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BREAKDOWN OF CHARACTERISTIC SOLUTIONS IN STEADY HYPERSONIC FLOWS OF DISSOCIATING GASES

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1. Introduction

The effect of non-linearity on the wave propagation has been the subject of great interest from a mathematical and physical point of view. The propagation of one-dimensional acceleration wave and its termination into a shock wave due to non-linear steepening has been extensively studied during the last decade by several researchers [1—5]. Jeffrey [6] developed the idea of jump discontinuities in a non-linear hyperbolic system of equations with two independent variables. Suhubi and Jeffrey [7], Varley [8] and Collins [9] studied the growth and decay of the one-dimensional acceleration wave in a variety of materials. Recently Ram and Pandey [10] solved the problem of the breakdown of an acceleration wave in a one-dimensional transient gas flow with a vibrational relaxation. The main academic interest of the present paper is to study the effect of dissociation on the breakdown of acceleration wave during the propagation. We assume a simple dissociating gas model of Lighthil [11] under the temperature range from 1000 K to 7000 K. In this temperature range when the dissociation is important, the contribution of energy from electronic excitation and ionization are both assumed negligibly small.

2. Intrinsic properties of wave motion

We now outline the analysis for the two-dimensional steady flow of a gas moving at a speed greater than M, the frozen speed ot sound. The equations ot motion governing the two-dimensional steady flow of dissociating gas are

$$\begin{aligned} & u\,\rho_x + v\,\rho_y + \rho\,(u_x + v_y) = 0, \\ & \rho\,u u_x + \rho\,v v_x + p_x = 0, \\ & \rho\,v v_y + \rho\,u v_x + p_y = 0, \\ & u p_x + v p_y + \gamma_e\,p\,(u_x + v_y) + F\,(p, \rho, \alpha) = 0, \\ & u \alpha_x + v \alpha_y = 3\,(1+\alpha)\,F\,(p, \rho, \alpha)/\{3p - \rho D\,(1+\alpha)^2\}, \end{aligned} \tag{2.1}$$

where

$$F\,(p, \rho, \alpha) = 4 \rho D^2 K_y\,\{3p - \rho D\,(1+\alpha)^2\}\,\{\rho_d\,(1-\alpha)\exp\,(-T_d/T) - \rho\alpha^2\}/3R^2 T_d^2$$

and a subscript will, in general, denote the partial differentiation unless stated otherwise. Here p, ρ, α, u, v, T_d, ρ_d, R, k_d, D and γ_e respectively denote the gas pressure, the density, the degree of dissociation, x and y components of velocity, the characteristic temperature, the characteristic density, the gas constant, the reaction rate constant, the dissociation energy per unit mass and the effective exponent of heat. In the present model the gas is moving along a plane wall at hypersonic speed and encounters a smooth compressive corner. If s and n are the measure of distances along and normal to the streamline and θ is the angle of deflection of the streamline from a suitable reference direction then the equation (2.1) can be transformed into intrinsic co-ordinates (s, n) in the following forms:

$$\begin{aligned} &q\rho_s + \rho q_s + \rho q\theta_n = 0, \\ &\rho q q_s + p_s = 0, \\ &\rho q^2\theta_s + p_n = 0 \\ &\rho\,(C_e^2 - q^2)\, q_s + \rho C_e^2\, q\, \theta_n + F(p, \rho, \alpha) = 0, \\ &q\alpha_s = \{3F(p, \rho, \alpha)\,(1+\alpha)\}\,|\,\{3p - \rho D\,(1+\alpha)^2\}, \end{aligned} \tag{2.2}$$

where

$$C_e^2 = (\gamma p/\rho), \quad q^2 = u^2 + v^2.$$

It has been observed that the variation of ρ_d over the temperature range for dissociation is very slight. Hence for practical purposes the useful simplification of regarding ρ_d as a constant should lead to negligible errors.

Now combining the above set of equations, we have

$$U_s + AU_n + B = 0, \tag{2.3}$$

where U and B are the column matrices and A is a square matrix of order 5 given by: and

$$U = \begin{bmatrix} p \\ q \\ \theta \\ \rho \\ \alpha \end{bmatrix} \quad A = \begin{bmatrix} 0 & 0 & \rho q^2 C_e^2/(q^2 - C_e^2) & 0 & 0 \\ 0 & 0 & -qC_e^2/(q^2 - C_e^2) & 0 & 0 \\ 1/\rho q^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \rho q^2/(q^2 - C_e^2) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$B = \begin{bmatrix} qF/(q^2 - C_e^2) \\ -F/(q^2 - C_e^2)\,\rho \\ 0 \\ F/q\,(q^2 - C_e^2) \\ -3F(1+\alpha)/q\{3p - \rho D(1+\alpha)^2\} \end{bmatrix}$$

The system (2.3) admits discontinuities which propagate along the forward moving characteristics. The system (2.3) is quasilinear with five real characteristics. The eigen values of the matrix A are

$$\lambda^1 = (M_e^2 - 1)^{-\frac{1}{2}}, \quad \lambda^2 = -(M_e^2 - 1)^{-\frac{1}{2}}$$

$$\lambda^3 = \lambda^4 = 0 = \lambda^5, \quad M_e = q/C_e.$$

And the corresponding eigen vectors are

$$L^1 = [(M_e^2 - 1)^{-\frac{1}{2}} \quad \rho q \quad \rho q^2 \quad C_e^2 \quad 0]$$

$$L^2 = [-(M_e^2 - 1)^{-\frac{1}{2}} \rho q \quad \rho q^2 \quad C_e^2 \quad 0]$$

$$L^3 = [1 \quad \varphi q \quad 0 \quad 0 \quad 0]$$

$$L^4 = [0 \quad 0 \quad 0 \quad 1 \quad 0], \quad L^5 = [0 \quad 0 \quad 0 \quad 0 \quad 1].$$

In order to study the characteristic properties of the wave motion we introduce two characteristic co-ordinates $s'(s, n)$ and $\Phi(s, n)$ such that Φ = constant represents a wavefront and $s' = s$. The leading forward characteristic front can be represented by $\Phi(s, n) = 0$. Any flow property $f(s, n)$ is continuous across a characteristic wave front $\Phi(s, n) = 0$, but f_s and f_n undergo finite jumps across it. The transformation $(s, n) \to (s', \phi)$ is non-singular provided the Jacobian of transformation $J = n\phi = 1/\phi_n$ is non zero. Let us consider an open region R' bounded by two characteristics $\Phi(s, n) = 0$ and $\xi(s, n) = 0$ such that no charachteristics issuing from the origin enter this open region R'. We assume that U remains smooth in R' at least for a finite time during which the transformation is non-singular. Transforming the quasi-linear system (2.3) into new co-ordinates (Φ, s') and multiplying it by L^j, such that $b^j = L^j B$, we get,

$$L^j \{n_\Phi U_{s,} + (\lambda - \lambda^1) U_\Phi\} + n_\varphi b^j = 0 \tag{2.4}$$

which yields,

$$L^j U_{s'} + b^j = 0, \quad (\lambda = \lambda^1). \tag{2.5}$$

If $C^{(\Phi)}$ represents the wavefront trace, the lower suffix Φ denotes differentiation along the normal to $C^{(\Phi)}$ and the lower suffix s' denotes differentiation along $C^{(\Phi)}$. The boundary conditions at the wavefront $\Phi(s, n) = 0$ are

(i) U is continuous: $[U] = 0$

(ii) $U_{s'}$ is continuous: $[U_{s'}] = 0$

(iii) U_Φ is discontinuous: $[U_\Phi] = \Pi(s')$,

(iv) n_Φ is discontinuous: $[n_\Phi] = R(s')$,

where the bracket [2] denotes the jump in Z across a wavefront $\Phi(s, n) = 0$, i.e.

$$[Z]_{\phi=0+}^{\phi=0-} = Z(0^-, s') - Z(0^+, s').$$

From the definition of $R(s')$ we observe that

$$R(s') + (n_\Phi)_0 = (n_\Phi)_{\Phi=0-} = 0. \tag{2.6}$$

If U_0 corresponds to the constant state condition ahead of the wave front, we have

$$B(U_0) = 0.$$

Taking jump in (2.4) across $\Phi = 0$ and using the constant state condition and the boundary conditions, we have

$$L^j{}_0 \, \Pi \, (s') = 0, \qquad (\lambda \mp \lambda^1). \tag{2.7}$$

Differentiating (2.5) with respect to Φ and evaluating at the wave front $\Phi = 0$, we get

$$L^1{}_0 \, \Pi s' + [\nabla_u (L^1 B)]_0 \, \Pi \tag{2.8}$$

where ∇_u denotes the gradient operator with respect to the component of the vector U.

The equation of an outgoing characteristic can be written as

$$n_s' = \lambda^1. \tag{2.9}$$

Differentiating (2.9) with respect to Φ at any point in the open region R' and allowing this to tend to a point on the wavefront $\Phi(s, n) = 0$ and using the jump conditions, we get

$$R_{s'} = [\nabla_u (\lambda^1)]_0 \, \Pi.$$

which provides that

$$R = \bar{R} + \int_0^s [\nabla_u (\lambda^1)]_0 \, \Pi \, ds', \tag{2.10}$$

where $\bar{R} = \lim_{s' \to 0.} R.$

We would expect the solution to breakdown after some finite critical distance s which can be determined by the interaction of two characteristics. At such a critical point the Jacobian of the Transformation must vanish and therefore, we have

$$R(s') + (n_\Phi)_0 = 0.$$

Hence the critical distance s_c is given by

$$1 + \int_0^{s_c} [\nabla_u (\lambda^1)]_0 \, \Pi / \bar{n}_\Phi) \, ds' = 0. \tag{2.11}$$

Equation (2.7) implies for the present model that

$$\Pi_4 = 0 = \Pi_5, \qquad \Pi_1 + \rho_0 \, q_0 \, \Pi_2 = 0 \tag{2.12}$$

$$\rho_0 \, q_0 \, \Pi_2 + \rho_0 \, q_0^2 \, \Pi_3 = (M_e^2 - 1)_0^{\frac{1}{2}} \, \Pi_1. \tag{2.13}$$

Using (2.12) in (2.8) we obtain an equation for Π in the form

$$\Pi_{2s}' + \Pi_2 \, \{M_e^2 (M_e^2 - 1)^{\frac{1}{2}} - (M_e^2 - 1)\}_0 \, \{M_e (\partial F / \partial p) \, pq - F. \, C_e\}_0 / \{2 M_e^2 \, (M_e^2 - 1)^{3/2} \cdot C_e p \cdot q\}_0. \tag{2·14}$$

Thus the jump discontinuities $\Pi = [U_\Phi]$ across $\Phi(s, n) = 0$ can be given by

$$\Pi = \bar{\Pi}_2 \exp(-C_1 s) \; [-q_0 \rho_0 \quad 1 \quad 2/q_0 \quad 0 \quad 0]. \tag{2.15}$$

where

$$C_1 = \{M_e^2 (M_e^2 - 1)^{\frac{1}{2}} - (M_e^2 - 1)\}_0 \{M_e (\partial F/\partial p) pq - F \cdot C_e\}_0 / \{2M_e (M_e^2 - 1)^{3/2} \cdot C_e pq\}_0$$

is a constant of the constant state. Now differentiating equation (2.9) with respect to Φ at any point in the domain R' and evaluating on the wavefront $\Phi(s, n) = 0$, we get

$$n_{\Phi s'} = -2M_{e_0} \Pi_2 / (M_e^2 - 1)_0 \tag{2.16}$$

The equation (2.14) provides that

$$q_{\Phi s'} = C_1 \Pi_2. \tag{2.17}$$

The amplitude $a(s)$ of the two-dimensional acceleration wave can be defined by

$$a(s) = [q_n] = \Pi_2 / n_\Phi. \tag{2.18}$$

Differentiating (2.18) with respect to s and making use of (2.17) and (2.16) we obtain

$$da(s)/ds + C_1 a - 2(M_e)_0 a^2 / (M_e^2 - 1)_0 = 0. \tag{2.19}$$

This is the fundamental growth equation governing the propagation of the wave with amplitude $a(s)$. The solution of (2.19) is a characteristic solution representing the amplitude of the characteristic wave front so long as it remains bounded.

3. The global behaviour of solutions

The solution of (2.19) is of the form

$$a(s) = \exp(-C_1 s)\, a(0)\, (M_e^2 - 1)_0 C_1 / \{M_e^2 - 1) C_1 - 2a(0) M_e (1 - e^{-C_1 s})\}_0 \tag{3.1}$$

where $a(0)$ is the value of $a(s)$ at $s = 0$. The solution (3.1) shows that if a (0) $< C_1 (M_e{}^2 - 1)_0 / 2 M_{e0}$, the wave amplitude will monotonically decay and tend to zero as $s \to \infty$. On the other hand if $a(0) > C_1 (M^2 - 1)_0 / 2 M_0$, there exists a critical value s_c given by:

$$C_1 \cdot s_c = \log \{2a(0) M_e / (2a(0) M_e - C_1 M_e^2 + C_1)\}_0 \tag{3.2}$$

such that the wave amplitude $a(s)$ increases without limit and becomes unbounded at $s = s_C$. Consequently the characteristic solutions will breakdown at the stage $s = s_C$ and a shock-type discontinuity will appear automatically. In the case of $a(0) = C_1 (M_e^2 - 1)_0 / 2 M_{e0}$, the wave will ultimately stabilize as a wave of constant amplitude. It is also concluded that the effect of dissociation is to delay the shock formation.

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RASPAD KARAKTERISTIČNIH REŠENJA U STACIONARNOM HIPERZVUČNOM STRUJANJU DISOCIRANIH GASOVA

U radu se proučava jedan sistem kvazi-linearnih hiperboličkih jednačina koje opisuju stacionarno ravansko hiperzvučno strujanje disociranog gasa. Određena je brzina rasprostiranja talasa ubrzanja, a problem narastanja i opadanja talasa je rešen analitički. Određen je kritični trenutak kada se talas ubrzanja raspada na udarni talas usled nelinearnog povećanja nagiba. Istraživano je globalno ponašanje amplitude talasa i dobijeni su neki interesantni rezultati.

РАСПАДЕНИЕ ХАРАКТЕРИСТИЧЕСКИХ РЕШЕНИЙ СТАЦИОНАРНЫХ ГИПЕРЗВУКОВЫХ ТЕЧЕНИЯХ ДИССОЦИРОВАННЫХ ГАЗОВ

Изучается система квазилинейных гиперболических уравнений, описывающая стационарное плоское гиперзбукове течение диссоциированного газа. Определена скорость распространения волны ускорения, а задача нарастания и опадания волны решена аналитически. Определен критический момент, когда волна ускарения распадется в ударную волну вследствие нелинейного увеличения стремммины. Исследовано глобальное поведение амплитуды волны и получены некторые интоересные результаты.

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