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# ON BRACHISTOCHRONIC MOTIONS OF NON-CONSERVATIVE DYNAMICAL SYSTEMS

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Brachistochronic motions of non-conservative systems, with different limitations on acting forces, have been considered by several authors.

Euler was the first to solve the problem of the brachistochronic motion of a particle in a medium with resistance depending on particle's velocity (cf [1], p. 241). In 1956. R. Stojanović [2] considered the brachistochronic motion of a time-independent holonomic dynamical system in a field of non-conservative time-independent and velocity-independent forces. Ashby et al. ([3], 1975.), studied the problem of finding the curve of the most rapid descent of a bead sliding along a wire, from one fixed point to another, under the simultaneous influence of gravity and Coulomb friction.

Finally, in 1979., Đ. Đukić [4] considered brachistochronic motions of non-conservative systems with generalized non-conservative forces that can depend on time, on generalized coordinates and velocities, although not in an arbitrary way, but under a given condition.

We shall study, in this paper, the brachistochronic motion of a holonomic, stationary non-conservative dynamical system, considering that acting forces can in an arbitrary way depend on time, generalized coordinates and generalized velocities of the system.

Obtained results contain the results of above quoted authors in some cases, whereas they differ from them in some other cases.

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Let us consider a holonomic, time-independent, non-conservative system with n degrees of freedom. Generalized coordinates, which determine the position of the system at any moment $t$, are the $q^i$ [1]. The kinetic energy $T$ of the system is a homogeneous quadratic function of generalized velocities $\dot{q}^i$:

$$T=\frac{1}{2}\, a_{ij}\, \dot{q}^i\, \dot{q}^j \quad {}^{2)} \tag{1}$$

where coefficients $a_{ij}$ depend only on $q^i$: $a_{ij} = a_{ij}(q^k)$.

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1) The Latin indices take the values from 1 to n.

2) The repeated indices imply summation.

Let us assume that besides potential forces with potential

$$\Pi = \Pi\,(q^i), \tag{2}$$

the system is also acted on by generalized non-conservative forces $Q_i$, which can be arbitrary functions of time, generalized coordinates and velocities:

$$Q_i = Q_i\,(t,\, q^k,\, \dot{q}^k). \tag{3}$$

Let the considered system start from a given initial position $A$, with given values $q^i_{(0)}$ at $t = t_0$, $t_0$ being fixed,

$$A: \quad q^i\,(t_0) = q^i{}_{(0)}, \tag{4}$$

and reach a final position $B$, with given values $q^i_{(1)}$ at $t = t_1$, where $t_1$ is not fixed,

$$B: \quad q^i\,(t_1) = q^i{}_{(1)}. \tag{5}$$

The time the system needs to move from $A$ to $B$ is determined by the integral

$$I = \int_{t_0}^{t_1} dt. \tag{6}$$

If we take the requirement for $I$ to have a minimal value, i.e. for the system to move brachistochronically, the trajectories of the system will have a determined configuration, and control forces $u_i$ are to be added to given forces acting on the system, in order to make such a motion possible.

Our aim is, first, to obtain differential equations of the brachistochronic motion of the system, and, second, to determine the control forces $u_i$. We assume that with such a motion the law of conservation of energy has the same form as in the case of the motion under given forces, i.e.

$$\dot{T} + \dot{\Pi} = Q_i\,\dot{q}^i, \tag{7}$$

wherefrom we conclude that control forces satisfy the condition

$$u_i\,\dot{q}^i = 0. \tag{8}$$

Let us take, now, as an independent variable the first coordinate $q^1$, instead of $t$. Remaining coordinates and time will be functions of $q^1$. Primed quantities will stand for derivatives with respect to $q^1$, so that we shall write, for instance[3],

$$q^{\alpha\prime} = \frac{dq^\alpha}{dq^1}. \tag{9}$$

Writing the kinetic energy of the system in the form

$$T = \frac{1}{2}\left( a_{11} + 2a_{1\alpha}\,\frac{\dot{q}^\alpha}{\dot{q}^1} + a_{\alpha\beta}\,\frac{\dot{q}^\alpha}{\dot{q}^1}\,\frac{\dot{q}^\beta}{\dot{q}^1}\right)(\dot{q}^1)^2, \tag{10}$$

[3]) The Greek indices take the values $2, 3, \ldots, n$.

and introducing a function $G$:

$$G(q^1, q^\alpha, q^{\alpha\prime}) = a_{11} + 2a_{1\alpha}\, q^{\alpha\prime} + a_{\alpha\beta}\, q^{\alpha\prime}\, q^{\beta\prime}, \tag{11}$$

we obtain from (10)

$$dt = \sqrt{\frac{G}{2T}}\, dq^1. \tag{12}$$

Then, we can write (6) in the form

$$I = \int_{q^1_{(0)}}^{q^1_{(1)}} \sqrt{\frac{G}{2T}}\, dq^1. \tag{13}$$

Further, using (12), we get for (7)

$$T' + \left(\frac{\partial \Pi}{\partial q^\alpha} - \bar{Q}_\alpha\right) q^{\alpha\prime} + \frac{\partial \Pi}{\partial q^1} - \bar{Q}_1 = 0, \tag{14}$$

where we put

$$\bar{Q}_i = Q_i\left(q^1, q^\alpha, \sqrt{\frac{2T}{G}}, \sqrt{\frac{2T}{G}}\, q^{\alpha\prime}, t\right). \tag{15}$$

Let us remark, finally, that (12) yields

$$t' - \sqrt{\frac{G}{2T}} = 0. \tag{16}$$

Now, our problem is to minimize the functional (13), subject to the two conditions (14) and (16). This variational problem is equivalent to the minimization of a new functional

$$\bar{I} = \int_{q^1_{(0)}}^{q^1_{(1)}} \bar{F} dq^1, \tag{17}$$

with integrand

$$\bar{F}(q^1, q^\alpha, t, T, q^{\alpha\prime}, t'\ T', \lambda, \mu) =$$
$$= \sqrt{\frac{G}{2T}} + \lambda\left[T' + \left(\frac{\partial \Pi}{\partial q^\alpha} - \bar{Q}_\alpha\right) q^{\alpha\prime} + \frac{\partial \Pi}{\partial q^1} - \bar{Q}_1\right] + \mu\left(t' - \sqrt{\frac{G}{2T}}\right), \tag{18}$$

where the multipliers $\lambda$ and $\mu$ are functions of $q^1$ to be determined.

The conditions of extremality lead to equations:

$$\mu' + \lambda\left(\frac{\partial \bar{Q}_\alpha}{\partial t}\, q^{\alpha\prime} + \frac{\partial \bar{Q}_1}{\partial t}\right) = 0 \tag{19}$$

$$\lambda' - (1-\mu)\frac{\partial}{\partial T}\sqrt{\frac{G}{2T}} + \lambda\left(\frac{\partial \bar{Q}_\alpha}{\partial T}\, q^{\alpha\prime} + \frac{\partial \bar{Q}_1}{\partial T}\right) = 0 \tag{20}$$

$$\frac{d}{dq^1}\left[(1-\mu)\frac{\partial}{\partial q^{\alpha'}}\sqrt{\frac{G}{2T}}-\lambda\left(\frac{\partial \bar{Q}_\beta}{\partial q^{\alpha'}}q^{\beta'}+\frac{\partial \bar{Q}_1}{\partial q^{\alpha'}}+\bar{Q}_\alpha-\frac{\partial \Pi}{\partial q^\alpha}\right)\right]-$$

$$-(1-\mu)\frac{\partial}{\partial q^\alpha}\sqrt{\frac{G}{2T}}-\lambda\left[q^{\beta'}\frac{\partial}{\partial q^\alpha}\left(\frac{\partial \Pi}{\partial q^\beta}-\bar{Q}_\beta\right)+\frac{\partial}{\partial q^\alpha}\left(\frac{\partial \Pi}{\partial q^1}-\bar{Q}_1\right)\right]=0 \qquad (21)$$

$$\mu\,(q^1_{(1)})=0 \qquad (22)$$

$$\lambda\,(q^1_{(1)})=0. \qquad (23)$$

(19), (20) and (21) are Euler's equations for (18), which correspond to variables $t$, $T$ and $q^\alpha$. The end-conditions (22) and (23) appear as the consequence of the fact that the values of $t$ and $T$ are not fixed at the right end-point.

The n+3 functions $q^\alpha(q^1)$, $t(q^1)$, $T(q^1)$, $\lambda(q^1)$ and $\mu(q^1)$, can be determined by the integration of n—1 second-order differential equations (21), and four first-order differential equations (19), (20), (14) and (16). We have, for the determination of 2n+2 constants of integration, 2n conditions (4) and (5), plus two end-conditions (22) and (23).

The paths of brachistochronic motion are thus founded: by the inversion of the equation $t = t(q^1)$ we get $q^1 = q^1(t)$, and than, putting this in the functions $q^\alpha = q^\alpha(q^1)$, we obtain the remaining coordinates $q^\alpha$ as functions of $t$: $q^\alpha = q^\alpha(t)$.

Let us see now how we can attack the problem in another way. Taking $t$ as the independent variable, we are to minimize the integral (6), subject to the two conditions[4)]

$$\dot{T}+\frac{\partial \Pi}{\partial q^i}\dot{q}^i-Q_i\,\dot{q}^i=0 \qquad (24)$$

and

$$2T-a_{ij}\,\dot{q}^i\,\dot{q}^j=0. \qquad (25)$$

This means we have to minimize the functional

$$I_1=\int_{t_0}^{t_1}F_1\,dt, \qquad (26)$$

with the integrand

$$F_1(t, q^i, \dot{q}^i, T, \dot{T}, \lambda_1, \mu_1)=1+\lambda_1\left(\dot{T}+\frac{\partial \Pi}{\partial q^i}\dot{q}^i-Q_i\,\dot{q}^i\right)+\mu_1\left(2T-a_{ij}\,\dot{q}^i\,\dot{q}^j\right), \qquad (27)$$

where $\lambda_1(t)$ and $\mu_1(t)$ are Lagrange's multipliers.

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[4)] It is possible, in fact, to solve the problem by minimizing the integral (6) on one condition (24) only. But then the integrand $\bar{F}_1$ of the new functional would have the form $\bar{F}_1 = 1 + \bar{\lambda}_1(\dot{T}+\dot{\Pi}-Q_i\,\dot{q}^i) = 1+\bar{\lambda}_1\left(\frac{\partial T}{\partial q^i}\dot{q}^i+\frac{\partial T}{\partial \dot{q}^i}\ddot{q}^i+\frac{\partial \Pi}{\partial q^i}\dot{q}^i-Q_i\,\dot{q}^i\right)$, and depend on second-order derivatives of $q^i(t)$, so that Euler's equations would have the enlarged form (cf [1], p. 293). In order to avoid cumbersome calculations, we consider here the kinetic energy $T$ as a function of time on equality with $q^i(t)$. That is the reason for the additional limitation (25), which represents a non-holonomic constraint in $q^i$ and $T$.

Formulating Euler's equations for (27), we get

$$2\mu_1 - \dot{\lambda}_1 = 0 \tag{28}$$

and

$$\frac{d}{dt}\left(2\mu_1 \frac{\partial T}{\partial \dot{q}^i}\right) - 2\mu_1 \frac{\partial T}{\partial q^i} - \dot{\lambda}_1 \frac{\partial \Pi}{\partial q^i} + \frac{d}{dt}\left[\lambda_1 \left(Q_i + \frac{\partial Q_j}{\partial \dot{q}^i} \dot{q}^j\right)\right] - \lambda_1 \frac{\partial Q_j}{\partial q^i} \dot{q}^j = 0. \tag{29}$$

The upper limit of integral (26) not being fixed, and function $T(t)$ not having prescribed value on the right end-point, we have also the end-conditions in the form

$$\left[1 + \lambda_1 \frac{\partial Q_j}{\partial \dot{q}^i} \dot{q}^i \dot{q}^j + 4\mu_1 T\right]_{t=t_1} = 0 \tag{30}$$

and

$$\lambda_1(t_1) = 0. \tag{31}$$

Eliminating the multiplier $\mu_1$ by (28), we shall write (29) and (30) in the form

$$\frac{d}{dt}\left(\dot{\lambda}_1 \frac{\partial T}{\partial \dot{q}^i}\right) - \dot{\lambda}_1 \frac{\partial}{\partial q^i}(T + \Pi) + \frac{d}{dt}\left[\lambda_1 \left(Q_i + \frac{\partial Q_j}{\partial \dot{q}^i} \dot{q}^j\right)\right] - \lambda_1 \frac{\partial Q_j}{\partial q^i} \dot{q}^j = 0 \tag{29'}$$

and

$$\left[1 + \lambda_1 \frac{\partial Q_j}{\partial \dot{q}^i} \dot{q}^i \dot{q}^j + 2\lambda_1 T\right]_{t=t_1} = 0. \tag{30'}$$

After integration of n+2 equations (29′), (24) and (25), we get $q^i$, $T$ and $\lambda_1$ as functions of time. Conditions (4), (5), (30′) and (31) make it possible to determine the constants of integration. Finally, $\lambda_1(t)$ being determined, it is easy to obtain from (28) the second multiplier $\mu_1(t)$.

We shall show now that it is possible to obtain equations (29′) starting from (19)—(21). To do that, let us find, first, starting from (11), (12) and (15), derivatives

$$\mu' = \frac{d\mu}{dq^1} = \frac{d\mu}{dt} t' = \dot{\mu} \sqrt{\frac{G}{2T}}$$

$$q^{\alpha\prime} = \dot{q}^\alpha \sqrt{\frac{G}{2T}}$$

$$\lambda' = \dot{\lambda} \sqrt{\frac{G}{2T}}$$

$$\frac{\partial}{\partial T} \sqrt{\frac{G}{2T}} = -\frac{1}{2T} \sqrt{\frac{G}{2T}}$$

$$\frac{\partial \bar{Q}_j}{\partial T} = \frac{1}{2T} \frac{\partial Q_j}{\partial \dot{q}^i} \dot{q}^i$$

$$\frac{\partial}{\partial q^{\alpha\prime}} \sqrt{\frac{G}{2T}} = \frac{1}{2} \sqrt{\frac{2T}{G}} \cdot \frac{1}{2T} \frac{\partial G}{\partial q^{\alpha\prime}} = \frac{1}{2T} \frac{\partial T}{\partial \dot{q}^\alpha} \tag{32}$$

$$\frac{\partial \bar{Q}_i}{\partial q^{\alpha'}} = -\frac{1}{G}\sqrt{\frac{G}{2T}}\,\frac{\partial T}{\partial \dot{q}^{\alpha}}\,\frac{\partial Q_i}{\partial \dot{q}^{j}}\,\dot{q}^{j} + \frac{\partial Q_i}{\partial \dot{q}^{\alpha}}\sqrt{\frac{2T}{G}}$$

$$\frac{\partial}{\partial q^{i}}\sqrt{\frac{G}{2T}} = \frac{1}{2}\sqrt{\frac{2T}{G}}\,\frac{1}{2T}\,\frac{\partial G}{\partial q^{i}} = \frac{1}{2T}\sqrt{\frac{G}{2T}}\,\frac{\partial T}{\partial q^{i}}$$

$$\frac{\partial \bar{Q}_i}{\partial q^{j}} = \frac{\partial Q_i}{\partial q^{j}} + \frac{\partial Q_i}{\partial \dot{q}^{1}}\,\frac{\partial \dot{q}^{1}}{\partial q^{j}} + \frac{\partial Q_i}{\partial \dot{q}^{\beta}}\,\frac{\partial \dot{q}^{\beta}}{\partial q^{j}} = \frac{\partial Q_i}{\partial q^{j}} - \frac{1}{2T}\,\frac{\partial T}{\partial q^{j}}\,\frac{\partial Q_i}{\partial \dot{q}^{k}}\,\dot{q}^{k}$$

Then from (19), (20) and (21) we have, using (32),

$$\dot{\mu} + \lambda\,\frac{\partial Q_i}{\partial t}\,\dot{q}^{i} = 0, \tag{19'}$$

$$\dot{\lambda} + \frac{1}{2T}(1-\mu) + \frac{\lambda}{2T}\,\frac{\partial Q_j}{\partial \dot{q}^{i}}\,\dot{q}^{i}\,\dot{q}^{j} = 0, \tag{20'}$$

and

$$\frac{d}{dt}\left\{-\left[\frac{1}{2T}(1-\mu) + \frac{\lambda}{2T}\,\frac{\partial Q_j}{\partial \dot{q}^{i}}\,\dot{q}^{j}\,\dot{q}^{i}\right]\frac{\partial T}{\partial \dot{q}^{\alpha}}\right\} + \frac{d}{dt}\left[\lambda\left(Q_{\alpha} + \frac{\partial Q_j}{\partial \dot{q}^{\alpha}}\dot{q}^{j}\right)\right] -$$

$$-\dot{\lambda}\,\frac{d\Pi}{\partial q^{\alpha}} + \left[\frac{1}{2T}(1-\mu) + \frac{\lambda}{2T}\,\frac{\partial Q_j}{\partial \dot{q}^{i}}\dot{q}^{j}\,\dot{q}^{i}\right]\frac{\partial T}{\partial q^{\alpha}} - \lambda\frac{\partial Q_j}{\partial q^{\alpha}}\,\dot{q}^{j} = 0 \tag{21'}$$

After some calculus, using (20′) and (24), we get from (19′):

$$\frac{d}{dt}\left(\dot{\lambda}\,\frac{\partial T}{\partial \dot{q}^{1}}\right) - \dot{\lambda}\,\frac{\partial}{\partial q^{1}}(T + \Pi) + \frac{d}{dt}\left[\lambda\left(Q_1 + \frac{\partial Q_j}{\partial \dot{q}^{1}}\,\dot{q}^{j}\right)\right] - \lambda\,\frac{\partial Q_j}{\partial q^{1}}\,\dot{q}^{j} = 0, \tag{33}$$

while it is easy to obtain from (20′) and (21′)

$$\frac{d}{dt}\left(\dot{\lambda}\frac{\partial T}{\partial \dot{q}^{\alpha}}\right) - \dot{\lambda}\frac{\partial}{\partial q^{\alpha}}\,(T + \Pi) + \frac{d}{dt}\left[\lambda\left(Q_{\alpha} + \frac{\partial Q_j}{\partial \dot{q}^{\alpha}}\,\dot{q}^{j}\right)\right] - \lambda\,\frac{\partial Q_j}{\partial q^{\alpha}}\,\dot{q}^{j} = 0. \tag{34}$$

Putting in (22) multiplier $\mu$, found from (20), we get

$$\left[1 + \lambda\,\frac{\partial Q_j}{\partial \dot{q}^{i}}\,\dot{q}^{j}\,\dot{q}^{i} + 2\dot{\lambda}\,T\right]_{t = t_1} = 0.$$

It is thus obvious that equations (33) and (34) are identical with (29′).

At the end, we shall give a brief exposure of still another way to solve our problem.

It is a fact that the time the system needs to travel from $A$ to $B$ depends on control forces $u_i$, as well as on the paths of particles, i.e. on coordinates $q^{i}(t)$. The dependence of the functional (6) on control forces was, of course, taken in account up to now by using the condition (8) for these forces in the equations of constraints (14), resp. (24).

However, if we demand that control forces *appear* in the functional, the condition (8) must be considered as a constraint which must appear in the functional with Lagrange's multiplier. Further conditions are then given through the Lagrange's equations of motion

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}^i}-\frac{\partial T}{\partial q^i}=-\frac{\partial \Pi}{\partial q^i}+Q_i+u_i, \tag{35}$$

which represent non-holonomic constraints, relating $q^i(t)$ to $u_i(t)$. Finally, in order to avoid second-order derivatives $\ddot{q}^i$ in the functional, which appear as a consequence of the left-hand side in (35), let us introduce generalized impulses:

$$p_i=\frac{\partial T}{\partial \dot{q}^i}, \tag{36}$$

on which the new functional depends together with $q^i$ and $u_i$.

Equations (35) then take the form

$$\dot{p}_i-\frac{\partial T}{\partial q^i}+\frac{\partial \Pi}{\partial q^i}-Q_i-u_i=0, \tag{37}$$

and we, to relations (8) and (37), add equations

$$p_i-a_{ij}\,\dot{q}^j=0, \tag{38}$$

which follow from (36).

Now we formulate the problem:

Determine the conditions that minimize

$$I=\int_{t_0}^{t_1} dt \tag{6}$$

with conditions expressed by non-holonomic constraints

$$u_i\,\dot{q}^i=0, \tag{8}$$

$$\dot{p}_i-\frac{\partial T}{\partial q^i}+\frac{\partial \Pi}{\partial q^i}-Q_i-u_i=0, \tag{37}$$

$$p_i-a_{ij}\,\dot{q}^j=0. \tag{38}$$

The multiplier rule implies that the orbit is an extremal for the integrand

$$F_2(t,q^i,p_i,u_i,\dot{q}^i,\dot{p}_i,\lambda_2,\mu^i,\nu^i)=1+\lambda_2\,u_i\,\dot{q}^i+\mu^i\left(\dot{p}_i-\frac{\partial T}{\partial q^i}+\frac{\partial \Pi}{\partial q^i}-Q_i-u_i\right)+$$
$$+\nu^i\,(p_i-a_{ij}\,\dot{q}^j), \tag{39}$$

where $\lambda_2$, $\mu^i$ and $\nu^i$ are the Lagrange's multipliers.

It is easy to obtain first 2n Euler's equations for the integrand (39), corresponding to variables $u_i$ and $p_i$:

$$\lambda_2\,\dot{q}^i-\mu^i=0 \tag{40}$$

$$\dot{\mu}^i - \nu^i = 0. \tag{41}$$

Eulerian equations corresponding to coordinates $q^i$ are

$$\frac{d}{dt}\left(\lambda_2 u_i - \mu^j \frac{\partial^2 T}{\partial q^j \partial \dot{q}^i} - \mu^j \frac{\partial Q_j}{\partial \dot{q}^i} - \nu^j a_{ji}\right) + \mu^j \frac{\partial^2 T}{\partial q^j \partial q^i} - \mu^j \frac{\partial^2 \Pi}{\partial q^j \partial q^i} + \mu^j \frac{\partial Q_j}{\partial q^i} + \\ + \nu^k \frac{\partial a_{kj}}{\partial q^i} \dot{q}^j = 0. \tag{42}$$

Finally, next end-conditions appear as conditions of extremality

$$\left[1 + \nu^i p_i - \mu^i \left(\dot{p}_i - 2 \frac{\partial T}{\partial q^i}\right) + \mu^j \frac{\partial Q_j}{\partial \dot{q}^i} \dot{q}^i\right]_{t = t_1} = 0 \tag{43}$$

and

$$\mu^i (t_1) = 0. \tag{44}$$

After a rather long calculus (using equations (40), (41), (8), (37) and (38)), whose details we shall omit, equations (42) can be reduced to the form

$$\frac{d}{dt}\left(\dot{\lambda}_2 \frac{\partial T}{\partial \dot{q}^i}\right) - \dot{\lambda}_2 \frac{\partial}{\partial q^i}(T + \Pi) + \frac{d}{dt}\left[\lambda_2 \left(Q_i + \frac{\partial Q_j}{\partial \dot{q}^i} \dot{q}^j\right)\right] - \lambda_2 \frac{\partial Q_j}{\partial q^i} \dot{q}^j = 0, \tag{42'}$$

while condition of transversality (43) yield

$$\left[1 + \lambda_2 \frac{\partial Q_j}{\partial \dot{q}^i} \dot{q}^j \dot{q}^i + 2\dot{\lambda}_2 T\right]_{t = t_1} = 0. \tag{43'}$$

Having in mind the fact that condition (44) is equivalent to (31), it becomes clear that equations (42′) and (29′) are identical.

Control forces $u_i$ are obtained from equations (35), taking into consideration equations (29):

$$u_i = 2\left(\frac{\partial \Pi}{\partial q^i} - Q_i\right) - \frac{1}{\dot{\lambda}_1}\left[\frac{d}{dt}\left(\lambda_1 \frac{\partial Q_j}{\partial \dot{q}^i} \dot{q}^j\right)\right] - \lambda_1 \frac{\partial Q_j}{\partial q^i} \dot{q}^j + \ddot{\lambda}_1 \frac{\partial T}{\partial \dot{q}^i} + \lambda_1 \dot{Q}_i. \tag{45}$$

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Let us examine, finally, the form the differential equations of brachistochronic motion will take in several particular cases.

a) *The case of time-independent non-conservative forces:* $Q_i = Q_i(q^k, \dot{q}^k)$

In this case the integrand (27) does not contain $t$ explicitly, and thus we have (cf. [1], p. 216)

$$F_1 - \frac{\partial F_1}{\partial \dot{q}^i} \dot{q}^i - \frac{\partial F_1}{\partial \dot{T}} \dot{T} = C, \tag{46}$$

where $C$ is a constant. Substituting derivatives $\frac{\partial F_1}{\partial \dot{q}^i}$ and $\frac{\partial F_1}{\partial \dot{T}}$ in (46), we get

$$1 + \lambda_1 \frac{\partial Q_j}{\partial \dot{q}^i} \dot{q}^j \dot{q}^i + 2\dot{\lambda}_1 T = C. \tag{47}$$

From (47) and (30′) it follows that $C = 0$, i.e.

$$1 + \lambda_1 \frac{\partial Q_j}{\partial \dot{q}^i} \dot{q}^j \dot{q}^i + 2\dot{\lambda}_1 T = 0. \tag{47'}$$

Hence, differential equations of brachistochronic motion are in this case determined by (29′), the multiplier $\lambda_1(t)$ satisfying the equation (47′) and the end-condition (31).

b) *The case of dissipative forces, with the dissipation function of Rayleigh* $\Phi = \frac{1}{2} b_{ij}(q^k) \dot{q}^i \dot{q}^j$

This case is somewhat more special than a), because non-conservative forces are then

$$Q_i = -\frac{\partial \Phi}{\partial \dot{q}^i} = -b_{ij} \dot{q}^j. \tag{48}$$

The equations (29′) take the form

$$\frac{d}{dt}\left(\dot{\lambda}_1 \frac{\partial T}{\partial \dot{q}^i}\right) - \dot{\lambda}_1 \frac{\partial}{\partial q^i}(T + \Pi) - \frac{d}{dt}\left(2\lambda_1 \frac{\partial \Phi}{\partial \dot{q}^i}\right) + 2\lambda_1 \frac{\partial \Phi}{\partial q^i} = 0 \tag{29''}$$

and (47) becomes

$$\dot{\lambda}_1 - \frac{\Phi}{T}\lambda_1 + \frac{1}{2T} = 0. \tag{47''}$$

c) *Generalized non-conservative forces depend only on coordinates* $q^i : Q_i = Q_i(q^k)$

The equations (29′) and (47) are then still more simplified:

$$\frac{d}{dt}\left(\frac{1}{2T}\frac{\partial T}{\partial \dot{q}^i}\right) - \frac{1}{2T}\frac{\partial}{\partial q^i}(T + \Pi) - \lambda_1\left(\frac{\partial Q_i}{\partial q^j} - \frac{\partial Q_j}{\partial q^i}\right)\dot{q}^j + \frac{1}{2T} Q_i = 0, \tag{29'''}$$

$$\dot{\lambda}_1 = -\frac{1}{2T}. \tag{47'''}$$

d) *The dynamical system is conservative, i.e.* $Q_i = 0$

The differential equations of brachistochronic motion take the well-known simple form

$$\frac{d}{dt}\left(\frac{1}{2T}\frac{\partial T}{\partial \dot{q}^i}\right) - \frac{1}{2T}\frac{\partial}{\partial q^i}(T + \Pi) = 0. \tag{29^{IV}}$$

The motion of the system is then determined from (29′), with conditions (4) and (5).

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## SUR LE MOUVEMENT BRACHISTOCHRONIQUE DES SYSTEMES DYNAMIQUES NON-CONSERVATIFS

Résumé

Les équations différentielles du mouvement brachistochronique des systèmes dynamiques non-conservatifs sont déduites ainsi que l'expression des forces de contrôle qui déterminent un tel mouvement.

## O BRAHISTOHRONOM KRETANJU NEKONZERVATIVNIH MEHANIČKIH SISTEMA

Izvod

Izvode se diferencijalne jednačine brahistohronog kretanja holonomnog, skleronomnog, nekonzervativnog sistema, a takođe se određuju i kontrolne sile koje, zajedno sa datim silama, ostvaruju takvo kretanje sistema. Pri tome se uzima (jednačina (8)), da je rad kontrolnih sila na elementarnom stvarnom pomeranju sistema jednak nuli.

Problem se prvo rešava tako što se funkcional (6), koji predstavlja vreme kretanja sistema iz date početne konfiguracije (4) u datu krajnju konfiguraciju (5), transformiše na oblik (13), pri čemu je umesto nezavisno promenljive $t$ uvedena nova nezavisno promenljiva $q^1$. Varijacioni zadatak se svodi na određivanje uslova ekstremalnosti novog funkcionala (17), sa podintegralnom funkcijom (18), koji dovode do Ojlerovih jednačina (19) — (21) i prirodnih graničnih uslova (22) i (23). Jednačine (19) — (23), uz (4) i (5) određuju brahistohrono kretanje sistema.

Dalje se pokazuje da se do istog rezultata dolazi i određivanjem uslova ekstremalnosti netransformisanog funkcionala (6), uz veze (24) i (25). U tome slučaju podintegralna funkcija funkcionala koji se minimizira ima oblik (27), a jednačine koje se dobijaju rešavanjem problema su (28) — (31).

Najzad, pokazuje se da je moguće rešiti problem i uvođenjem kontrolnih sila u funkcional koji se minimizira. Tada se određuju uslovi ekstremalnosti funkcionala (6) uz ograničenja (8), (37) i (38), što dovodi do Ojlerovih jednačina (40) — — (42), uslova transverzalnosti (43) i graničnog uslova (44). Pokazuje se da se dobivene jednačine svode na one koje su rezultat prethodno izloženih postupaka rešavanja postavljenog problema.

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