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# A NEW APPROACH FOR DEVELOPMENT OF FINITE ELEMENTS

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## 1. Introduction

In the current practice the most common approach in the development of finite elements is energetic. The minimum potential energy variational principle is the most often applied approach. The stiffness matrix of an element is developed by variation of the total potential energy.

The deformation shape function is required to be compatable. This requirement sometimes is difficult to be fulfilled, (for instance in the shell problems). Can be shown that, even in the case of compatable deformation shape function, there are some approximations and neglections. Any variational principle involves certain approximations, whish, quite often, is not clear of what character they are. The comptuation of the potential energy involves two or three dimensional integration. Those integrations, for instance in the case of curved isoparametric elements, can be quite rough, and can lead to big errors.

All those difficulties can be overcamed by the approach which in short is described in this paper.

## 2. The concept

The matrix equation of equilibrium of a certain problems is as follows

$$[K]\{\delta\} = \{P\} \tag{1}$$

where $[K]$ is the stiffness matrix, $\{\delta\}$ — vector matrix of the unknown nodal deformations, and $\{P\}$ is the matrix of the external load.

The stiffness matrix can be understood as a matrix of influence coefficients, which define the influence of the unit nodal deformations on the nodal forces. It seems logical to define these coefficients by application of unit nodal deformations, giving unit value to one of the nodal deformations, while holding the others equal to zero (Fig. 1).

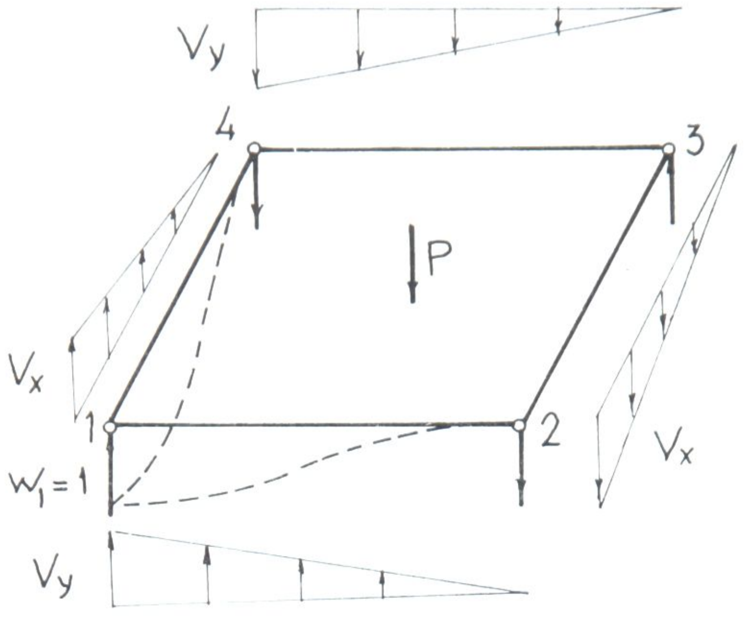


Fig. 1

On Fig. 1 is represented a plate bending element, on which a deformation $W_1 = 1$ is given. Due to that deformation on the boundaries of the element there apprear shear forces. In the case of linear variation of those forces, as on Fig. 1, it is easy to find their distribution on the nodes $1-4$. So distributed forces at the nodes actually give the stiffness coefficients $K_{11}$, $K_{12}$, $K_{13}$ and $K_{14}$, corresponding to the nodal deflections $W_1 - W_4$. In that way, without any integration, one can gat some of the stiffness coefficients.

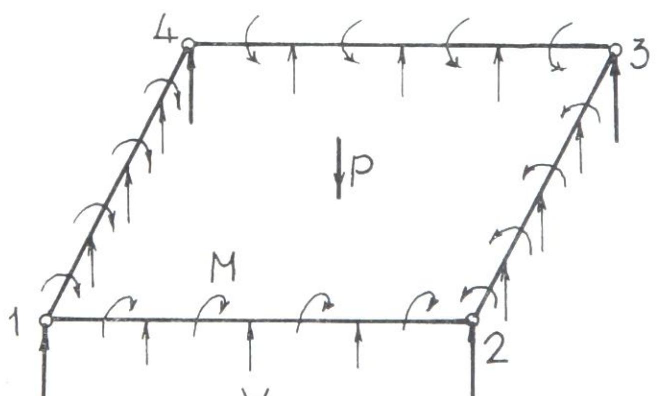


Fig. 2

Along the element boundaries (Fig. 2) there are boundary forces, which are in equilibrium with the external forces P. The finite elements are interconected at the boundary nodes, and the final equations define the equlibrium of the forces at those points. The task is to find the boundary forces and distribute them to the nodes, (Fig. 2).

The boundary forces actually are defined by the nodal deformations, as were the shear forces due to $W_1 = 1$ on Fig. 1. Instead of giving unit deformations for all nodal deformations separately, it is more ocnvinient to define a complete field deformation due to nodal deformations at once. That field of deformation can be described for instance by a simple polinomial,

$$W = a_1 + a_2 x + a_3 y + a_0 x^2 + \dots \tag{2}$$

That is so called deformation shape function. It should be considered as a possible given deformation. The opinion that this function is an approximate solution of the problem is not quite all right, and has led to some conclusions, which are not allways correct. Following the standard procedure in the finite element method the coefficients $a_1$ are expressed in terms of the nodal deformations.

Knowing the physical meaning of the equations which have to be developed, it is easy to develop them. For instance, in the problem of plate bending, when the unknows („degrees of freedom") are the nodal deformations $W$, $\partial W/\partial x$, $\partial W/\partial y$, those equations are the equilibrium equations of the normal forces $V$ and moments $Mx$ and $My$. So, by the third and second derivatives of the deformation shape function (2), the boundary cuts are determined.

The distribution of the boundary cuts on the particular nodes can be as follows. The deformation field, for instance the deformation due to $W_1 = 1$ (Fig. 1), represents influence surface for the normal force $V$ at the node 1. The deformation lines 1—2 and 1—4 represent influence lines vor $V_1$ also. Therefore, by multiplication of the shear forces with the corresponding deformations $w$, the concentrated force $V_1$ is computed. That force is equal to the stiffness,

$$K_{11} = \int_0^b V_x \cdot W_{14} \cdot dy + \int_0^a V_y \cdot W_{12} \cdot dx \tag{3}$$

This equation can be understood as a work of the shear forces due to $W_1 = 1$, on the deformation due to $W_1 = 1$ also.

The distribution of the external load $P$ (Fig. 1), to the element nodes, can be by use of the influence surface, i.e. the deformation shaep function. It is defined by the product of $P$ with the deformation under the force. That product can be understood as a work of the external load on the given deformation.

The total work defines the so called functional. The functional in the problem of bending of plates is as follows,

$$\Pi_p = \int_s Mn \frac{\partial w}{\partial s} ds - \int_s V \cdot w \cdot ds + \iint p \cdot w \cdot dx \cdot dy \tag{4}$$

In that way the work of the internal forces, i.e. the potential energy $U$, is computed by linear integration only, instead of area integration. In general, for any problem the functional can be described as follows,

$$\Pi_p = \int_s F_s \cdot w_s \cdot ds + \int_v p \cdot w \cdot dv \tag{5}$$

where the first integral is a boundary integral of the all boundary forces $F_s$, on the corresponding boundary deformations $w_s$, and the second integral is a volume integral of the external load $p$, on the corresponding deformations $w$.

The substitution of the potential energy $U$ with the work of the boundary cuts,

$$U = \int_{s} F_s \cdot w_s \cdot ds \tag{6}$$

easily can be explained by the well known principle of the classical mechanics: the work of the internal forces is equal to the work of the external forces. In this case, in the development of the stiffness matrix, the boundary forces for the element are external forces. In order this statement to be true, the deformation shape function (2) should be simple, describing deformations which are due to given nodal deformations. The functional (5) can be developed mathematically also.

The application of a functional of the form of expression (5) is simple. Nevertheless, it is better to apply the previously described procedure, by gviing unit deformations. In that way the physical meaning of an operation is much easy to follow.

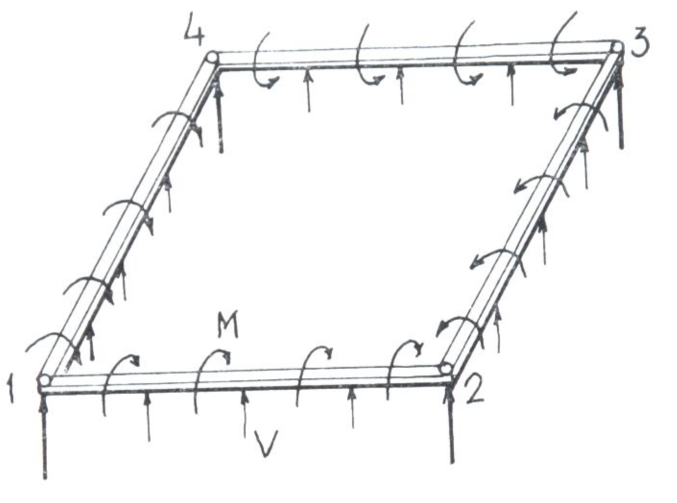


Fig. 3

The described procedure for development of the element matrix can be explained in the following way (Fig. 3) .The element can be imaged as a beam grid interconected at the nodes. The forces acting on the beams are determined on the base of the deformation shape function. Those forces have to be transfered to the supports (nodes). The forces at a node, for instance node 1, which a function of the nodal deformations, define the stiffness coefficients in a rwo, i.e. the stiffness coefficients $K_{1i}$.

The distribution of the forces acting on the beams to the nodes can be on the base of the beam theory. In the case of stiffness elements the beams should be considered as built up at the nodes. The most easy way of distribution of those forces to the nodes is the use of the deformation boundary lines due to unit nodal deformation (as on Fig. 1), which, as was mentioned, represent influence lines. In the clasical slope deflection method and three moments rule, instead of using tables, in a simmilar way we compute the load terms.

In the case of three dimensional elements simmilar procedure can be used. In that case the beam grid is three dimensional.

By the application of the concept described, or the functional (5), some additional terms, which are not present in the potential energy, are derived. In that way an improved stiffness matrix is developed. The stiffness matrix correction which in that way is derived, is a matrix with summ of the coefficients in a row equal to zero. In the limit, when the size of the elements becomes small enought, the contribution of that stiffness correction tends to zero. Therefore, the present elements, without this correction, give converging results. However, the inclusion of the stiffness matrix correction is very important. With the corrected stiffness elements one can get certain accuracy with much less computation that with the elements without stiffness correction. What is important, is that the results obtained with so improved elements allways are reliable.

In the development of finite elements on the base of the approach described here, by giving unit deformations or unit forces, a curious thing hase been noticed. Some terms of the stiffness or flexibility matrix are not simmetric. That is,

$$K_{ij} \neq K_{ij} \tag{7}$$

Some of them could have even different signes. The stiffness coefficeints derived by the application of the potential energy actually are mean value, equal to $(K_{ij} + K_{ji})/2$. However, the diagonal stiffness coefficients usually are dominant, and they are very good, even they can be exact [4]. Therefore, the results derived with average, simmetric matrix, are good. Nevertheless, some results derived with the unsimmetric element matrix show slightly improved accuracy.

The deformation shape function does not have to be compatable. By the use of boundary integration in the computation of the potential energy (5) the effect of the boundary "discontinuity" is taken into account. In that way the problem of the compatibility actually does not represent any problem.

Some adventages the approach has in the development of the curved isoparametric elements. More about it shall be given later on.

The deformation shape function must not be considered as an approximate solution of the problem. It is a very bad base for computation of the stresses. The stress matrix should be derived from the stiffness matrix and the effect of the distributed load. Some preliminary results in the problem of plate bending show much improvement in the stresses determined in that way.

## 3. Application

The approach presented we have applied for ddevelopment of a mixed element for analysis of plate bending [3]. In that way several improvements were obtained. The accuracy the element gives is very good, particulary the accuracy of the stresses. Now development of stiffness elements for two dimensional analysis is in progres. Preliminary results are promising, particulary the results for stresses.

The present concept in the development of isoparametric elements, with more ore less Gauss points of integration, is questionable. A correctly developed element, with all factors influencing the element taken into account, the best results should give at the node points, not at the Gauss points.

An improved isoparametric element can be developed by application of the concept presented here (Fig. 4). The Gauss points now are along the boundaries.

The line integration which has to be applied should be much better than area integration which has been used. ·uch one curved mixed element for plate bending analysis is under development. The element is simple one, with midside nodes only for definition of the geometry.

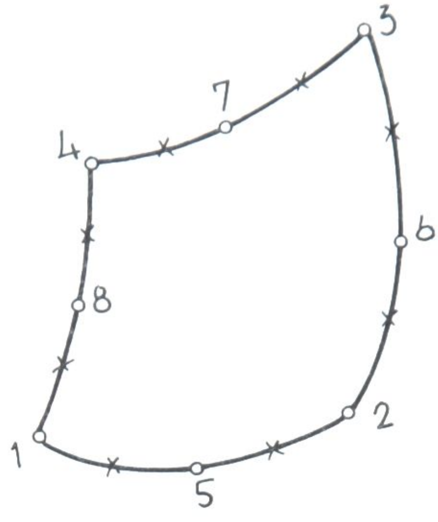


Fig. 4

The concept has advantages in the solution of shell problems. In the development of shell elements it is very difficult to find a compatable deformation shape function, which at the same time can describe rigid body motion also. With the approach presented here this problem is easily overcamed. A simple mixed shell elements based on this approach is under development.

The approach shortly outlined in the previous text should find application in the solution of all types of problems, two and three dimensional. The present two and three dimensional elements should be revised, although they give converging results. The reasons for this sugestion were given in the previous chupter of this paper.

## 4. Conclusions

A new approach for development of finite elements for solid continua was described. The approach is based on the soun l principles of the classical mechanics. The potential energy is substituted with the work of the boundary forces. In that way improved element matrixes are derived.

The proposed concept has some advantages over the present concept. It is a simple one and gives better, allways reliable results. The deformation shape function does not have to be compatable.

The concept should be applied for solution of all kinds of problems. Improved elements, particulary curved isoparametric elements, should be developed.

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## НОВЫЙ ПОДХОД ДЛЯ РАЗВИТИЯ КОНЕЧНИХ ЭЛЕМЕНТОВ

Р е э ю м е

Потенциальная энергия в элементе заменяется работой сил на границах элемента (6). Таким образом вклюочаются дополнительные члены, которие в потенциальной знергии не содержатся, и получаетстя улучшенная матрица жесткости.

Функция деформаций в элементе не должна быт компатибильной. На основании этой функции определяются силы на границах элемента (Рис 3.). Элемент можно разсматривать как бы связание узли балками. Эти силы переносятся в узловых точках и получаются коефициенты матрицы жесткости.

Физическое значение проблеммы всегда надо иметь в виду. Рекомендуется применение теоремы об одиночной деформации и одиночной силы. Таким образом для некоторых коефициентов жесткости получаются $K_{ij} \neq K_{ji}$.

Результатыб полученние элементами развитих по предложенному подходу лучшие, и всегда их можно брать с доверием. Поэтому надо получить лучших элементов по предложенному концепту для решения всех типов задач. Это особенно существенно для изопараметрических элементов.

## НОВ ПРИОД ЗА РАЗВИВАЩЕ НА КОНЕЧНИ ЕЛЕМЕНТИ

И з в о д

Потенциалната енергија во елементот се заменува со работата на пресечните сили на границите од елементот. На тој начин се вклучуват и некои дополнителни членови, кои во потенцијалната енергија не се содржат, и се добива подобрена матрица на елементот.

Функцијата на деформациите во елементот не мора да биде компатибилна. На основание на таа функција се определуват пресечните сили на границите од елементот (Сл. 3). Елементот може да се смета како гредна скара. Тие сили се пренесуват во јазловите точки и се добиват коефициентите од матрицата на крутоста.

Физичкиот смисол на задачата секогаш треба да се има во вид. Се препорачува примената на теоремата за единечна деформација и единечна сила. На тој начин за некои коефициенти на матрицата на елементот се добива $K_{ij} \neq K_{ji}$.

Добиените резултати со елементи развиени по предложениот приод се подобри, и секогаш можат да се примат со доверба. Затоа по предложениот приод треба да се развиат подобрени елементи за решение на сите типови задачи. Тоа нарочно важи за изопараметриjските елементи.

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