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ON RADIALLY PROPAGATING ROTATING SOLITARY WAVES

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1. Introduction

Solitary waves — weakly nonlinear and weakly dispersive waves of permanent form, the existence of which is based on the balance between two competing effects, the effect of nonlinearity and the effect of dispersion, distorting the shape of a wave differently, represent the most characteristic feature of the nonlinear wave theory. That is why the Korteweg-de Vries (K-dV) equation which describes the propagation of solitary waves is readily said to be the basic equation of this theory, although within the scope of it some other nonlinear equations arise too, and play a very important role in investigations of various phenomena of nonlinear wave theory, as for example Benjamin-Ono equation, nonlinear Schrödinger equation, sine-Gordon equation etc. Originally derived to describe long gravity waves on the free surface of a liquid, the K-dV equation turned out to govern many other wave forms in fluids, longitudinal vibrations of an unharmonic discrete -mass string, magnetohydrodinamic and ion-acoustic waves in a cold plasma, pressure waves in liquid-gas buble mixtures etc. (for a more detailed account of problems described by the K-dV equation, see [1] and [2]). Many modified forms of the K-dV equation arising by including some additional effects, such as for example, the presence of any kind of unhomogeneity in the direction of wave propagation, dissipation etc. are known in the literature too. However, two-dimensional problems are mostly treated. Relatively small attention has been paid in tha literature to three-dimensional problems, including axisymmetric ones, until recently. Axisymmetric solitary waves seem to have been first considered by Maxon & Viecelly [3] within the context of ion-acoustic waves. For water waves, the first K-dV equation for radi ally propagating solitary waves, so-called cylindrical K-dV equation, was derived by Miles [4], who at the same time showed that this equation possessed the similarity solutions of the same type as the classical K-dV equation. A similarity solution of this equation was found by Cumberbatch [5] as well. Both papers are referred to a constant depth of water. The effect of variable depth on the propagation of axisymmetric solitary waves was analyzed by Chwang & Wu [6]. The governing equations were numerically solved for an ingoing solitary wave, whereby its „reflection on itself" and further propagation from the center were included into the analysis.

In this paper we wish to extend the current researches in this field to the case of rotating solitary waves, i.e. of solitary waves emerging on the free surface of a rotating liquid, which, as we know well has the form of a paraboloid. We will assume that the liquid rotates slowly, so that the Rossby number, defined by means of the velocity of infinitesimal gravity waves, is high enough. We will distinguish between three characteristic cases and will derive the evolution equations of the K-dV type and discuss the effect of rotation on the propagation of solitary waves in all of them. Using a known perturbation procedure we will determine the effect of variable depth of liquid on the amplitude, the length and the speed of a solitary wave and by that, contribute to the theory of these equations even when the rotation of the whole system is not present. A separate article will be dedicated to similarity solutions.

2. The derivation of evolution equations

We will consider the problem according to Fig. 1. Long gravity waves propagate on the paraboliodal free surface of an inviscid liquid rotating about the vertical z-axis with constant angular velocity ω. The depth of water is variable and determined by $h(r)$, where r is the cylindrical coordinate (the origin lies on the free surface). The problem is axisymmetric, but all three velocity components are

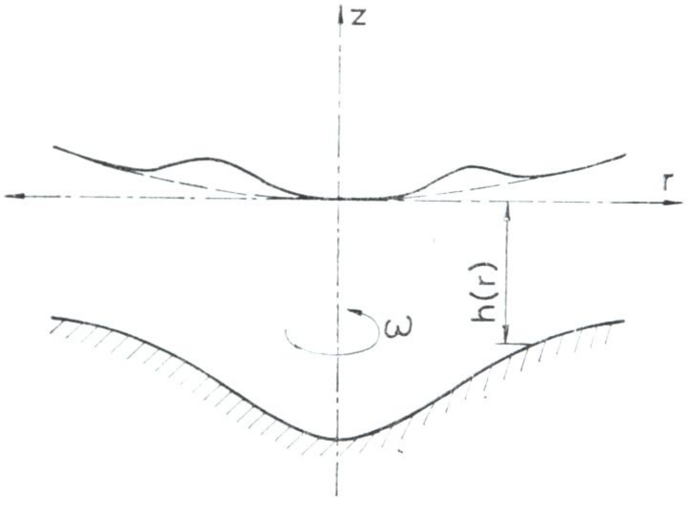

Fig. 1

present — the circular velocity is affected by Coriolis force. It is described by Euler equations, continuity equation, boundary condition at the bottom and kinematic and dynamic boundary conditions on the free surface. If they are written in non-dimensional form by means of the following scales: h_0 — for all lengths (h_0 is a representative depth of water), $\sqrt{gh_0}$ — for velocities (g is acceleration due to gravity), $h_0/\sqrt{gh_0}$ — for time and $\rho g h_0$ — for pressure (ρ_0 is constant density of the liquid), they will be respectively:

$$u_t + uu_r + wu_z - \frac{v^2}{r} - \frac{2}{R_o} v = -p_r$$

$$v_t + uv_r + wv_z + \frac{uv}{r} + \frac{2}{R_o} u = 0$$

$$w_t + uw_r + ww_z = -p_z$$

$$(ru)_r + (rw)_z = 0 \tag{1}$$

$$z = -h(r): \qquad w = -u \frac{dh}{dr}$$

$$z = \frac{1}{2R_0^2} r^2 + \zeta(t, r): \qquad w = \zeta_t + u\left(\frac{1}{R_0^2} r + \zeta_r\right)$$

$$p = \zeta.$$

In (1), in addition to already mentioned notations, t is the time, u, w and v are perturbed velocities in r, z and circular directions respectively, p is the perturbed pressure, $\zeta(t, r)$ is the perturbation of the free surface and $R_0 = \sqrt{gh_0}/\omega\, h_0$ is the Rossby number defined by means of the velocity $\sqrt{gh_0}$ of linear long gravity waves. In what follows we will suppose that the liquid rotates relatively slowly, so that the Rossby number is high and will express it by means of the (small) amplitude parameter $\varepsilon > 0$ in the following form:

$$\frac{1}{R_o} = \Omega\, \varepsilon^s, \ \Omega = 0(1), \ s > 0. \tag{2}$$

In the derivation of the K-dV type of equations from (1), we will apply the procedure which is now commonly used in the literature, i.e. we will introduce far--field coordinates:

$$\tau = \varepsilon^{1/2} \left\{ \int_{r_0}^{r} \frac{dr}{c(\xi)} - t \right\}, \ \xi = \varepsilon^{3/2} r. \tag{3}$$

In this way we implicitly assume that the wave propagates with the variable velocity $c(\xi)$ which in this case, due to the elevation of the free surface imposed by the rotation, can vary even for a constant depth of water and which is to be determined from the analysis. The variations of this velocity and other quantities characterizing the wave are however slow — they vary on the scale of the coordinate ξ, which is introduced in order to express such variations explicitly. The depth of water is slowly varying too, i.e. $h = h(\xi)$, $h' = 0(1)$. At the moment $t = 0$ the pick of the wave will be in $r = r_0$. Since we consider weakly nonlinear waves, we will expand all perturbed quantities in powers of ε:

$$\zeta = \sum_{n=1}^{\infty} \zeta^{(n)}(\tau, \xi)\, \varepsilon^n, \ u = \sum_{n=1}^{\infty} u^{(n)}(\tau, \xi, z)\, \varepsilon^n,$$

$$w = \sum_{n=1}^{\infty} w^{(n)}(\tau, \xi, z)\, \varepsilon^{n+1/2}, \quad p = \sum_{n=1}^{\infty} p^{(n)}(\tau, \xi, z)\, \varepsilon^n. \tag{4}$$

The series for the circular velocity will be prescribed by the momentum balance in circular direction:

$$v = \sum_{n=1}^{\infty} v^{(n)} (\tau, \xi, z) \ \varepsilon^{n+s-1/2}. \tag{4}$$

Using far-field coordinates (3), inserting expressions (2), (4) and (4') in (1) and arranging them in powers of ε, the equations and the boundary conditions for each order of ε can now be obtained in the usual way, but, for the sake of simplicity, we will not write them here. Before we quote only the main results, we will preliminary discuss the effect of rotation on the free surface at the Rossby number defined by (2). The free surface on which the kinematic and the dynamic boundary conditions are to be satisfied is:

$$z = z_0 \ \varepsilon^{2s-3} + \sum_{n=1}^{\infty} \zeta^{(n)} (\tau, \xi) \ \varepsilon^n,$$

where $z_0 = \frac{1}{2} \Omega^2 \xi^2$. The first and the second term in this expression represent the elevations of the free surface due to rotation and due to waves respectively. Obviously for $s \geqslant 5/2$ the elevation due to rotation will be much less than the one due to waves, for $s = 2$ both elevations will be of the same order of magnitude, while for $s \leqslant 3/2$ the elevation due to rotation prevails. That is why we will separately investigate three characteristic cases: $s = 5/2$, $s = 2$ and $s = 3/2$. In the first two of them, in using boundary conditions on the free surface, perturbed quntities are to be expanded in series about $z = 0$, while in the third case they are to be expanded about $z = z_0$, but we will not go into this procedure in detail.

The equations for the first approximation ($n = 1$) for $s = 5/2$ and $s = 2$ give $c^2 = h$, while $\xi^{(1)}$ remains undetermined. A secularity condition appearing in the course of finding the solution of the equations for the second approximation ($n = 2$) has the form of the desired equation for $\zeta^{(1)}$. For $s = 5/2$ this equation is:

$$\left(c' + \frac{c}{\xi}\right) \zeta^{(1)} + 2c\zeta^{(1)}_{\xi} + \frac{3}{h} \zeta^{(1)} \zeta^{(1)}_{\tau} + \frac{h}{3} \zeta^{(1)}_{\tau\tau\tau} = 0, \tag{5}$$

and represents the K-dV equation for long gravity waves propagating radially over an uneven bottom. This equation can be readily derived from the system of equations used by Chwang & Wu [6] and, of course, it reduces to the equation derived by Miles [4] for an even bottom. Consequently, in this case the rotation of the whole system does not affect either the speed or the shape of the wave. It however induces a circular velocity component of order $O(\varepsilon^3)$ which can be obtained from the corresponding momentum equation:

$$v^{(1)} = \frac{2\Omega}{c} \int_{\pm\infty}^{\tau} \zeta^{(1)} \ d\tau.$$

For $s = 2$ the following evolution equation is obtained in the same way:

$$\left(c' + \frac{c}{\xi}\right) \zeta^{(1)} + 2c \ \xi^{(1)}_{\xi} + \frac{z_0}{h} \xi^{(1)}_{\tau} + \frac{3}{h} \zeta^{(1)} \zeta^{(1)}_{\tau} + \frac{h}{3} \zeta^{(1)}_{\tau\tau\tau} = 0 \tag{6}$$

which contains the effect of rotation through the term $\frac{z_0}{h}\zeta^{(1)}_\tau$. It can be easily shown, however, that by means of a simple transformation of the coordinate τ:

$$\tilde{\tau} = \tau - \int_{\xi_0}^{\xi} \gamma(\xi)\, d\xi,$$

where $\xi_0 = \varepsilon^{3/2} r_0$ and $\gamma(\xi) = z_0/2\,ch$ this term can be removed from the equation (6), whereby it then reduces to the previous equation (5). The rotation of the whole system in this case obviously affects only the speed of waves. In order to investigate this effect more deeply, we will write the expression for τ in the following way:

$$\tilde{\tau} = \varepsilon^{1/2}\left\{\int_{r_0}^{r} \frac{dr}{c\left[1+\varepsilon\frac{z_0}{2R}+O(\varepsilon^2)\right]} - t\right\}.$$

Hence, the correction of the speed of waves due to rotation is small: $\varepsilon\, z_0\, c/2\,h$ and is of the same sign as the speed c. Consequently, the rotation of the whole system will accelerate outgoing ($c = \sqrt{h}$) as well as ingoing ($c = -\sqrt{h}$) waves. For an even bottom ($h =$ const.), a strong dependence of this correction on the distance from the axis of rotation is noteworthy, since: $z_0 = \frac{1}{2}\,\Omega^2\xi^2$. For a solitary wave this correction will be of the same order of magnitude as the correction of the linear speed c due to nonlinearity.

For $s = 3/2$ the equations for the first approximation give: $c^2 = z_0 + h$. Hence, the rotation in this case affects the speed of waves in the linear approximation. It will affect the shape of waves as well, for the equation for $\zeta^{(1)}$ is now:

$$\left(c' + \frac{c}{\xi}\right)\zeta^{(1)} + 2c\,\xi^{(1)}_\xi + \frac{3}{z_0+h}\,\zeta^{(1)}\,\zeta^{(1)}_\tau + \frac{z_0+h}{3}\,\zeta^{(1)}_{\tau\tau\tau} = 0. \qquad (7)$$

It is however of the same type as its counterparts (5) and (6), so that we will analyze here only some solutions of the equation (5), with a notice that all procedures which will be done can be simply applied on the example of the equation (7). For the sake of simplicity the superscript (1) will be dropped.

3. An exact solution of equation *(5)*

In a recent paper [7], Hirota managed to reduce so-called cylindrical K-dV equation (equation (5) for $h =$ const.) to the classical one using a suitable transformation of coordinates and made it possible one to obtain a series of exact solutions of this equation. Inspired by this paper we managed [8] to reach the same goal for the K-dV equation with variable coefficients describing the propagation of long gravity waves over an uneven bottom. It is not difficult to combine both results and to obtain an exact solution of equation (5). It will be only quotedhere. The reader can verify it by simply inserting it in equation (5):

$$\zeta = h^2\, u(X, \tau),\; X = \int_{\xi_0}^{\xi} c(\xi)\, d\xi,$$

whereby $u(X, \tau)$ satisfies the classical K-dV equation:

$$2u_X + 3uu_\tau + \frac{1}{3}\, u_{\tau\tau\tau} = 0,$$

provided: $h = (\xi_0/\xi)^{2/9}$ ($h(\xi_0) = 1$ has been arbitrarily chosen). It is well known that, according to the classical K-dV equation, a perturbation of the form:

$$u = \alpha_0\, sech^2\, \beta_0 \tau,\; \beta_0 = \sqrt{3\alpha_0/2},\; \alpha_0 > 0 \tag{8}$$

induced at $X = 0$ ($\xi = \xi_0$), evolves preserving its shape and propagating with the speed $\gamma_0 = \alpha_0/2$ i.e. as:

$$u = \alpha_0\, sech^2\, \beta_0 (\tau - \gamma_0 X),$$

which represents an one-solitary wave solution of this equation. We can now conclude that the same perturbation will evolve according to equation (5) as:

$$\zeta = \alpha_0 \left(\frac{\xi_0}{\xi}\right)^{4/9} sech^2\, \beta_0 (\tau - \gamma_0 X),$$

i.e. the length of the wave determined by β_0 will remain constant, while the amplitude and the speed of the wave will change. The amplitude will obviously decrease (increase) for outgoing (ingoing) waves as $\xi^{-4/9}$, while the change of the speed of the wave can be easily calculated by using the expression for X.

In the next section this solution will also be obtained as one of perturbed solutions of equation (5).

4. A perturbation analysis of equation *(5)*

If we assume that the wave moves very far from the axis of rotation, i.e. that ξ is great and that the bottom is varying a little slowlier than previously assumed, the first term in equation (5) will be much less than the remaining ones. In this case equation (5) will belong to the class of so-called perturbed K-dV equations. Two analytical methods for solving equations of this type are available in the literature. The first is a method of multiple scales applied first by Ott & Sudan [9] and the second one is a method based on the inverse scattering method, developed recently by Karpman & Maslov [10]. In the treatment of equation (5) we will apply here the method of multiple scales. In order to express the „distance" of the wave from the axis and the slow variation of the bottom explicitly, we will introduce a new slow coordinate: $X = \delta\, \xi$ in which δ must satisfy the following condition: $\varepsilon \ll \delta \ll 1$ so as not to invalidate the previous asymptotic scheme (the notation X used in this section has nothing in common with the same notation from the preceding one and is not to be mixed with it!). Now we will have: $h = h(X)$, $h'_X = 0(1)$. We will introduce the coordinate:

$$T = \tau - \int_{\xi_0}^{\xi} \gamma(\xi)\, d\xi$$

and expand ζ in powers of δ:

$$\zeta = \sum_{n=0}^{\infty} \zeta_n \delta^n .$$

Inserting this in (5) a sequence of equations for each order of ε can be easily obtained. The solution of the equation for ζ_0 for a perturbation of the form (8) will be:

$$\zeta_0 = \alpha(X)\, sech^2 \beta(X)\, T, \quad \beta^2 = 3\alpha/4h^2, \quad \gamma = \alpha/2hc, \tag{10}$$

where $\alpha(X)$ is arbitrary and remains undetermined at this stage of approximation. The relations between α, β and γ point out that the solitary wave keeps its structure in this case in the sense that these relations are the same as in the classical soliton. The solution of the equation for ζ_1 gives, in the form of a secularity condition, the necessary expression for the amplitude $\alpha(X)$:

$$X\, ch\, \alpha^{2/3} = \text{const.} \tag{11}$$

From (10) and (11) and for h = const. we can now obtain:

$$\alpha = \alpha_0 \left(\frac{\xi_0}{\xi}\right)^{2/3}, \quad \beta = \beta_0 \left(\frac{\xi_0}{\xi}\right)^{1/3}, \quad \gamma = \gamma_0 \left(\frac{\xi_0}{\xi}\right)^{2/3}.$$

The result for the amplitude is in full agreement with the so-called „spike" solution of the cylindrical K-dV equation [5] and also with the numerical results by Maxon & Viecelli [3]. Unfortunately, we could not compare this result with the numerical results by Chwang & Wu [6]! Namely, for an ingoing solitary wave they assumed that the pick of the wave was situated at $r_0 = 30$ (s. their Fig. 3). For $\varepsilon = 0{,}01$ this would correspond to our $\xi_0 = 0{,}3$ and for $\varepsilon = 0{,}1$ to $\xi_0 = 3$. These values are however not great enough for this theory to be applied. Also, in another example treated by them — the evolution of a solitary wave converging over a submerged conical island, the bottom was too steep (cf. their Fig. 6).

The expression (11) can be used to determine the shape of the bottom for which a solitary wave would keep one of its characteristics constant. For example, in order for the amplitude α to remain constant, h should vary according to:

$$h = \left(\frac{\xi_0}{\xi}\right)^{2/3},$$

while β and γ would simultaneously be:

$$\beta = \beta_0 \left(\frac{\xi_0}{\xi}\right)^{2/3}, \quad \gamma = \gamma_0 \frac{\xi}{\xi_0}.$$

For β = const. we would have:

$$h = h_o \left(\frac{\xi_0}{\xi}\right)^{2/9}, \quad \alpha = \alpha_0 \left(\frac{\xi_0}{\xi}\right)^{4/9}, \quad \gamma = \gamma_0 \left(\frac{\xi_0}{\xi}\right)^{1/9}$$

It is interesting that this solution coincides with the exact solution (9)! Finally, if one wishes to keep the speed of the wave constant (γ = const.), h should vary as:

$$h = h_0 \left(\frac{\xi_0}{\xi}\right)^{4/15},$$

while α and β would at the same time vary according to:

$$\alpha = \alpha_0 \left(\frac{\xi_0}{\xi}\right)^{2/5}, \quad \beta = \beta_0 \left(\frac{\xi}{\xi_0}\right)^{1/15}.$$

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UEBER DIE AXIALSYMMETRISCHEN ROTIERENDEN SOLITAEREN WELLEN

Zusammenfassung

Es wurde in der Arbeit gezeigt, dass die solitären Wellen — ein häufig und sehr charakteristisch Phänomen der nichtlinearen Wellentheorie, können unter bestimmten Bedingungen auch an der Oberfläche einer rotierenden Flüssigkeit existieren. Im Falle dass die Flüssigkeit relativ langsam rotiert, die Gleichungen des Korteweg-de Vries Types, die die Fortpflanzung dieser Wellen beschreiben, werden ausgeführt. Der Einfluss der Rotation auf ihre Fortpflanzung wird diskutiert und einige exacte und approximative Lösungen der Evolutionsgleichungen werden gefunden.

O OSNOSIMETRIČNIM ROTIRAJUĆIM SOLITARNIM TALASIMA

Izvod

U radu je pokazano da tzv. solitarni talasi — čest i veoma karakterističan fenomen nelinearne teorije talasa, mogu pod određenim uslovima da egzistiraju i na slobodnoj površini jedne rotirajuće tečnosti. U slučaju da tečnost rotira relativno sporo izvedene su jednačine tipa Korteweg-de Vries-a koje opisuju evoluciju ovih talasa. Prodiskutovan je uticaj rotacije na njihovo rasprostiranje i data su neka tačna i približna rešenja evolucionih jednačina.

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