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# NOETHER'S THEOREM IN THE LINEAR THEORY OF ELASTIC DIELECTRICS

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## 1. Introduction

Particular attention has been paid in continuum mechanics to „the path-independent integrals" recently. Apart from their inherent theoretical interest these integrals are of pratical importance in connection with the direct asymptotic analysis of geometrically induced singular stress concentrations, such as those occasioned by cracks and notches.

The way to drive them differs from author to author. It was shown in a recent paper by Knowles and Sternberg [1] that J-integral or the conservation law of Rice [2], as well as its three-dimensional analogue, can be generated systematically with the aid of a theorem due to Noether [3] on invariant variational principles in conjuction with the principle of stationary potential energy. It was shown that, within the context of linear isotropic, homogeneous elastostatics, the three conservation laws are complete in the sense that they are the only ones furnished by Noether's theorem.

Noether's theorem on variational principles invariant under a group of infinitesimal transformations was used by Fletcher [4] to obtain a class of conservation laws associated with linear elastodynamics.

It was shown in [1] and in later paper by Green [5], that analogous laws exist for finite deformations of homogeneous elastic materials, but the completeness of these conservation laws, within the framework of Noether's theorem, was not proved. More recently, Chen and Shield [6] investigated, on the basis of the framework of Noether's theorem, the completeness of the conservation laws for finite elastic deformations among the class of laws expressible as functionals linear in the strain energy W and its first derivatives with respect to the deformation gradients.

Aifantis [7] considered conservation laws for symmetric stressdiffusion fields surrounding line cracks.

Gurtin remarked in [8] that path-independent integrals can be derived for both dynamic and quasi-static linear viscoelasticity utilizing the convolution as the basic tool.

It is our purpose to extend the investigation of conservation laws to the linear theory of elastic dielectrics.

## 2. Noether's Theorem

Let

(2.1) $$\bar{\underset{\sim}{X}} = \bar{\underset{\sim}{X}}(\underset{\sim}{X}, \eta),$$

be a regular one-parameter family of transformation,

where

(2.2) $$\underset{\sim}{X} = \underset{\sim}{X}(\underset{\sim}{\xi}, \underset{\pi}{\tilde{W}}, \underset{\sigma}{\varphi}), \qquad \begin{matrix}(\pi = 1,2 \; \dots\dots \; m)\\(\sigma = 1,2 \; \dots\dots \; 1)\end{matrix}$$

$\underset{\sim}{\xi}(\xi_\alpha)$ — $(\alpha = 1, 2, \dots, n)$ — a point in regular Cartesian coordinates of $n$-dimensional Eucliden space $E_n$,

$\underset{\pi}{W}(\underset{\pi}{W_i})$ — $(i = 1, 2, \dots, p)$ — and $\underset{\sigma}{\varphi}$ — are arbitrary $m$ — vector fields of $p$ components and 1 scalar fields, respectively, defined and twice continuously differentiable on bounded, closed, regular region $R$ in $E_n$.

For $\eta = 0$ the transformations are required to be identical

2.3) $$\underset{\pi}{W} = \underset{\pi}{W}(\xi), \quad \underset{\sigma}{\varphi} = \underset{\sigma}{\varphi}(\xi), \quad \bar{\underset{\sim}{X}} = \underset{\sim}{X}.$$

Now we define a functional $\mathcal{L}$ for given fields $Y$ by the formula

(2.4) $$\mathcal{L}(\underset{\pi}{W}, \underset{\sigma}{\varphi}) = \int_R L(\underset{\sim}{Y})\, d\underset{\sim}{\xi},$$

where

(2.5) $$\underset{\sim}{Y} = \underset{\sim}{Y}(\underset{\sim}{\xi}, \underset{\pi}{W}, \nabla \underset{\pi}{W}, \underset{\sigma}{\varphi}, \nabla \underset{\sigma}{\varphi}).$$

The functional $\mathcal{L}$ in (2.4) is said to be invariant at $(\underset{\pi}{W}, \underset{\sigma}{\varphi})$ under transformation (2.1) if

(2.6) $$\int_{\bar{R}} L(\bar{\underset{\sim}{Y}})\, d\bar{\underset{\sim}{\xi}} = \int_R L(\underset{\sim}{Y})\, d\underset{\sim}{\xi},$$

for all sufficiently small values of $|\eta|$.

If, for a given $(\underset{\pi}{W}, \underset{\sigma}{\varphi})$,

(2.7) $$\frac{d}{d\eta}\left[\int_{\bar{R}} L(\bar{\underset{\sim}{Y}})\, d\bar{\xi}\right]_{\eta=0} = 0,$$

then $\mathcal{L}$ is said to be infinitesimally invariant at $(\underset{\pi}{W}, \underset{\sigma}{\varphi})$.

Evidently, if $\mathcal{L}$ is invariant at $(\underset{\pi}{W}, \underset{\sigma}{\varphi})$, then $\mathcal{L}$ is infinitesimaly invariant at $(\underset{\pi}{W}, \underset{\sigma}{\varphi})$.

Now we state a restricted version of Noether's Theorem 1: Let $R$ be a domain in $E$, and suppose $(\underset{\pi}{W}, \underset{\sigma}{\varphi})$ satisfy the Euler-Lagrange equations:

(2.8) $$L, \underset{\pi}{W_i} - \frac{\partial}{\partial \xi_\alpha} L, \underset{\pi}{W_{i,\alpha}} = 0,$$

(2.9) $$L, \underset{\sigma}{\varphi} - \frac{\partial}{\partial \xi_\alpha} L, \underset{\sigma}{\varphi_{,\alpha}} = 0,$$

Then $\mathcal{Z}$ in (2.4) is infinitesimally invariant at $(\underset{\pi}{W}, \underset{\sigma}{\varphi})$ under transformations (2.1) for every bounded regular subregion $R$ of $R$ if satisfies

$$\frac{\partial}{\partial \xi^{\alpha}}\left[L(\underset{\sim}{Y})\,\alpha_{\alpha}+L,\underset{\pi}{W}_{i,\alpha}(\underset{\sim}{Y})\,\underset{\pi}{p}_i+L,\underset{\sigma}{\varphi}_{,\alpha}\,\underset{\sigma}{q}\right]=0 \tag{2.10}$$

where

$$\alpha\equiv\left(\frac{\partial\bar{\xi}}{\partial\eta}\right)_{\eta=0},\quad \beta\equiv\left(\frac{\partial\bar{W}}{\partial\eta}\right)_{\eta=0},\quad \underset{\sigma}{\gamma}\equiv\left(\frac{\partial\underset{\sigma}{\varphi}}{\partial\eta}\right)_{\eta=0}$$

$$\underset{\pi}{p}_i=\underset{\pi}{\beta}_i-\underset{\pi}{W}_{i,\alpha}\,\alpha_{\alpha},\quad \underset{\sigma}{q}=\underset{\sigma}{\gamma}-\underset{\sigma}{\varphi}_{,\alpha}\,\alpha_{\alpha}. \tag{2.11}$$

Further, (2.10) is equivalent to the conservation law in integral form asserting that

$$\int_{\partial R}[L(\underset{\sim}{Y})\,\alpha_{\alpha}+L,\underset{\pi}{W}_{i,\alpha}(\underset{\sim}{Y})\,\underset{\pi}{p}_i+L,\underset{\sigma}{\varphi}_{,\alpha}\,\underset{\sigma}{q}]\;n_{\alpha}(\xi)\,d\xi=0, \tag{2.12}$$

for every $\partial R$ that is the boundary of a regular subregion of $R$, provided $\underset{\sim}{n}$ is the outward unit normal vector of $\partial R$.

The proof of this theorem can be found in [1], [4], [10]. We used the following notations in this paper:

$$L_{,\alpha}=\frac{\partial L}{\partial\xi_{,\alpha}},\quad L,\underset{\pi}{W}_i=\frac{\partial L}{\partial\underset{\pi}{W}_i},\quad L,\underset{\pi}{W}_{i\,\alpha}=\frac{\partial L}{\partial\underset{\pi}{W}_{i,\alpha}} \tag{213}$$

$$L,\underset{\sigma}{\varphi}=\frac{\partial L}{\partial\underset{\sigma}{\varphi}},\quad L,\underset{\sigma}{\varphi}_{,\alpha}=\frac{\partial L}{\partial\underset{\sigma}{\varphi}_{,\alpha}}$$

provided the foregoing differentiations are used. By $L/\alpha$ we denote the partial derivative of $L$ with respect to $L,\alpha$ to distinguish it from the total derivative $L,\alpha$.

## 3. The classical, linear theory of elastic dielectrics

R. D. Mindlin [9] in a paper devoted to elasticity, piezoelectricity and crystal lattice dynamics, gave the Euler equations which, for classical linear theory of elastostatic dielectrics in the absence of body forces, read:

$$\begin{aligned} &T_{ij,i}=0,\\ &E_{ij,i}+E_j{}^L-\varphi_{,j}=0\\ &-\varepsilon_0\,\varphi_{,jj}+\pi_{j,j}=0,\end{aligned} \tag{3.1}$$

where the above quantities represent:

$T_{ij}$ — stress tensor; $E_{ij}$ — local electric stress tensor; $E^L{}_j$ — electric field; $\varphi$ — electric potential; $\pi_j$ — polarization vector; $u_{i,j}$ — displacement vector; $e_{ij}$ = $= u_{(i,j)}$ — infinitesimal strain tensor.

The corresponding constitutive quantities are given by

$$T_{ij}=\frac{\partial W^L}{\partial e_{ij}},\qquad E_i{}^L=\frac{\partial W^L}{\partial \pi_i},\qquad E_{ij}=\frac{\partial W^L}{\partial \pi_{j,i}}, \tag{3.2}$$

where the stored energy of deformation and polarization $W^L$ is given by

$$W^L=\frac{1}{2}A_{ij}\,\pi_i\,\pi_j+\frac{1}{2}B_{ijkl}\,\pi_{j,i}\,\pi_{l,k}+\frac{1}{2}C_{ijkl}\,e_{ij}\,e_{kl}+D_{ijkl}\,\pi_{j,i}\,e_{kl} \tag{3.3}$$

If we define an electric enthalpy

$$H=W^L(e_{ij},\pi_i,\pi_{ij})-\frac{1}{2}\varepsilon_0\,\varphi_{,i}\,\varphi_{,i}+\varphi_{,i}\,\pi_i, \tag{3.4}$$

then we can show that

$$\frac{\partial H}{\partial u_{i,j}}=\frac{\partial H}{\partial e_{ij}}=\frac{\partial W^L}{\partial e_{ij}},\quad \frac{\partial H}{\partial \pi_{j,i}}=\frac{\partial W^L}{\partial \pi_{j,i}},\quad \frac{\partial H}{\partial \pi_i}=\frac{\partial W^L}{\partial \pi_i}+\varphi_{,i}. \tag{3.5}$$

and the equations (3.1) can be written as

$$\left(\frac{\partial H}{\partial u_{j,i}}\right)_{,i}=0,\quad \frac{\partial H}{\partial \pi_j}-\left(\frac{\partial H}{\partial \pi_{j,i}}\right)_{,i}=0,\quad \left(\frac{\partial H}{\partial \varphi_{,i}}\right)_{,i}=0. \tag{3.6}$$

Since,

$$\frac{\partial H}{\partial x_i}=0,\quad \frac{\partial H}{\partial u_i}=0,\quad \frac{\partial H}{\partial \varphi}=0, \tag{3.7}$$

it follows that the relations (3.6) are the Euler-Lagrange equations.

Let $L$ in (2.4) be

$$L(\underset{\sim}{Y})=H(\underset{\sim}{Z}),\qquad \underset{\sim}{Z}\equiv(u_{i,j},\ \pi_i,\ \pi_{i,j}\ \varphi_{,i}), \tag{3.8}$$

Then, Noether's theorem holds and so does the conservation law (2.12) in the form

$$\int_S [H(Z)\,\alpha_j+T_{ij}\,p_i+E_{ij}\,q_i+(-\varepsilon_0\,\varphi_{,j}+\pi_j)r]\,n_j\,da=0, \tag{3.9}$$

where

$$\xi_i=x_i,\ \ u_i=\underset{1}{W_i},\ \ \pi_i=\underset{2}{W_i},\ \ \varphi=\underset{1}{\varphi},\ \ \beta_i=\underset{1}{\beta_i},\ \ \gamma_i=\underset{2}{\beta_i}$$

$$\gamma=\underset{1}{\gamma},\ \ p_i=\underset{1}{p_i}\ \ q_i=\underset{2}{p_i},\ \ r=\underset{1}{q} \tag{3.10}$$

The actual form of the law (3.10) depends upon the existence of the corresponding family of transformations (2.1) or (2.11). Thus it is our next task to determine of this family of transformations.

## 4. Inverse Noether's Theorem

Theorem. Suppose the elastic dielectric material is isotropic and let $\mathcal{L}$ be the functional

$$\mathcal{L}(\underset{\sim}{\psi}) = \int_V H(Z)\, d\underset{\sim}{x}, \tag{4.1}$$

where $V$ is a bounded regular region, $H$ is given by (3.4), and $\psi = (u, \pi, \varphi)$. The $\mathcal{L}$ is infinitesimally invariant at $\psi$ under transformations

$$\bar{\psi} = \bar{\psi}(\psi, \eta), \tag{4.2}$$

for every $\psi$, wich do not depend on the material property of body and which satisfy the equations (3.2), if $\underset{\sim}{\psi}$ satisfy

$$\bar{\underset{\sim}{\psi}} = \underset{\sim}{\psi} + \underset{\sim}{\Phi}(\underset{\sim}{\psi})\, \eta + 0(\eta^2), \tag{4.3}$$

where

$$\underset{\sim}{\Phi}(\underset{\sim}{\psi}) = \left(\frac{\partial \bar{\underset{\sim}{\psi}}}{\partial \eta}\right)_{\eta=0} = (\underset{\sim}{\alpha}, \underset{\sim}{\beta}, \underset{\sim}{\gamma}, \delta) \tag{4.4}$$

$$\alpha_i = a_i, \quad \beta_i = \varepsilon_{ijk}\, x_j\, \Phi_k + b_i, \quad \gamma_i = 0, \quad \delta = c \tag{4.5}$$

and $a_i$, $b_i$, $c$, $\Phi_i$ are arfiitrary constants and $\varepsilon_{ijk}$ are the components of the three--dimensional alternating tensor.

Proof of Theorem 2:

According to the Theorem 2, the relations (3.1-6) and (3.9) hold. The local balance law of (3.9) can be written, fiy the use of (3.2) and (3.6), in the form

$$\frac{\partial}{\partial x_j}\left[H(Z)\,\alpha_j + \frac{\partial W^L}{\partial e_{ij}}\, p_i + \frac{\partial W^L}{\partial \pi_{j,i}}\, q_i + \frac{\partial H}{\partial \varphi_{,j}}\, r\right] = 0. \tag{4.5}$$

Using (3.3-6) in (4.6), after longer calculations and rearrangements of the terms in this expression, we obtain

$$\begin{aligned}
&C_{ijpq}\, e_{pq}\,(\beta_{j,i} - u_{j,k}\,\alpha_{k,i} + \frac{1}{2}\, e_{ij}\,\alpha_{s,s}) + D_{ijpq}\,[\pi_{q,p}\,(\beta_{j,i} - u_{j,k}\,\alpha_{k,i} + e_{pq}\,\alpha_{s,s}) + \\
&\qquad + e_{pq}\,(\gamma_{j,i} - \pi_{j,k}\,\alpha_{k,i})] + \\
&B_{ijpq}\,(\gamma_{j,i} - \pi_{j,k}\,\alpha_{k,i} + \frac{1}{2}\,\pi_{j,i}\,\alpha_{s,s}) + \\
&A_{ij}\,\pi_i\,(\gamma_i + \frac{1}{2}\,\pi_{j,i}\,\alpha_{s,s}) + \\
&-\varepsilon_0\,\varphi_{,i}\,(\delta_{,i} - \varphi_{,k}\,\alpha_{k,i} + \frac{1}{2}\,\varphi_{,i}\,\alpha_{s,s}) + \\
&\pi_j\,(\delta_{,i} - \varphi_{,k}\,\alpha_{k,i}) + \varphi_{,i}\,\gamma_j + \pi_i\,\varphi_{,i}\,\alpha_{s,s} = 0
\end{aligned} \tag{4.7}$$

Since $\alpha$, $\beta$, $\gamma$, $\delta$ are independent of $\pi_{i,j}$ $u_{i,j}$ and $\varphi_{,i}$ it follows that the linear, quadratic, and cufiic parts in $\pi_{j,i}$ $u_{i,j}$ and $\varphi_{,i}$ of expresion (4.7) must vanish separately. Moreover, we require, according to Theorem 2, that the transformations (2.1) do not depend on material properties of a fiody. From a mathematical point of view this is equivalent to the requirement that the terms along the coefficients $C_{ijpq}$, $D_{ijpq}$, $B_{ijpq}$, $A_{ij}$ and $\varepsilon_0$ in (4.7), must vanish separately, i.e.

$$C_{ijpq}\, e_{pq}\, (\beta_{j,i} - u_{j,k}\,\alpha_{k,i} + \frac{1}{2}\, e_{ij}\,\alpha_{s,s}) = 0 \tag{4.8}$$

$$D_{ijpq}\, [\pi_{q,p}\,(\beta_{j,i} - u_{j,k}\,\alpha_{k,i} + e_{pq}\,\alpha_{s,s}) + e_{pq}\,(\gamma_{j,i} - \pi_{j,k}\,\alpha_{k,i})] = 0 \tag{4.9}$$

$$B_{ijpq}\, (\gamma_{j,i} - \pi_{j,k}\,\alpha_{k,i} + \frac{1}{2}\,\pi_{j,i}\,\alpha_{s,s}) = 0 \tag{4.10}$$

$$A_{ij}\, (\gamma_i + \frac{1}{2}\,\pi_j\,\alpha_{s,s}) = 0 \tag{4.11}$$

$$-\varepsilon_0\,\varphi_{,i}\,(\delta_i - \varphi_{,k}\,\alpha_{k,i} + \frac{1}{2}\,\varphi_{,i}\,\alpha_{s,s}) = 0 \tag{4.12}$$

$$\pi_i\,(\delta_{,i} - \varphi_{,k}\,\alpha_{k,i}) + \varphi_{,j}\,\gamma_j + \pi_i\,\varphi_{,i}\,\alpha_{s,s} = 0 \tag{4.13}$$

From (4.13), if follows that

$$\gamma_j = 0 \tag{4.14}$$

fiecause $\gamma_j$ is the coefficient along the linear part of (4.13) in $\varphi_{,j}$ and (4.13) must vanish without restriction on $\varphi_{,j}$ and $\pi_i$. The linear term of (4.12) in $\varphi_{,i}$ gives

$$\delta_{,i} = 0 \tag{4.15}$$

During further discussion we shall use the explicit form of the following isotropic tensors:

$$\begin{aligned} C_{ijpq} &= \lambda\delta_{ij}\,\delta_{pq} + \mu\,(\delta_{ip}\,\delta_{iq} + \delta_{iq}\,\delta_{jp}) \\ D_{ijpq} &= d_1\,\delta_{ij}\,\delta_{pq} + d_2\,(\delta_{ip}\,\delta_{jq} + \delta_{iq}\,\delta_{jp}) \\ B_{ijpq} &= b_1\,\delta_{ij}\,\delta_{pq} + b_2\,\delta_{ip}\,\delta_{jq} + b_3\,\delta_{iq}\,\delta_{jp} \\ A_{ij} &= a_0\,\delta_{ij} \end{aligned} \tag{4.16}$$

and the expression

$$\begin{aligned} \alpha_{s,r} &= \alpha_{s/r} + \alpha_{s,u_t}\,u_{t,r} + \alpha_{s,\pi_t}\,\pi_{t,r} + \alpha_{s,\varphi}\,\varphi_{,r} \\ \beta_{s,r} &= \beta_{s/r} + \beta_{s,u_t}\,u_{t,r} + \beta_{s,\pi_t}\,\pi_{t,r} + \beta_{s,\varphi}\,\varphi_{,r} \\ \gamma_{s,r} &= \gamma_{s/r} + \gamma_{s,u_t}\,u_{t,r} + \gamma_{s,\pi_t}\,\pi_{t,r} + \gamma_{s,\varphi}\,\varphi_{,r} \\ \delta_{,r} &= \delta_{/r} + \delta_{,u_t}\,u_{t,r} + \delta_{,\pi_t}\,\pi_{t,r} + \delta_{,\varphi}\,\varphi_{,r} \end{aligned} \tag{4.17}$$

From (4.17) and (4.15) we ofitain

$$\delta = C \tag{4.18}$$

where $c$ is an arfiitrary real constant of integration.

With the help of (4.16), (4.14) and the requirement that a polarization vector $\pi_i$ (xist i.e. $\pi_i \pi_j \neq 0$, from (4.11) there follows

(4.19) $$\alpha_{s,s} = 0$$

or

$$\alpha_{s/s} = 0$$

$$\alpha_{s,u_t} = 0$$

(4.20) $$\alpha_{s,\pi_t} = 0$$

$$\alpha_{s,\varphi} = 0$$

according to $(4.17)_1$.

Using (4.13), (4.14), (4.15), (4.19) and (4.20) we ofitain

(4.21) $$\alpha_{k,i} = 0,$$

which, on account of (4.17) and (4.20) is equivalent to

(4.22) $$\alpha_{k/i} = 0.$$

From (4.20) and (4.22) we have

(4.23) $$\alpha_i = a_i,$$

where $a_i$ is an arfiitrary constant.

The expressions (4.10-13) are identically satisfied because of (4.14), (4.15) and (4.20), while (4.8) and (4.9) give

(4.24) $$C_{ijpq}\, e_{pq}\, \beta_{j,i} = 0$$

(4.25) $$D_{ijpq}\, \pi_{q,p}\, \beta_{j,i} = 0$$

From (4.25) we ofitain

(4.26) $$D_{ijpq}\, \beta_{j,i} = 0$$

Returning $(4.16)_2$ and $(4.17)_2$ we see that

(4.27) $$\beta_{(i/j)} = 0, \quad \beta_{i,u_j} = 0, \quad \beta_{i,\pi_j} = 0, \quad \beta_{i,\varphi} = 0.$$

The solution of this system of differential equations reads

(4.28) $$\beta_i = \Phi_{ij}\, x_j + b_i,$$

where $\Phi_{ij} = \Phi_{ji}$ and $b_i$ are arbitrary constants.

If we denote

(4.29) $$\Phi_{ij} = \varepsilon_{ijk}\, \Phi_k$$

then (4.28) can fie written as

(4.30) $$\beta_i = \varepsilon_{ijk}\, x_j\, \Phi_k + b_i$$

## 5. Conservation laws

It can be seen from (4.5) that four families of transformations which correspond to the arfiitrary values of four sets of constants $a_i$, $b_i$, $c$ and $\Phi_i$ exist. To these transformations the following conservation laws correspond:

I). $$b_i \neq 0, \quad a_i = \Phi_i = 0, \quad c = 0.$$

The corresponding family of transformations is

(5.1) $$\bar{x}_i = x_i, \quad \bar{u}_i = u_i + b_i \eta \quad \bar{\pi}_i = \pi \quad \bar{\varphi} = \varphi$$

which represents the rigid fiody translation, and

(5.2) $$\int_R T_{ij} \, n_j \, da = 0$$

II). $$\Phi \neq 0, \quad a_i = b_i = 0, \quad c = 0$$

(5.3) $$\bar{x}_i = x_i, \quad \bar{\pi}_i = \pi_i, \quad \bar{u}_i = u_i + \varepsilon_{ijk} \, x_j \, \Phi_k \eta, \quad \bar{\varphi} = \varphi$$

which represents the rigid fiody rotation, and

(5.4) $$\int \varepsilon_{ikl} \, x_k \, T_{ij} \, n_j \, da = 0$$

III). $$c \neq 0 \quad a_i = b_i = \Phi_i = 0$$

(5.5) $$\bar{x}_i = x_i, \quad \bar{\pi}_i = \pi_i, \quad \bar{u}_i = u_i, \quad \bar{\varphi} = \varphi + c_\eta$$

which represents a family of scale changes, and

(5.6) $$\int_S (-\varepsilon_0 \, \varphi_{,j} + \pi_j) \, n_j \, da = 0$$

In the local form the integrals (5.2) and (5.6) give the balance equations $(3.1)_1$ and $(3.1)_3$.

IV). $$a_i \neq 0, \quad b_i = \Phi_i = 0; \quad c = 0,$$

(5.7) $$\bar{x}_i = x_i + a_i \eta; \quad \bar{\pi}_i = \pi_i; \quad \bar{u}_i = u; \quad \bar{\varphi} = \varphi$$

which represents the coordinate translations, and

(5.8) $$\int [H \delta_{jk} - T_{ij} \, u_{i,k} - E_{ij} \, \pi_{i,k} - (-\varepsilon_0 \, \varphi_{,j} + \pi_j) \, \varphi_{,k}] \, n_j \, da = 0.$$

This integral is a new conservation law and represents the generalization of the corresponding conservation law of Rice [2] in the case of linear elastic dialectric.

### REFERENCES

[1] Knowles J. K., Sternberg E., *On a class of conservation laws in linearized and finite elasticity*, Arch. Rat. Mech. Anal. 44, 187—211 (1972).

[2] Rice J. R., *A path independent integral and the approximate analysis of strain concetrations by notches and cracks*, Journal of Applied Mechanics, 35, 379 (1968).

[3] Noether E., *Invariante Variationsprobleme*, Ges. Wiss. Göttingen, 2, 235 (1918).

[4] Fletcher D. C., *Conservation laws in linear elastodynamics*, Arch. Rat. Mech. Anal. 60, 329—353, (1976).

[5] Green A. E., *On some general formulae in finite elastostatics*, Arch. Rat. Mech. Anal. 50, 73—80, (1973).

[6] Chen F. H. K. and Shield R. T., *Conservation laws in elasticity*, ZAMP, 28, 123, (1977).

[7] Aifantis E. C. and Gerberich W. W., Cacture Vo 3, ICF4, Waterloo, 1977.

[8] Gurtin M. E., *On a path-independent integral for elastodynamics*, Int. J. Fracture, 12, 643—645, (1976).

[9] Mindlin R. D., *Elasticity, piezoelasticity and crystal lattice dynamics*, Journal of Elasticity, 2, 217—283, (1972).

[10] Gelfand I. M. and Fomin V., *Calculus of Varitions*, Prentice-Hall, Englewood Cliffts, New Jersey (1963).

## NOETHERS THEOREME IN DER LINEAREN THEORIE DER ELASTICHEN DIALEKTRIK

Zusammenfassung

Es wird in der Arfieit die Noethersch Theoreme gegefien und zwar in der Form, welche an der Theorie der elastischen Dialektrik angewendet wird. Durch diese Theoreme werden die Familien der Transfoımation und die denen entsprechenden Gesatze der Konservation zur linearen Theorie der elastischen Dialektrik bestimmt.

## NETEROVA TEOREMA U LINEARNOJ TEORIJI ELASTIČNOG DIJALEKTRIKA

Izvod

U poslednje vreme u mehanici kontinuuma posebna pažnja je posvećena određivanju integrala nezavisnih od putanje. Interesovanje za ove integrale nije samo teorijske prirode već i praktičnog značaja s obzirom na njihovu primenu kao na primer u problemima loma i prskotina. Iz tih razloga u ovom radu se daje proširenje primene Neterove teoreme na linearnu teoriju elastičnog dijalektrika u polju elastičnog potencijala. Izvode se odgovarajuće familije transformacija i njima odgovarajući zakoni konzervacije ili integrali nezavisni od putanje. U opštem slučaju dobijaju se četiri zakona konvervacije od kojih je jedan nov.

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