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FIRST STRAIN GRADIENT THEORY OF THERMOELASTICITY OF POROUS SOLID

by

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1. Introduction

In the present work the first strain gradient theory of thermoelasticity of porous solids is studied. The effects of variation of the solid volume fraction (one minus porosity) are considered in the present theory. Based on the thermodynamic consideration a set of constitutive equations are derived and the basic equations of motion and heat transfer are obtained and discussed:

The linearlized equations for the propagation of small disturbances are also investigated. It is observed that the shear wave is decoupled from the porosity and irrotational waves; however, the latter two waves are coupled even in the absence of thermal effects. The dispersion relations are obtained and briefly discussed.

Laws of motion

The basic laws of motion of a continuum with microstructures as derived for example in [1, 2]

Conservation of Mass

$$\frac{\partial \rho}{\partial t} + (\rho\, v_j)_{,j} = 0 \tag{2.1}$$

Balance of Linear Momentum

$$\tau_{jk,j} + \rho f_k = \rho\, \ddot{u}_k \tag{2.2}$$

Balance of Angular Momentum

$$\mu_{ij,i} + e_{ijk}\, \tau_{ki} + \rho\, C_j = 0 \tag{2.3}$$

Balance of equilibrated force

$$h_{i,i} + g + \rho l = \rho k \ddot{\nu} \tag{2.4}$$

Conservation of Energy

$$\rho\, \dot{e} = \bar{\tau}_{jk}\, d_{jk} + \mu^D_{ij}\, \dot{\bar{K}}_{ij} + \bar{\bar{\mu}}_{ijk}\, \dot{\bar{\bar{K}}}_{ijk} + h_i \dot{\nu}_{,i} - g\, \dot{\nu} + q_{i,i} + \rho h \tag{2.5}$$

Entropy Inequality

$$\rho\, \dot{\eta} - (q_k/\theta)_{,k} - \rho h/\theta \geq 0. \tag{2.6}$$

In these equations ρ is the mass density of the porous solid; k the equilibrated inertia; $v_k = \dot{u}_k$ the velocity vector, u_k the displacement vector; τ_{jk} the stress tensor; μ_{ij} the couple stress tensor; f_k the body force per unit mass; e_{ijk} the alternating tensor; C_j the body couple per unit mass; h_i the equilibrated stress vector; υ the solid volume fraction which is equal to one minus porosity; l the equilibrated force per unit mass; g the internal unit mass; the absolute temperature; q_i the heat flux vector pointing outward; η the entropy density per unit mass; μ_{ij} the deviatoric part of the couple stress tensor; μ_{ijk} the symmetrized double stress tensor. The stress tensor τ_{ij} is given by

$$\bar{\tau}_{jk} = \tau_{(jk)} + \bar{\bar{\mu}}_{ijk,i}, \tag{2.7}$$

where $\tau_{(jk)}$ is the symmetric part of τ_{jk}. The strain tensor ε_{ij}, the deformation rate tensor d_{ij} and the microstrain tensors K_{ij} and K_{ijk} are defined by

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) = \varepsilon_{ji}, \quad d_{ij} = \frac{1}{2}(v_{i,j} + v_{j,i}) = d_{ji} \tag{2.8}$$

$$\bar{K}_{ij} = \frac{1}{2} e_{jlk}\, u_{k,li} = \text{Gradient of rotation } (\bar{K}_{ii} = 0) \tag{2.9}$$

$$\bar{\bar{K}}_{ijk} = \frac{1}{3}(u_{k,ij} + u_{i,jk} + u_{j,ki}) = \bar{\bar{K}}_{jki} = \bar{\bar{K}}_{kij} = \bar{\bar{K}}_{kji}$$
$$= \text{Symmetrized second gradient of displacement.} \tag{2.10}$$

The equations of balance of linear and angular momenta (2.2) and (2.3) could be combined. Evaluating the antisymmetric part of the stress tensor from equation (2.3) and substituting the result into equation (2.2), it follows that

$$\tau_{(jk),j} - \frac{1}{2}\mu^D_{il,ij}\, e_{jkl} + \rho f_k - \frac{1}{2} e_{jkl}\, (\rho C_l)_{,j} = \rho\, \ddot{u}_k. \tag{2.11}$$

From the definition of the solid volume fraction it follows that

$$\rho = \rho_0\, \upsilon, \tag{2.12}$$

where ρ_0 is the mass density of the solid material. If ρ_0 is constant, that is, the solid being incompressible, from equation (2.1) it follows that

$$\frac{\partial \upsilon}{\partial t} + (\nu\, v_j)_{,j} = 0 \tag{2.13}$$

Introducing the Helmholtz free energy ψ

$$\psi = \varepsilon - \theta\, \eta \tag{2.14}$$

and eliminating ρh between equations (2.5) and (2.6), another form of the Clausius-Duhem inequality is found, i.e.

$$-\rho(\dot{\psi}+\dot{\theta}\,\eta)+\bar{\tau}_{jk}\,d_{jk}+\mu^{D}_{ij}\,\dot{\bar{K}}_{ij}+\bar{\bar{\mu}}_{ijk}\,\dot{\bar{\bar{K}}}_{ijk}+h_i\,\dot{\nu}_{,i}-g\,\dot{\nu}+\frac{1}{\theta}q_k\,\theta_{,k}\geq 0. \quad (2.15)$$

Throughout this paper the regular Cartesian tensor notation has been employed with superposed dot indicating the material time derivative and indices following a comma denoting partial differentiations.

3. Constitutive equations

The following set of constitutive equations is now proposed:

$$\psi=\psi\,(\theta,\nu,\nu_{,i},\varepsilon_{ij},\bar{K}_{ij},\bar{\bar{K}}_{ijk}),\quad \bar{\tau}_{ij}=\bar{\tau}_{ij}\quad(\ldots.),$$
$$\mu^{D}_{ij}=\mu^{D}_{ij}\ (\ldots.),\quad \bar{\bar{\mu}}_{ijk}=\bar{\bar{\mu}}_{ijk}\ (\ldots.),\quad h_i=h_i\quad(\ldots.),$$
$$g=g\ (\ldots.),\quad q_i=q_i\quad(\ldots.),\quad \eta=\eta\ (\ldots.), \quad (3.1)$$

where the principle of equipresent is employed and the dependent constitutive variables are assumed to be functions of the same set of constitutive independent variables.

Evaluating $\dot{\psi}$ from equation (3.1) and substituting in equality (2.15), the result is

$$-\rho\left(\frac{\partial\psi}{\partial\theta}+\eta\right)\dot{\theta}+\left(\bar{\tau}_{k1}-\rho\frac{\partial\psi}{\partial\varepsilon_{k1}}+\rho\frac{\partial\psi}{\partial\nu_{,k}}\,\nu_{,1}\right)d_{k1}+\rho\frac{\partial\psi}{\partial\nu_{,k}}v_{,1}\ v_{(k,l)}$$
$$+\left(\mu^{D}_{ij}-\rho\frac{\partial\psi}{\partial\bar{K}_{ij}}\right)\dot{\bar{K}}_{ij}+\left(\bar{\bar{\mu}}_{ijk}-\rho\frac{\partial\psi}{\partial\bar{\bar{K}}_{ijk}}\right)\dot{\bar{\bar{K}}}_{ijk}$$
$$+\left(h_k-\rho\frac{\partial\psi}{\partial\nu_{,k}}\right)\dot{\nu}_{,k}-\left(g+\rho\frac{\varrho\psi}{\partial\nu}\right)\dot{\nu}+\frac{1}{\theta}q_k\,\theta_{,k}\geq 0 \quad (3.2)$$

where $v_{(k,l)}$ is the antisymmetric part of the velocity gradient tensor, and the identity

$$\frac{d}{dt}\,\nu_{,k}=\dot{\nu}_{,k}-\nu_{,j}\,v_{j,k} \quad (3.3)$$

has been employed.

The entropy inequality (3.2) must hold for all independent variations of $\dot{\theta}$, d_{kl}, $v_{(k,l)}$, $\dot{\bar{K}}_{ij}$, $\dot{\bar{\bar{K}}}_{ijk}$, $\dot{\nu}_{,k}$ and $\dot{\nu}$. It then follows that $\rho\frac{\partial\psi}{\partial\nu_{,k}}\,\nu_{,l}$ is a symmetric tensor and

$$\eta=-\frac{\partial\psi}{\partial\theta}, \quad (3.4)$$

$$\bar{\tau}_{k1} = -\rho \frac{\partial \psi}{\partial \nu_{,k}} \nu_{,1} + \rho \frac{\partial \psi}{\partial \varepsilon_{k1}} \tag{3.5}$$

$$\bar{\mu}^D_{ij} = \rho \frac{\partial \psi}{\partial \bar{K}_{ij}} \tag{3.6}$$

$$\bar{\bar{\mu}}_{ijk} = \rho \frac{\partial \psi}{\partial \bar{\bar{K}}_{ijk}}, \tag{3.7}$$

$$h_k = \rho \frac{\partial \psi}{\partial \nu_{,k}}, \tag{3.8}$$

$$g = -\rho \frac{\partial \psi}{\partial \nu}, \tag{3.9}$$

and inequality (3.2) reduces to

$$\frac{1}{\theta} q_k \, \theta_{,k} \geq C. \tag{3.10}$$

Equations (3.7) — (3.9) are the general constitutive equations for a poro-elastic solid.

For a homogeneous isotropic media, the most general form of a positive definite Helmholtz free energy function which leads to a set of linear constitutive equations is

$$\rho\psi = \rho\psi_c - \rho \; \eta_o T - \frac{\rho \; \beta(\nu)}{2T_o} T^2 - \gamma(\nu)\varepsilon_{ii} T + \frac{1}{2} \lambda \varepsilon_{ii} \, \varepsilon_{jj} + 2\mu\varepsilon_{ij}\varepsilon_{ij}$$

$$+ 2\bar{d}_1 \, \bar{K}_{ij} \, \bar{K}_{ij} + 2\bar{d}_2 \, \bar{K}_{ij} \, \bar{K}_{ji} + \frac{3}{2} \bar{a}_1 \, \bar{\bar{K}}_{iij} \, \bar{\bar{K}}_{kkj} + \bar{a}_2 \bar{\bar{K}}_{ijk} \bar{\bar{K}}_{ijk}$$

$$+ \bar{f} e_{ijk} \, K_{ij} \, K_{k11} + \alpha(\nu) \, \nu_{,k} \, \nu_{,k} + \frac{1}{2} a_o \nu^2 + b_o \, \nu \, \varepsilon_{ii} \tag{3.11}$$

where

$$T = \theta - T_0 \tag{3.12}$$

and T_0 is the temperature of the natural state.

Employing expression (3.11) for the free energy into equations (3.4) — (3.9) the explicit constitutive equations are obtained

$$\eta = \eta_o + \beta \; T/T_o + \gamma \; \varepsilon_{ii}/\rho, \tag{3.13}$$

$$\bar{\tau}_{pq} = (\lambda \, \varepsilon_{ii} + b_o \; \nu - \gamma T) \; \delta_{pq} + 2\mu\varepsilon_{pq} - 2\alpha \; \nu_p \; \nu_{,q} \tag{3.14}$$

$$\mu^D_{pq} = 4\bar{d}_1 \, \bar{K}_{pq} + 4\bar{d}_2 \, \bar{K}_{qp} + \bar{f} e_{pqi} \, \bar{\bar{K}}_{ijj}, \tag{3.15}$$

$$\bar{\bar{\mu}}_{pqr} = \bar{a}_1 \, (\bar{\bar{K}}_{iir} \; \delta_{pq} + \bar{\bar{K}}_{iip} \; \delta_{qr} + \bar{\bar{K}}_{iiq} \; \delta_{rp}) + 2\bar{a}_2 \; \bar{\bar{K}}_{pqr}$$

$$+ \frac{1}{3} \bar{f} \; \bar{K}_{ij} \; (\delta_{pq} \; e_{ijr} + \delta_{qr} \; e_{ijp} + \delta_{rp} \; e_{ijq}), \tag{3.16}$$

$$h_k = 2\alpha\, \nu_{,k}, \tag{3.17}$$

$$g = -\frac{d\alpha}{d\nu}\nu_{,k}\, \nu_{,k} + \frac{d\gamma}{d\nu}\varepsilon_{ii}\; T - \frac{1}{2T_o}\,\frac{d\beta}{d\nu}\, T^2 - a_o\, \nu - b_o\, \varepsilon_{ii} \tag{3.18}$$

Note that in the derivation of equation (3.18), ρ and υ are treated as independent variables.

Employing constitutive equations (3.14) — (3.16) into equation (2.11) it follows that

$$\rho\, \ddot{\underset{\sim}{u}} = + b_o\, \nabla\nu - \nabla(\gamma T) - 2\nabla\cdot(\alpha\nabla\nu\nabla\nu) + (\lambda + 2\mu)\,(1 - l_l^2\nabla^2)\,\nabla\nabla\, \underset{\sim}{u}$$
$$-\mu\,(1 - l_2^2\nabla^2)\,\nabla\times\nabla\times\underset{\sim}{u} + \rho\, \underset{\sim}{f} + \frac{1}{2}\nabla\times(\rho\, \underset{\sim}{C}) \tag{3.19}$$

The use of constitutive equations (3.17) and (3.18) into equation (2.4) yields

$$\rho k\ddot{\nu} = 2\,\nabla\cdot(\alpha\nabla\nu) - \frac{d\alpha}{d\nu}\,\nabla\nu\cdot\nabla\nu + \frac{d\gamma}{d\nu}\nabla\cdot\underset{\sim}{u}\; T - \frac{1}{2T_o}\,\frac{d\beta}{d\nu}\, T^2 - \alpha_o\,\nu - b_o\,\nabla\cdot\underset{\sim}{u} + \rho\, l \tag{3.20}$$

To find the equation of heat transfer, we first notice that the equation of conservation of energy (2.5) by the use of relationships (2.14), and (3.4) — (3.9) becomes

$$\rho\, \theta\, \dot{\eta} = \nabla\cdot\underset{\sim}{q} + \rho\, h. \tag{3.21}$$

Assuming a simple Fourier law for heat conduction,

$$\underset{\sim}{q} = k(\upsilon)\;\nabla T \tag{3.22}$$

and employing constitutive equation (3.13), equation (3.21) reduces to

$$\rho\,\theta\,\frac{d}{dt}\,(\beta\, T/T_o + \gamma\,\nabla\cdot\underset{\sim}{u}/\rho) = \nabla\cdot(k\,\nabla\, T) + \rho\, h \tag{3.23}$$

which is a generalization for heat conduction equation.

The following restrictions are imposed on the coefficients [3]

$$\mu \geq 0, \quad 3\lambda + 2\mu \geq 0, \quad -\bar{d}_1 \leq \bar{d}_2 \leq \bar{d}_1,$$
$$\bar{a}_2 \geq 0, \quad 5\bar{a}_1 + 2\bar{a}_2 \geq 0, \quad 5\bar{f}^2 \leq 6(\bar{d}_1 - \bar{d}_2)\,(5\bar{a}_1 + 2\bar{a}_2),$$
$$a_o \geq 0, \quad b_0^2 \leq \lambda$$
$$a \geq 0, \quad \beta \geq 0, \quad \gamma \geq 0, \quad \varkappa \geq 0 \tag{3.24}$$

with

$$l_1^2 = (3\bar{a}_1 + 2\bar{a}_2)/(\lambda + 2\mu) \geq C$$
$$l_2^2 = (3\bar{d}_1 + \bar{a}_1 + 2\bar{a}_2 - \bar{f})/3\mu \geq 0 \tag{3.25}$$

Equations (3.19), (3.20) and (3.23) together with equation (2.1) form six equations for the six unknown $\underset{\sim}{u}$, υ, T and ρ.

4. Linearized equations of motion

Within the limits of linear theory, assuming that the material moduli α, β, γ and k are constant and the variations of temperature, density and solid volume fraction and displacement are small, equations (3.19), (3.20) and (3.23) in the absence of body force, equilibrated force and body couple become

$$\rho \ddot{\underset{\sim}{u}} = -\gamma \nabla T + b_0 \nabla \nu + (\lambda + 2\mu)(1 - l_1^2 \nabla^2) \nabla\nabla \cdot \underset{\sim}{u}$$
$$-\mu (1 - l_2^2 \nabla^2) \nabla \times \nabla \times \underset{\sim}{u}, \tag{4.1}$$

$$\rho k \ddot{\nu} = 2\alpha \nabla^2 \nu - a_0 \nu - b_0 \nabla \cdot \underset{\sim}{u}, \tag{4.2}$$

$$\rho \beta \dot{T} + \gamma T_0 \nabla \cdot \dot{\underset{\sim}{u}} = k \nabla^2 T + \rho h. \tag{4.3}$$

Equations (4.1) — (4.3) are five equations for finding five infinitesimal unknowns $\underset{\sim}{u}$, υ and T.

In the absence of heat source, introducing the scalar and vector potentials φ and A so that

$$\underset{\sim}{u} = \nabla \varphi + \nabla \times \underset{\sim}{A}, \tag{4.4}$$

equation of heat transfer becomes

$$\rho \beta \dot{T} + \gamma T_0 \nabla^2 \dot{\varphi} = k \nabla^2 T. \tag{4.5}$$

Equation (4.1) in terms of potentials decouples into two equations,

$$\rho \ddot{\varphi} = -\gamma T + b_0 \nu + (\lambda + 2\mu)(1 - l_2^1 \nabla^2) \nabla^2 \varphi, \tag{4.6}$$

$$\rho \ddot{\underset{\sim}{A}} = \mu (1 - l_2^2 \nabla^2) \nabla^2 \underset{\sim}{A}. \tag{4.7}$$

Equation (4.2) in terms of potential becomes

$$\rho k \ddot{\nu} = 2\alpha \nabla^2 \nu - a_0 \nu - b_0 \nabla^2 \varphi. \tag{4.8}$$

Equations (4.5) — (4.8) are a set of couples wave equations and heat transfer in the poroelastic media.

In the absence of heat transfer, dropping the thermal effects, the equations of wave propagation in a poroelastic medium become

$$\ddot{\varphi} = C_1^2 (1 - l_1^2 \nabla) \nabla^2 \varphi + \bar{b}_0 \nu, \tag{4.9}$$

$$\ddot{\underset{\sim}{A}} = C_2^2 (1 - l_2^2 \nabla^2) \nabla^2 \underset{\sim}{A}, \tag{4.10}$$

$$\ddot{\nu} = C_3^2 \nabla^2 \nu - \bar{a}_0 \nu - \frac{\bar{b}_0}{k} \nabla^2 \varphi. \tag{4.11}$$

The speeds of irrotational wave (P-wave) C_1, shear wave C_2 and porosity wave C_3 are defined by

$$C_1^2 = \frac{\lambda + 2\mu}{\rho},$$

$$C_2^2=\frac{\mu}{\rho},$$

$$C_3^2=\frac{2\alpha}{\rho k}. \tag{4.12}$$

It is observed that the shear wave is decoupled from the other two and that it satisfies a dispersion relation

$$\omega^2=C_2^2\ (1+l_2^2\ K^2)\ K^2 \tag{4.13}$$

where ω is the frequency and K is wave vector. The irrotational wave and the porosity wave are coupled and they jointly satisfy the following dispersion relation:

$$[C_1^2\ (1+l_1^2\ K^2)\ K^2-\omega^2]\ (C_3^2\ K^2+\overline{a}_0-\omega^2)=\frac{\overline{b}_0^2\ K^2}{k}. \tag{4.14}$$

The amplitude of the porosity wave υ_0 is related to that of the irrotational wave Φ_0 according to

$$\nu_0=\frac{C_1^2\ (1-l_1^2\ K^2)-\omega^2}{\overline{b}_0}\varphi_0. \tag{4.15}$$

When l_1 and l_2 are taken to be zero, the first strain gradient effects are eliminated and equations (4.5) — (4.8) govern the thermoelasticity of porous media in the absence of couple and double stresses. Equations (4.9) — (4.11) then become the corresponding wave propagation equations in poroelastic media. Equations (4.13) — — (4.15) then become simply,

$$\omega^2=C_2^2\ K^2, \tag{4,16}$$

$$(C_1\ K^2-\omega^2)\ (C_3^2\ K^2+\overline{a}_0-\omega^2)=\overline{b}_0^2\ K^2/k, \tag{4.17}$$

$$\nu_0=\frac{C_1^2\ K^2-\omega^2}{\overline{b}_0}\varphi_0, \tag{4.18}$$

Equations (4.16) and (4.17) give the dispersion relations for the shear wave and coupled irrotational and porosity waves. It will be observed that the coupled irrotational and porosity waves are dispersed. The relationship between the amplitudes now is given by equation (4.18).

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ERSTE DEHNUNGSANSTEIGSTHEORIE DER THEROMELASTIZITÄT VON PORQSEN KQRPERN

Zusammenfassung

Eine erste Dehnungsansteigstheorie für die thermoelastizität des porösen Körpers wird formuliert. In dieser Theorie ist der Effect der Abänderung der Volumenfraktion auch enthalten. Die Materialgleichungen werden abgeleitet und die Grundgleichungen von Bewegung und Wärmeübertragung werden geschrieben und diskutiert. Die Ausbreitung der kleinen Störungen nird untersucht und die Streuungs-relationen für die harmonische Wellenausbreitungen werden dargelegt.

GRADIJENTNA TEORIJA TERMOELASTIČNOSTI POROZNIH SREDINA

Izvod

Formulisana je gradijentna teorija termoelastičnosti poroznih sredina. U predloženoj teoriji uveden je efekt promene zapremine čvrste frakcije. Izvedene su konstitutivne jednačine kao i osnovne jednačine kretanja i provođenja toplote i izvršena njihova diskusija. Izučen je problem prostiranja malih poremećaja i postavljene disperzione relacije za prostiranje harmonijskih talasa.

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