TEORIJSKA I PRIMENJENA MEHANIKA 7, pp. 61—65, 1981, UDK 531

SIMILARITY SOLUTIONS FOR RADIALLY PROPAGATING ROTATING LONG GRAVITY WAVES

Vladan D. Djordjević and *Miloš Pavlović*

(Received February 20, 1981)

It was shown in the preceding paper [1] that the evolution equation, describing nonlinear long gravity waves propagating radially on the free surface of a slowly rotating liquid of variable depth $h(\xi)$, in all three cases considered had the same form:

$$\left(c' + \frac{c}{\xi}\right)\zeta + 2c\,\zeta_\xi + \frac{3}{c^2}\,\zeta\zeta_\tau + \frac{c^2}{3}\,\zeta_{\tau\tau\tau} = 0. \tag{1}$$

In this equation of the Korteweg-de Vries (K-dV) type, $\zeta(\tau, \xi)$ is the function which describes the free surface, τ is the time, $\xi > 0$ is the distance from the rotation axis and $c(\xi)$ is the speed of infinitesimal waves. In cases where the elevation of the free surface due to waves is much greater than or is of the same order of magnitude as the elevation due to rotation, it was shown that: $c^2 = h$. These cases will be referred to here as I and II, respectively. In the case, referred to as III, where the elevation due to waves is much less than that due to rotation: $c^2 = z_0 + h$, where $z_0 = \frac{1}{2}\,\Omega^2\xi^2$ and $\Omega = 0(1)$ is a constant. Also, in case II the „time":

$$\tau - \int_{\xi_0}^{\xi} \frac{z_0}{2ch}\,d\,\xi,$$

should stand instead of τ in equation (1), where ξ_0 is a constant, which can be arbitrarily chosen in the following way: $h(\xi_0) = 1$. In the same paper [1], equation (1) was shown to possess a solution in the form of a solitary wave and a perturbation analysis of this equation was carried out too, whereby many approximate solutions in the form of slowly varying solitary waves were shown to exist.

It is well known that equations of the K-dV type, in addition to solitary waves solutions, possess many other solutions as well, among them similarity solutions in the form of an exponentially decreasing front of the wave followed by an oscillatory tail. The first similarity solution of the classical K-dV equation was found by Berezin ε, Karpman [2] and was analyzed in detail by Rosales [3]. Miles [4] extended these results to so-called „cylindrical" K-dV equation. It was recently shown that similarity solutions with an algebraically decaying wave-front exist too [5] and that the equations of the K-dV type with variable coefficients, which within the context of water waves describe the propagation of long gravity waves over an uneven bottom, possess similarity solutions as well [6]. In this papper we construct the similarity solutions for equation (1) and analzyze them numerically.

If a similarity solution of equation (1) is presumed in the form:

$$\zeta=\alpha(\xi)\, f(\theta), \quad \theta=\tau/\beta(\xi), \tag{2}$$

$f(\theta)$ can be easily shown to satisfy the following ordinary differential equation:

$$f'''+Cff'-A\,\theta f'+Bf=0, \tag{3}$$

where A, B and C are arbitrary constants, provided that following relations hold:

$$\alpha=\frac{C}{9}\frac{c^4}{\beta^2}, \quad \beta^3=\frac{A}{2}\int^{\xi} c(\xi)\, d\xi+\beta_0^3,$$

$$c^2=\left(\frac{\xi}{\xi_0}\right)^s, \; 3A(9s+2)=2(2A+B)(s+2), \tag{4}$$

in which β_0 and s are arbitrary constants too and the lower limit of the integral for β is to be conveniently chosen in each individual case of variable depth of liquid and remains unspecified so far. From the expression for c we now can have:

$$h(\xi)=\left(\frac{\xi}{\xi_0}\right)^s,$$

in cases I and II and:

$$h(\xi)=\left(1+\frac{1}{2}\Omega^2\xi_0^2\right)\left(\frac{\xi}{\xi_0}\right)^s-\frac{1}{2}\Omega^2\xi^2,$$

in case III and can conclude that similarity solutions of equation (1), determined by equation (3), are possible for a variety of diferent depths of liquid. Some characteristic depths in cases I and II are presented in Fig. 1.

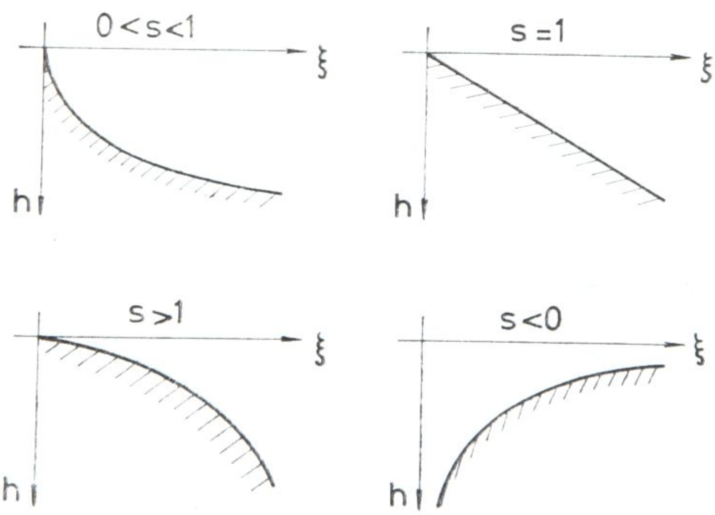

Fig. 1

Equation (3) can be transformed into a more convenient form:

$$F''' + 9FF' - \theta_a F' + bF = 0, \tag{5}$$

where:

$$\theta_a = \frac{A\theta}{\sqrt[3]{A^2}}, \qquad f = \frac{9}{C}\sqrt[3]{A^2}\, F(\theta_a), \qquad b = \frac{B}{A}, \qquad A \neq 0.$$

The corresponding relation between b and s is:

$$b = \frac{3}{2}\,\frac{9s+2}{s+2} - 2. \tag{6}$$

For $s = 0$ (constant depth!), $b = -1/2$ and equation (5) reduces to the equation analytically and numerically treated by Miles [4], while for $s = -2/9$, $b = -2$, and equation (5) has the form of the equation describing similarity solutions for the classical K-dV equation. Consequently, from the point of view of similarity solutions, there is an analogy between the classical K-dV equation and our equation (1) for $s = -2/9$. For $A = 0$ ($s = -2$), $\beta(\xi) = $ const., and the similarity solution has a form in which the variables are separated. This solution appears as a special similarity solution for the classical K-dV equation with variable coefficients [6] too and will not be discussed here.

We solved equation (5) numerically for a series of different values of b. A. an initial condition we used the asymptotic behaviour of the function F as $\theta_a \to \infty$. Namely, if the wave-front is supposed to decay exponentially for $\theta_a \to \infty$, the non-linear term can be neglected in (5) and the wave-front can be described by the approximate equation:

$$F''' - \theta_e F' + b\,F = 0. \tag{7}$$

The solution of this linear equation in cases which were treated in the literature ($b = -2$ and $b = -1/2$) can be expressed in terms of Airy functions. For an arbitrary b, however, its solution cannot be expressed by means of special functions. That is why we made use here of the WKB method to obtain the desired front of the wave. The physical optics approximation gives (details of this analysis can be found in [6]):

$$F \sim a\,\theta_a^{-\frac{1}{2}\left(\frac{3}{2}+b\right)} \exp\left(-\frac{2}{3}\,\theta_a^{\frac{3}{2}}\right), \tag{8}$$

where a is an arbitrary constant usually referred to as the amplitude parametar. The main result in the previous papers [3] and [4] was just connected with some characteristic values of this parameter. It was shown namely that the wave-tail could be either oscillatory or non-oscillatory depending on a. Moreover, the solution in the non-oscillatory case can develop singularities. Therefore, only results which give the oscillatory tail are of practical importance. These values of a which separate the oscillatory case from a non-oscillatory one are called critical values and are denoted by a_k. In the numerical treatment of equation (5) with the initial condition (8), we paid special attention to the critical values of a and their dependence on b and, on the basis of the obtained results, we were able to construct the diagram in Fig. 2. In the striped area the wave-tail is non-oscillatory. It will be noted that for $b > -1/2$ both positive and negative values of a_k exist, while for $b > -1/2$ only negative and absolutely very small values are present. Thus, for $b > -1/2$ the tail will be oscillatory for every $a > 0$.

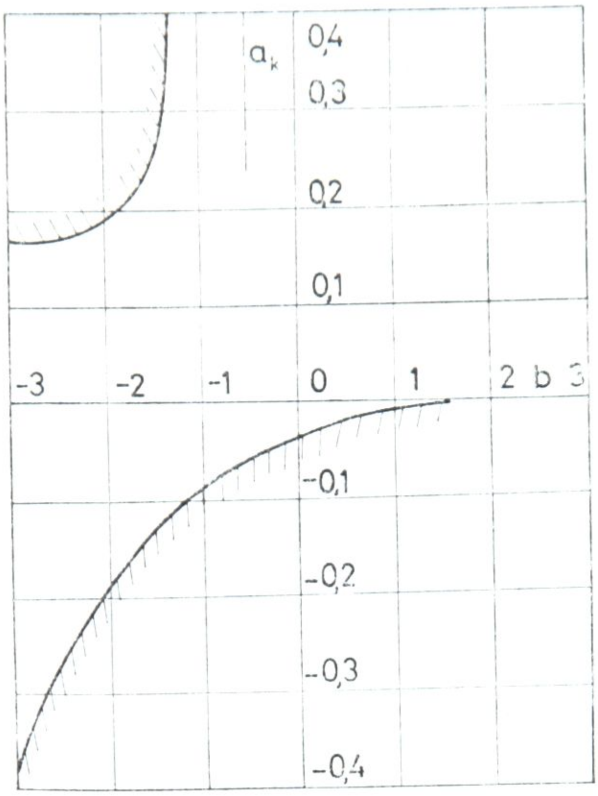

Fig. 2

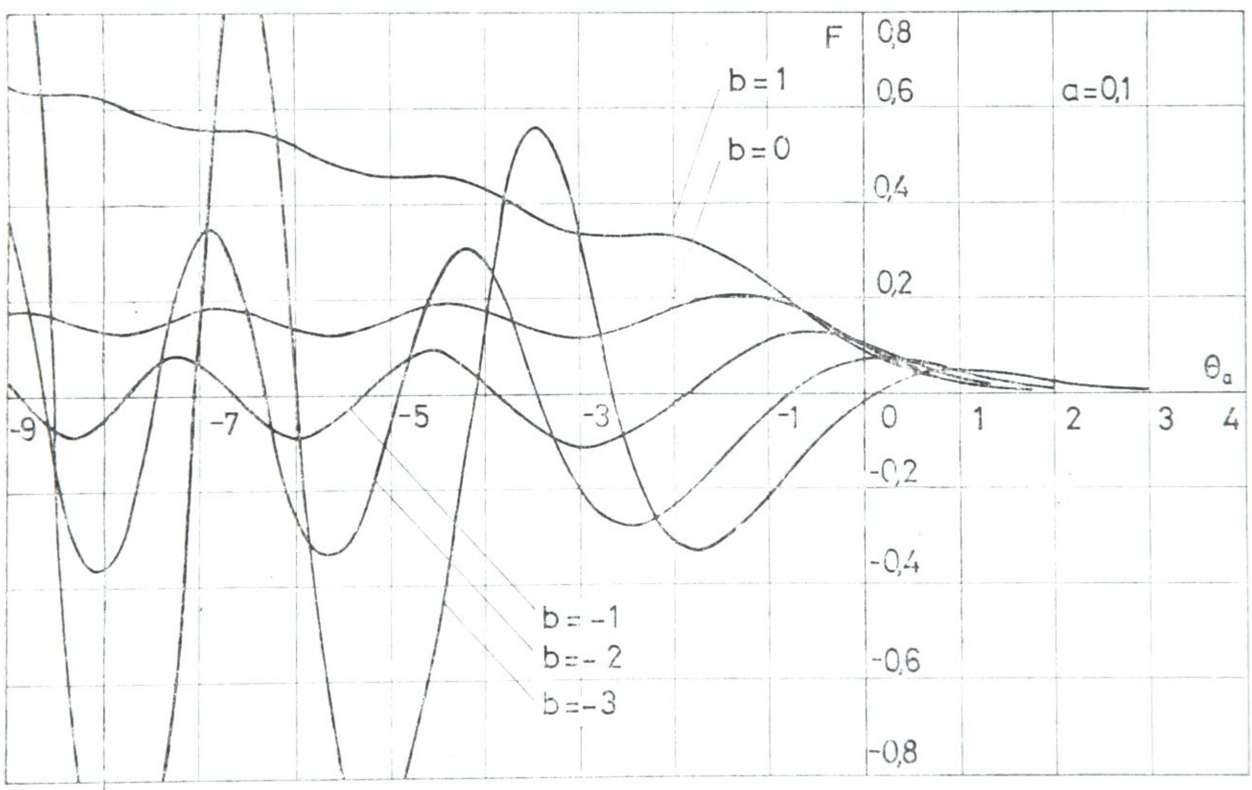

Fig. 3

From a series of numerical results only a few most characteristic ones were ipcked up and presented in Fig. 3. A strong dependence of the form of the oscillatory tail on b, i.e. on the form of variable depth, is remarkable. For $b = -1$ the amplitudes of oscillations slowly decrease, while for $b = -2$ they slowly increase. For $b = -3$ this increase is very clearly expressed. The results for $b = -1/2$ were not quoted, since they coincide with the Miles' results [4]. The minimums of oscillations lie on the θ_a axis for $b = -1/2$. For $b = 0$ these minimums are shifted upward and oscillations take place along a horizontal straight line. The tendency of shifting upward continues for $b > 0$ and oscillations take place along certain inclined straight lines.

REFERENCES

[1] Djordjević V. D., *On radially propagating rotating solitary waves*, Teorijska i primenjena mehanika 7, pp 49-57/1981)

[2] Berezin Y. A. & Karpman V. J., *Theory of non-stationary finite amplitude waves in a low-density plasma*, Soviet Phys. JETP 19, pp. 1265—1271 (1964).

[3] Rosales R. R., *The similarity solution for the Korteweg-de Vries equation and the related Painleve transcendent*, Proc. Roy. Soc. A. 361, pp. 265—275 (1978).

[4] Miles J. W., *An axisymmetric Boussinesq wave*, J. Fluid Mech. 84, part 1, pp. 181—191 (1978).

[5] Johnson R. S., *Water waves and Korteweg-de Vries* equations, J. Fluid Mech. 97, part 4, pp. 701—719 (1980).

[6] Djordjević V. D., *Similarity solutions of theKorteweg-de Vries equation with variable coefficients, submitted for publication in Int.* J. Non-linaar Mech.

AEHNLICHE LOESUNGEN DER AXIALSYMMETRISCHEN ROTIERENDEN LANGEN SCHWEREWELLEN

Zusammenfassung

Es wird in der Arbeit gezeigt dass die Gleichungen, die Fortpflanzung langer Schwerewellen auf der Oberfläche einer rotierenden Flüssigkeit beschreiben, die ähnlichen Lösungen für eine Reihe von verschiedenen Tiefen der Flüssigkeit besitzen. Die Gleichung der ähnlichen Lösungen wurde numerisch, für den Fall dass die mit der WKB Methode bestimmte Wellenfront exponential abnimmt, integriert. Die kritischen Werte des Amplitudenparameters wurden erhalten.

SLIČNA REŠENJA OSNOSIMETRIČNIH ROTIRAJUĆIH DUGIH GRAVITACIONIH TALASA

Izvod

U radu je pokazano da jednačine koje opisuju rasprostiranje dugih gravitacionih talasa na slobodnoj površini jedne rotirajuće tečnosti poseduju slična rešenja za niz različitih dubina tečnosti. Jednačina sličnih rešenja je numerički integraljena u slučaju kada front talasa, određen WKB metodom opada eksponencijalno. Dobijene su kritične vrednosti amplitudnog parametra.

Vladan D. Djordjević i
Miloš Pavlović
Mašinski fakultet
27 marta 80
11000 Beograd
Yugoslavia