coordinate system $\bar{x}^{\alpha}$ also

$$\bar{\vartheta}^{\alpha\beta} = \vartheta^{\varphi\rho} \frac{\partial \bar{x}^{\alpha}}{\partial x^{\rho}} \frac{\partial \bar{x}^{\beta}}{\partial x^{\varphi}} \tag{8}$$

Further, we need the mixed components of the expansion tensor, but we shall express them in the following form

$$\bar{\vartheta}^{\alpha\rho} = \bar{g}^{\rho\beta} \bar{\vartheta}^{\alpha}_{\beta} \tag{9}$$

which enable us to find $\bar{\vartheta}^{\alpha}_{\beta}$. Using (3), (5), (8) and (9) we obtain

$$\begin{aligned}
\bar{\vartheta}^{1}_{1} &= \frac{1}{g^{11}} \left[\vartheta^{11} - 2\, \frac{\xi^{1}}{\xi^{4}}\, \vartheta^{14} + \left(\frac{\xi^{1}}{\xi^{4}}\right)^{2} \vartheta^{22} \right] \left(\frac{\partial \psi}{\partial r}\right)^{2} \\
\bar{\vartheta}^{2}_{2} &= \frac{1}{g^{22}}\, \vartheta^{22} \\
\bar{\vartheta}^{3}_{3} &= \frac{1}{g^{33}}\, \vartheta^{33}
\end{aligned} \tag{10}$$

In the coordinate system $\bar{x}^{\alpha}$ other components are zero.

Using (2), (5) and (7) we obtain $\bar{\vartheta}^{1}_{1}$ depending on ϑ_{11} only

$$\bar{\vartheta}^{1}_{1} = \frac{[1 + (\xi^{1}/\xi^{4})^{2}]^{2}}{g^{11} + (\xi^{1}/\xi^{4})^{2}\, g^{44}}\, g^{11} g^{11} \vartheta_{11} \tag{11}$$

We note that in the coordinate system $\bar{x}^{\alpha}$

$$\bar{\vartheta}^{2}_{2} = \bar{\vartheta}^{3}_{3}$$

We shall investigate only the case

$$\bar{\vartheta}^{2}_{2} = k = \text{const} \tag{12}$$

It implies the following equation for μ

$$\frac{d\mu}{dr} = \frac{(a + R)\, k\, r}{\left(a + R\sqrt{1 - \frac{r^{2}}{R^{2}}}\right) \sqrt{\left(1 - \frac{r^{2}}{R^{2}}\right)\left[4\left(1 - \frac{r^{2}}{R^{2}}\right) + k^{2} r^{2} \gamma\right]}}$$

This equation has the solution

$$\mu = P\, ln \frac{Rx + Aa - \sqrt{A^{2} a^{2} + k^{2} R^{4}}}{Rx + Aa + \sqrt{A^{2} a^{2} + k^{2} R^{2}}}$$

Where we used, for brevity

$$P = \frac{-kR(a+R)}{\sqrt{A^2 a^2 + k^2 R^4}}, \qquad A = \sqrt{4 - k^2 R^2} = \text{const}$$

$$x = \sqrt{A^2 a^2 \left(1 - \frac{r^2}{R^2}\right)} + \sqrt{4 A^2 + k^2 r^2}, \qquad 4 - k^2 R^2 > 0$$

The function μ is positive when

$$0 < k < \frac{2}{R}$$

Now we shall investigate the component ϑ_1^1. Using (6), (7), (9) and (12) we find ξ_1 and substitute it in (11). The sign of that expression depend on ϑ_{11} only. We obtein for ϑ_{11}

$$\vartheta_{11} = \left\{ \frac{1}{1 - \frac{r^2}{R^2}} - \frac{\frac{r^2}{R}\left[1 + \frac{k^2 r^2}{4\left(1 - \frac{r^2}{R^2}\right)}\right]}{\left(a + R\sqrt{1 - \frac{r^2}{R^2}}\right)\left(1 - \frac{r^2}{R^2}\right)^{3/2}} \right\}$$

This expression is positive and independant on a and k, for $0 > r > \frac{\sqrt{2}}{2R}$. When k is small (sllow expansion) the domain of r making ϑ_{11} positive is wider. For example, when $kR = 1$, ϑ_{11} is positive for $0 > r > 0{,}8R$. But, when r is close to R influence of k is overwhelming and ϑ_{11} is negative.

We see therefrom a possibility of finding in the modified de Sitter universe, of world lines analogous to the well-known world lines of the Lemaître transformation [3]

$$\bar{t} = t + \frac{1}{2} R \, ln \left(1 - \frac{r^2}{R^2}\right)$$

for the classical de Sitter metric, mentioned at the beginning of the paper. In that case expansion was constant and automatically isotropic, i.e. equal in every spacelike direction.

REFERENCES

[1] В.К. Мальцев, М.А. Марков *К вопросу о сингулярности в модели Де Ситтера*; Труды Физ. Инст. Лебедева, Т 96 (1977)

[2] I. Lukačević *On Relative Deformation in Relativistic Kinematics*, Publ. Inst. Math. N. S. 19 (33) 1975.

[3] R. C. Tolman *Relativity, Thermodymics and cosmology*, Oxford; Clarendon Press, 1950.

LE TENSEUR D'EXPANSION DANS LA METRIQUE MODIFIEE DE DE SITTER

R é s u m é

Le modèle cosmologique de De Sitter était publie en 1917. La densité de la matière y est egale à zero et c'est le modele de l'univers „vide". A propos de ce modèle, Einstein a suggeré que la matière était repandue sur une sphere. Cette idée était réalisée par Mal'cev et Markov soixante ans plus tard [1].

On considère ici le tenseur de l'expansion dans le modèle de De Sitter ainsi modifié. Le tenseur de l'expansion est défini par

$$\vartheta_{\alpha\beta} = \mathfrak{L}_{\xi} (g_{\alpha\beta} + \xi_{\beta} \xi_{\alpha})$$

ou ξ_{α} est le vecteur orienté dans le temps, $g_{\alpha\beta}$ est le tenseur métrique de l'espace considéré et $\mathfrak{L}_{\xi}$ est la derivation de Lie. Dans un systeme de coordonnées convenable on trouve des conditions qui déterminent une expansion positive.

TENZOR EKSPANZIJE U MODIFIKOVANOJ DE SITEROVOJ METRICI

I z v o d

Kosmološki model De Sitera objavljen je 1917. godine. U tom modelu gustina materije jednaka je nuli i zato je to model „prazne" vasione. Povodom tog modela Ajnštajn je nagovestio mogućnost da je materija raspoređena na jednoj sferi. Tu ideju realizovali su V.K. Maljcev i M.A. Markov šezdeser godina kasnije [1].

U ovom radu ispituje se tenzor ekspanzije u tako modifikovanoj De Siterovoj metrici. Tenzor ekspanzije definiše se kao

$$\vartheta_{\alpha\beta} = \mathfrak{L}_{\xi} (g_{\alpha\beta} + \xi_{\alpha} \xi_{\beta})$$

gde je ξ_{α} jedinični vektor vremenskog tipa, $g_{\alpha\beta}$ je metrički tenzor posmatranog prostora, a $\mathfrak{L}_{\xi}$ je Liov izvod. U odnosu na pogodno izabran koordinatni sistem nalaze se uslovi koji određuju pozitivnu ekspanziju.

Dragi Radojević, Matematički institut
Knez Mihailova 35, 11000 Beograd