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ON OPTIMALITY AND DUALITY IN NONSMOOTH VECTOR FRACTIONAL CONTINUOUS-TIME PROGRAMMING: STRENGTHENED CONDITIONS FOR MIXED-AFFINE MODELS

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Abstract. We investigate a nonsmooth vector fractional continuous-time programming problem with inequality-type phase constraints, motivated by applied-mechanics contexts in which performance is quantified by ratio-type, time-accumulated indices under pointwise-in-time operational constraints, including abrasive machining and grinding, quasi-steady aircraft cruise efficiency, and energy-aware gait scheduling in legged robotics. We derive saddle-point and Karush-Kuhn-Tucker type necessary optimality conditions for properly efficient solutions by combining a continuous-time Slater-type condition with a regularity requirement for convex inequality systems, and we also establish a sufficient saddle-point optimality condition that holds in the convex framework. For models with mixed affine structure, we strengthen the theory beyond classical Slater-based frameworks by introducing two additional verifiable hypotheses, a solvability condition and a separation direction condition. These assumptions yield sharper multiplier conclusions, including nontriviality of multipliers associated with nonaffine constraints, and lead to refined optimality statements without auxiliary parameters. A key lemma is established and provides the main tool underlying these results. We then introduce a vector-valued Lagrangian and formulate a corresponding vector dual model. For the dual problem, we prove weak and strong duality results, including a strong duality theorem that guarantees the absence of a duality gap. Several examples illustrate how the assumptions can be verified and how the theoretical results apply through explicit multiplier constructions.

1. Introduction

A vector continuous-time programming problem involves minimizing a vector-valued integral functional subject to phase constraints that must hold almost everywhere in time. Over the past several decades, such problems have been studied extensively due to their central role in modeling time-dependent systems in

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economics, engineering, and control; see, e.g., [1–4]. In particular, a broad optimality and duality theory has been developed for vector continuous-time programming under various convexity and generalized convexity assumptions; see, e.g., [5–12]. In many applications, performance is naturally measured through *efficiency-type* criteria, that is, ratios of accumulated benefits to accumulated costs. This leads to *vector fractional continuous-time programming*, where several fractional indices must be optimized simultaneously. In contrast to the non-fractional case, the literature on vector fractional continuous-time programming is comparatively sparse; see, for example, [13–17]. The smooth version of a vector fractional continuous-time problem was studied in [13]. Subsequently, [14, 15] derived saddle-point type criteria and duality theorems for a class of *nonsmooth* vector fractional problems involving Volterra-type integral inequalities. It is important to emphasize that these papers impose the nonnegativity constraint $z(\cdot)$ for trajectories in $L_\infty(\Delta;\mathbb{R}^n)$. Moreover, in order to apply scalar nonsmooth continuous-time results from [18], Zalmai adopted restrictive hypotheses and constraint qualifications that can be difficult to verify in practice, and also worked with a special Volterra-type constraint structure. More recently, [16] examined a more general *smooth* vector fractional continuous-time programming model and obtained improved optimality conditions under new constraint qualifications that are demonstrably verifiable through examples.

1.1. Motivation. Vector fractional continuous-time models arise whenever engineering performance is assessed through *time-accumulated efficiency indices*, i.e., ratios of "useful effect" to "spent resources" measured over an operating horizon. In many mechanical systems the decision variables can be interpreted as *time-dependent set-points* (or scheduling signals) chosen in response to changing operating conditions, while no state dynamics are explicitly modeled. This makes the resulting formulation closer to continuous-time programming than to optimal control. Moreover, in practice the involved performance maps are frequently obtained from data sheets, calibrated look-up tables, or piecewise models, which naturally leads to convex but possibly nonsmooth integrands.

Model 1 (Abrasive machining / grinding efficiency). In grinding and related abrasive finishing processes, practitioners evaluate the performance using *specific energy* (energy per removed material volume) and *wear efficiency* (material removed per wheel wear), together with thermal or quality indicators; see, e.g., [19–21]. Let the time variable $\tau \in [0,T]$ parameterize a production horizon (or a pass along the workpiece) and let $z(\tau) := \big(v_s(\tau),\, v_w(\tau),\, a_e(\tau),\, d(\tau)\big) \in \mathbb{R}^n$ collect adjustable set-points such as wheel speed v_s, workpiece feed speed v_w, depth of cut a_e, and a dressing/conditioning parameter d. Denote by $Q(\tau, z(\tau))$ the (instantaneous) material removal rate (volume per unit time), by $P(\tau, z(\tau))$ the grinding power (energy per unit time), and by $W(\tau, z(\tau))$ a wheel-wear rate (wheel volume loss per unit time). A standard efficiency criterion is the *specific grinding energy*

$$J_1(z(\cdot)) := \frac{\int_0^T P(\tau, z(\tau))d\tau}{\int_0^T Q(\tau, z(\tau))d\tau},$$

i.e., total energy divided by total removed volume. Similarly, the inverse of the grinding ratio (wear per removed volume) can be written as

$$J_2(z(\cdot)) := \frac{\int_0^T W(\tau, z(\tau))d\tau}{\int_0^T Q(\tau, z(\tau))d\tau},$$

while a thermal damage proxy (e.g., heat flux or maximum temperature surrogate) $\Theta(\tau, z(\tau))$ can be incorporated as

$$J_3(z(\cdot)) := \frac{\int_0^T \Theta(\tau, z(\tau))d\tau}{\int_0^T Q(\tau, z(\tau))d\tau}.$$

The engineering requirement is to keep these indices low *simultaneously*, leading to a vector objective $(J_1(z(\cdot)), J_2(z(\cdot)), J_3(z(\cdot))) \to \inf$ and hence a Pareto notion of optimality. Phase constraints express process feasibility and quality requirements almost everywhere in time, for instance: power limits $P(\tau, z(\tau)) \leqslant P_{\max}$, burn avoidance $\Theta(\tau, z(\tau)) \leqslant \Theta_{\max}$, surface-finish constraints, chatter/vibration bounds, and machine-tool limitations, all written as inequalities $h_i(\tau, z(\tau)) \leqslant 0$ a.e. on $[0, T]$.

Example: Time-dependent surface-grinding model. We consider a simplified surface-grinding process over the time interval $\Delta = [0, 1]$. Let $z(\tau) = p(\tau) \in L_\infty([0, 1]; \mathbb{R})$ denote the commanded normalized contact pressure between the grinding wheel and the workpiece. In this model, the local contact geometry and workpiece condition vary along the grinding pass. We represent this variation by the term τ. Thus, the effective grinding intensity is $q(\tau, p) := p - \tau$. Mechanically, $q(\tau, p)$ represents the effective pressure available for material removal after compensating for the local position-dependent geometry or hardness variation of the workpiece. The instantaneous material removal rate is modeled by $Q(\tau, p) = p - \tau$. The instantaneous grinding power is modeled by $P(\tau, p) = 1 + p - \tau$. Here the constant term 1 represents an idle machine-power contribution, while $p - \tau$ represents the pressure-dependent cutting/grinding contribution. The wheel-wear rate is modeled by $W(\tau, p) = 2(p - \tau) - 1$. Thus, wheel wear increases with the effective grinding intensity. The mechanical operating constraints are $1 \leqslant p(\tau) - \tau \leqslant 2$ a.e. in $[0, 1]$. The lower bound guarantees a minimal material-removal intensity, while the upper bound represents a power/thermal/wheel-wear limitation. Equivalently, $h_1(\tau, p) = 1 + \tau - p \leqslant 0$, $h_2(\tau, p) = p - \tau - 2 \leqslant 0$.
Therefore, the corresponding vector fractional continuous-time programming problem is

$$\text{(G)} \qquad \begin{aligned} &\min(J_1(p(\cdot)), J_2(p(\cdot))) = \left(\frac{\int_0^1 (1 + p(\tau) - \tau)d\tau}{\int_0^1 (p(\tau) - \tau)d\tau}, \frac{\int_0^1 (2(p(\tau) - \tau) - 1)d\tau}{\int_0^1 (p(\tau) - \tau)d\tau} \right) \\ &\text{s.t.} \quad 1 + \tau - p(\tau) \leqslant 0 \quad \text{a.e. in } [0, 1], \\ &\qquad\; p(\tau) - \tau - 2 \leqslant 0 \quad \text{a.e. in } [0, 1], \\ &\qquad\; p(\cdot) \in L_\infty([0, 1]; \mathbb{R}). \end{aligned}$$

The denominator $\int_0^1 (p(\tau) - \tau)d\tau$ represents the total removed material volume in normalized units. The first numerator $\int_0^1 (1 + p(\tau) - \tau)d\tau$ represents the total

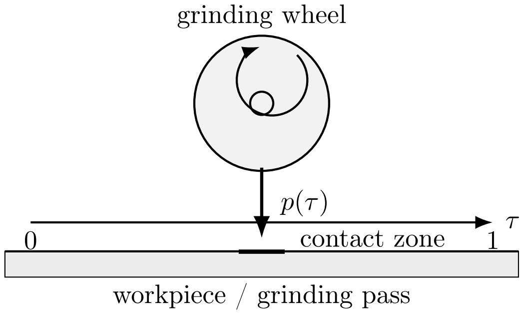


FIGURE 1. Schematic representation of the time-dependent surface-grinding model.

grinding energy. Hence, $J_1(p(\cdot))$ is the specific grinding energy, i.e., energy per removed material volume. The second numerator $\int_0^1 (2(p(\tau)-\tau)-1)d\tau$ represents the total wheel wear. Therefore, $J_2(p(\cdot))$ represents wheel wear per removed material volume, i.e., the inverse of a grinding-ratio type efficiency index.

Thus, the problem describes the fundamental engineering compromise between reducing energy consumption per removed material volume and reducing wheel wear during the grinding process.

Model 2 (Aeronautics: cruise eco-efficiency without explicit dynamics). A classical aircraft-performance quantity is *specific air range* (distance per fuel), equivalently *fuel per distance* when posed as a minimization; see, e.g., [22–24]. Consider a quasi-steady cruise segment over a horizon $\tau \in [0, T]$, and let $z(\tau) := (V(\tau), h(\tau), M(\tau))$ denote scheduled set-points such as true airspeed V, altitude h, and/or Mach number M. Let $r_f(\tau, z(\tau))$ be a fuel-use rate (fuel mass per unit time) and let $V(\tau)$ represent airspeed. Define the accumulated air distance

$$D(z(\cdot)) := \int_0^T V(\tau)d\tau$$

and total fuel

$$F(z(\cdot)) := \int_0^T r_f(\tau, z(\tau))d\tau.$$

Minimizing fuel per distance is then

$$J_1(z(\cdot)) := \frac{\int_0^T r_f(\tau, z(\tau))d\tau}{\int_0^T V(\tau)d\tau}.$$

If, in addition, one wishes to reduce environmental impact, a second fractional component can be added, e.g., a NO_x emission rate $r_{NO_x}(\tau, z(\tau))$:

$$J_2(z(\cdot)) := \frac{\int_0^T r_{NO_x}(\tau, z(\tau))d\tau}{\int_0^T V(\tau)d\tau},$$

and possibly a third component for a noise proxy or an engine-health proxy, again normalized by distance. The feasibility region is governed by phase constraints representing the flight envelope and engine limits, such as stall margin, maximum Mach, thrust/temperature margins, and structural load constraints, all naturally written as $h_i(\tau, z(\tau)) \leqslant 0$ a.e. Importantly, in this viewpoint, τ indexes a mission segment and $z(\tau)$ is a scheduled operating point; one can formulate the problem without explicitly introducing a state equation, while still capturing time-varying atmosphere, mass/configuration envelopes, and operational constraints. With convex/nonsmooth approximations (e.g., envelope or piecewise-affine fits) for rates and constraints, the model falls into the nonsmooth convex framework.

Model 3 (Robotics: energy-aware gait scheduling in legged locomotion). In legged robotics, a widely used efficiency measure is the (total/mechanical) *cost of transport*, i.e., energy (or power) normalized by weight and distance; see, e.g., [25–28]. Let $\tau \in [0, T]$ denote a traversal interval and let $z(\tau)$ collect gait parameters and set-points, for example, step frequency, duty factor, step length, body height, compliance settings, or controller gains. Let $P_{\text{elec}}(\tau, z(\tau))$ be electrical power consumption and let $v_x(\tau, z(\tau))$ be forward speed. A distance-normalized energy criterion is

$$J_1(z(\cdot)) := \frac{\int_0^T P_{\text{elec}}(\tau, z(\tau))d\tau}{\int_0^T v_x(\tau, z(\tau))d\tau}.$$

Additional simultaneously optimized indices can include a wear/slip proxy $S(\tau, z(\tau))$ or a tracking/impact proxy $E(\tau, z(\tau))$, again normalized by distance:

$$J_2(z(\cdot)) := \frac{\int_0^T S(\tau, z(\tau))d\tau}{\int_0^T v_x(\tau, z(\tau))d\tau}, \quad J_3(z(\cdot)) := \frac{\int_0^T E(\tau, z(\tau))d\tau}{\int_0^T v_x(\tau, z(\tau))d\tau}.$$

Phase constraints encode actuator and stability limits that must hold throughout the motion, such as joint/torque bounds, friction-cone (often polyhedral) contact constraints, center-of-pressure or support-polygon stability margins, and power/temperature caps, all written as $h_i(\tau, z(\tau)) \leqslant 0$ a.e. This yields a multiobjective fractional continuous-time model whose solutions are naturally interpreted via proper efficiency: it excludes pathological trade-offs (e.g., "infinitely better energy at arbitrarily worse impacts") and supports meaningful multiplier and duality statements.

Our contribution. In this paper, we study a more general class of nonsmooth vector fractional continuous-time programming problems with inequality-type phase constraints, formulated in $L_\infty(\Delta; \mathbb{R}^n)$, where the nonnegativity requirement on $z(\cdot)$ is not imposed. We derive saddle-point optimality conditions under a Slater-type constraint qualification together with a regularity condition for convex inequality systems. A key lemma is formulated and proved, and it plays a pivotal role in establishing the subsequent multiplier and duality results. In comparison with the hypotheses imposed in, for example, [13–15], the assumptions used in this paper are less restrictive and are designed to be directly verifiable in concrete examples. Besides a Slater-type constraint qualification (SQ) and a regularity requirement (RC) for convex inequality systems, we employ two additional structural conditions tailored to our mixed-affine setting: the *solvability condition* (SC), which guarantees the existence of a trajectory satisfying the affine constraint block while strictly improving at least one fractional component, and the *separation direction condition* (SD), which provides a uniform descent direction separating the affine parts of the Lagrangian integrands. These conditions allow us to strengthen the multiplier statements, including nontriviality of the nonaffine multiplier block, and to obtain parameter-free optimality assertions in the mixed-affine framework. To the best of our knowledge, the resulting necessary optimality conditions are derived under some of the weakest and most practically checkable assumptions currently available for nonsmooth vector fractional continuous-time programming in $L_\infty(\Delta;\mathbb{R}^n)$. The remainder of the paper is organized as follows. Section 2 collects preliminaries and basic definitions. In Section 3 we introduce and discuss a Lagrangian-type function for the vector fractional continuous-time problem and establish saddle-point optimality conditions—results that, to the best of our knowledge, have not appeared in the literature for models formulated in $L_\infty(\Delta;\mathbb{R}^n)$. In Section 4 we define a vector Lagrangian dual model and prove weak and strong duality theorems.

2. Preliminaries

Let us consider the following vector fractional continuous-time problem:

$$\min_{z(\cdot)\in L_\infty(\Delta;\mathbb{R}^n)} \frac{\int_\Delta f(\tau,z(\tau))d\tau}{\int_\Delta g(\tau,z(\tau))d\tau} = \left(\frac{\int_\Delta f_1(\tau,z(\tau))d\tau}{\int_\Delta g_1(\tau,z(\tau))d\tau},\dots,\frac{\int_\Delta f_k(\tau,z(\tau))d\tau}{\int_\Delta g_k(\tau,z(\tau))d\tau}\right)$$

$$\text{(VFCTP)} \qquad \text{s. t.} \;\; h(\tau,z(\tau)) := (h_1(\tau,z(\tau)),\dots,h_m(\tau,z(\tau))) \leqq 0, \;\; \text{a.e. in } \Delta,$$

where $f_j, g_j, h_i\colon \Delta\times\mathbb{R}^n\to\mathbb{R}$, $j\in J=\{1,\dots,k\}$, $i\in I=\{1,\dots,m\}$ are given functions where $\Delta=[0,T]\subset\mathbb{R}$. The following convention for equalities and inequalities will be used. If $p,q\in\mathbb{R}^k$, then

(i) $p=q \iff p_j=q_j,\ j=1,\dots,k,$
(ii) $p\leqq q \iff p_j\leqq q_j,\ j=1,\dots,k,$
(iii) $p\leqslant q \iff p\leqq q$ and $p\neq q,$
(iv) $p<q \iff p_j<q_j,\ j=1,\dots,k,$
(v) $p\nleqslant q$ is the negation of $p\leqslant q$.

For each $\tau\in\Delta$, $z_i(\tau)$ is the ith component of $z(\tau)\in\mathbb{R}^n$ and all integrals are in the sense of Lebesgue. Let Ω_P be the set of all feasible solutions of (VFCTP) i.e.,

$$\Omega_P=\{z(\cdot)\in L_\infty(\Delta;\mathbb{R}^n) : h(\tau,z(\tau))\leqq 0,\quad \text{a.e. in } \Delta\}.$$

We assume that functions $f(\tau,\cdot)$, $h(\tau,\cdot)$ are convex and continuous on $\mathbb{R}^n$, for a.e. $\tau\in\Delta$ and $g(\tau,\cdot)$ is concave and continuous on $\mathbb{R}^n$, for a.e. $\tau\in\Delta$. Also, we assume that functions $f(\cdot,z)$, $g(\cdot,z)$, $h(\cdot,z)$ are Lebesgue measurable for all $z\in\mathbb{R}^n$ and functions $f(\tau,z)$, $g(\tau,z)$, $h(\tau,z)$ are bounded for a.e. $\tau\in\Delta$ and for all bounded $z\in\mathbb{R}^n$. In some literature, these functions are also called Caratheodory functions. We use the standard Euclidean inner product $\langle\cdot,\cdot\rangle$. For $z(\cdot)\in\Omega_P$, we also assume that $\int_\Delta f(\tau,z(\tau))d\tau\geqq 0$ and $\int_\Delta g(\tau,z(\tau))d\tau>0$.

REMARK 2.1 (Mixed affine/convex structure). If there exist index sets $J_A \subseteq \{1, \ldots, k\}$ and $I_A \subseteq \{1, \ldots, m\}$ such that for each $j \in J_A$,

$$f_j(\tau, z) = \langle a_j(\tau), z\rangle + b_j(\tau), \quad g_j(\tau, z) = \langle c_j(\tau), z\rangle + d_j(\tau),$$

and for each $i \in I_A$,

$$h_i(\tau, z) = \langle p_i(\tau), z\rangle + q_i(\tau),$$

for a.e. $\tau \in \Delta$ and all $z \in \mathbb{R}^n$, we have a special case of (VFCTP). Of course, in that case for $j \in J_N := \{1, \ldots, k\} \setminus J_A$ the functions $f_j(\tau, \cdot)$ are convex and $g_j(\tau, \cdot)$ are concave, and for $i \in I_N := \{1, \ldots, m\} \setminus I_A$ the functions $h_i(\tau, \cdot)$ are convex.

Throughout this paper, all vectors are considered column vectors, and we denote transposition by $'$. Also, the minimization in (VFCTP) is in the sense of a proper efficient solution.

DEFINITION 2.1. A point $\hat{z}(\cdot) \in \Omega_P$ is said to be an efficient solution for (VFCTP) if there is no other $z(\cdot) \in \Omega_P$ such that

$$\frac{\int_\Delta f(\tau, z(\tau))d\tau}{\int_\Delta g(\tau, z(\tau))d\tau} \leqslant \frac{\int_\Delta f(\tau, \hat{z}(\tau))d\tau}{\int_\Delta g(\tau, \hat{z}(\tau))d\tau}.$$

DEFINITION 2.2. A point $\hat{z}(\cdot) \in \Omega_P$ is said to be a proper efficient solution for (VFCTP) if it is efficient and if there exists $M > 0$ such that, for each i,

$$\frac{\frac{\int_\Delta f_i(\tau,\hat{z}(\tau))d\tau}{\int_\Delta g_i(\tau,\hat{z}(\tau))d\tau} - \frac{\int_\Delta f_i(\tau,z(\tau))d\tau}{\int_\Delta g_i(\tau,z(\tau))d\tau}}{\frac{\int_\Delta f_j(\tau,z(\tau))d\tau}{\int_\Delta g_j(\tau,z(\tau))d\tau} - \frac{\int_\Delta f_j(\tau,\hat{z}(\tau))d\tau}{\int_\Delta g_j(\tau,\hat{z}(\tau))d\tau}} \leqq M$$

for some j such that

$$\frac{\int_\Delta f_j(\tau, z(\tau))d\tau}{\int_\Delta g_j(\tau, z(\tau))d\tau} > \frac{\int_\Delta f_j(\tau, \hat{z}(\tau))d\tau}{\int_\Delta g_j(\tau, \hat{z}(\tau))d\tau}$$

when $z(\cdot) \in \Omega_P$, and

$$\frac{\int_\Delta f_i(\tau, z(\tau))d\tau}{\int_\Delta g_i(\tau, z(\tau))d\tau} < \frac{\int_\Delta f_i(\tau, \hat{z}(\tau))d\tau}{\int_\Delta g_i(\tau, \hat{z}(\tau))d\tau}.$$

If $\hat{z}(\cdot)$ is a proper efficient solution of (VFCTP), we denote a proper efficient value $\hat{w} \in \mathbb{R}^k_+$ by

$$\hat{w} = \frac{\int_\Delta f(\tau, \hat{z}(\tau))d\tau}{\int_\Delta g(\tau, \hat{z}(\tau))d\tau}.$$

Here, $\mathbb{R}^k_+$ denotes the positive orthant of $\mathbb{R}^k$ and $e = (1, \ldots, 1)' \in \mathbb{R}^k$. Let $\Lambda^+ = \{\lambda \in \mathbb{R}^k : \lambda' e = 1,\ \lambda > 0\}$, $V = \{v(\cdot) \in L_\infty(\Delta; \mathbb{R}^m) : v(\tau) \geqq 0, \text{ a.e. in } \Delta\}$. Next, we establish a Slater-type constraint qualification within the continuous-time framework.

CONSTRAINT QUALIFICATION 1 (SQ). (Slater's constraint qualification) We say that continuous-time Slater's constraint qualification (SQ) is satisfied, if there exists a function $s(\cdot) \in L_\infty(\Delta; \mathbb{R}^n)$ such that

$$h(\tau, s(\tau)) < 0 \quad \text{a.e. in } \Delta. \tag{SQ}$$

Constraint qualification 2 (SC). (Solvability condition) Assume the mixed affine structure in the sense that $J_N = \emptyset$ and $I = I_A \cup I_N$, where for each $j \in J$,

$$f_j(\tau, z) = \langle a_j(\tau), z\rangle + b_j(\tau), \quad g_j(\tau, z) = \langle c_j(\tau), z\rangle + d_j(\tau),$$

and for each $i \in I_A$,

$$h_i(\tau, z) = \langle p_i(\tau), z\rangle + q_i(\tau),$$

for a.e. $\tau \in \Delta$ and all $z \in \mathbb{R}^n$. Let $\hat{w}$ be defined as above. We say that the solvability condition (SC) for problem (VFCTP) with mixed affine/convex structure is satisfied if there exists a function $\zeta(\cdot) \in L_\infty(\Delta; \mathbb{R}^n)$ such that

(SC) $\quad h_i(\tau, \zeta(\tau)) = \langle p_i(\tau), \zeta(\tau)\rangle + q_i(\tau) \leqslant 0$ a.e. in Δ for all $i \in I_A$

and $\quad \int_\Delta \big(f_\ell(\tau, \zeta(\tau)) - \hat{w}_\ell g_\ell(\tau, \zeta(\tau))\big)d\tau < 0$ for some $\ell \in J$.

Constraint qualification 3 (SD). (Separation direction) Assume the mixed affine structure in the sense that $J_N = \emptyset$ and $I = I_A \cup I_N$, where for each $j \in J$,

$$f_j(\tau, z) = \langle a_j(\tau), z\rangle + b_j(\tau), \quad g_j(\tau, z) = \langle c_j(\tau), z\rangle + d_j(\tau),$$

and for each $i \in I_A$,

$$h_i(\tau, z) = \langle p_i(\tau), z\rangle + q_i(\tau),$$

for a.e. $\tau \in \Delta$ and all $z \in \mathbb{R}^n$. Let $\hat{w}$ be defined as above. We say that (SD) holds if there exists a function $d(\cdot) \in L_\infty(\Delta; \mathbb{R}^n)$ such that for a.e. $\tau \in \Delta$,

(SD) $\quad \langle a_j(\tau) - \hat{w}_j c_j(\tau),\, d(\tau)\rangle < 0 \quad \forall j \in J, \qquad \langle p_i(\tau),\, d(\tau)\rangle \leqslant 0 \;\; \forall i \in I_A.$

3. Optimality conditions

In vector fractional programming, there is a well-established link between the solutions of a constrained programming problem and the points that fulfill the so-called saddle point optimality conditions. In this section, we broaden these findings by applying them to vector fractional continuous-time programming under convexity assumptions. To achieve this, we first introduce novel definitions for saddle points within a continuous-time framework. For fixed $\hat{w} \in \mathbb{R}^k_+$, we define the Lagrange-type function

$$\mathcal{L}_{\hat{w}} : L_\infty(\Delta; \mathbb{R}^n) \times \Lambda^+ \times L_\infty(\Delta; \mathbb{R}^m) \to \mathbb{R}$$

with respect to Problem (VFCTP) as

$$\mathcal{L}_{\hat{w}}(z(\cdot), \lambda, v(\cdot)) = \lambda' \int_\Delta (f(\tau, z(\tau)) - \hat{w} \circ g(\tau, z(\tau)))d\tau + \int_\Delta v'(\tau)h(\tau, z(\tau))d\tau,$$

where $\circ$ denotes Hadamard product of vectors.

Definition 3.1. A point $(\hat{z}(\cdot), \hat{\lambda}, \hat{v}(\cdot)) \in L_\infty(\Delta; \mathbb{R}^n) \times \Lambda^+ \times V$ is said to be a KKTSP (a Karush-Kuhn-Tucker saddle point) for (VFCTP) if $\hat{v}(\tau) \geqq 0$ a.e. in Δ and

$$\mathcal{L}_{\hat{w}}(\hat{z}(\cdot), \hat{\lambda}, v(\cdot)) \leqq \mathcal{L}_{\hat{w}}(\hat{z}(\cdot), \hat{\lambda}, \hat{v}(\cdot)) \leqq \mathcal{L}_{\hat{w}}(z(\cdot), \hat{\lambda}, \hat{v}(\cdot)), \tag{3.1}$$

for all $z(\cdot) \in L_\infty(\Delta;\mathbb{R}^n)$ and all $v(\cdot) \in V$ where

$$\hat{w} = \frac{\int_\Delta f(\tau,\hat{z}(\tau))d\tau}{\int_\Delta g(\tau,\hat{z}(\tau))d\tau} \in \mathbb{R}^k_+.$$

Let $\hat{z}(\cdot) \in \Omega_P$ be a proper efficient solution for (VFCTP). The following system will be referred to in the next theorem:

$$\begin{aligned}
\chi_l(\tau,z) :=& \int_\Delta (f_l(\tau,z) - \hat{w}_l g_l(\tau,z))d\tau < 0,\\
\chi_j(\tau,z) :=& \int_\Delta (f_l(\tau,z) - \hat{w}_l g_l(\tau,z))d\tau + M\int_\Delta (f_j(\tau,z) - \hat{w}_j g_j(\tau,z))d\tau < 0, j \neq l, j \in J,\\
\chi_i(\tau,z) :=& h_i(\tau,z) \leqq 0, \quad i \in I,\\
& z \in \mathbb{R}^n,\ \hat{w} \in \mathbb{R}^k_+,\ M > 0.
\end{aligned}$$

Let $L = I \sqcup J$ and

$$\mathcal{I}(\tau,z) = \Big\{p \in L : \chi_p(\tau,z) = \max_{k\in L}\{\chi_k(\tau,z)\}\Big\}, \quad \tau \in \Delta, \quad z \in \mathbb{R}^n.$$

Definition 3.2. [29] The regularity condition (RC) holds, if there exist $\bar{z}(\cdot) \in L_\infty(\Delta;\mathbb{R}^n)$, reals $R \geqq 0$ and $\alpha > 0$ such that for a.e. $\tau \in \Delta$ and for all $z \in \mathbb{R}^n$ with $\|z - \bar{z}(\tau)\| \geqq R$, there exists $e = e(\tau,z) \in \mathbb{R}^n$ with $\|e\| = 1$, satisfying

$$\text{(RC)} \qquad \langle \partial_z \chi_p(\tau,z), e\rangle \geqq \alpha \quad \forall p \in \mathcal{I}(\tau,z).$$

The following lemma plays a key role in proving the main result in this section.

Lemma 3.1. *If $\hat{z}(\cdot)$ is a proper efficient solution of (VFCTP) and $f_j(\tau,\cdot)$, $h_i(\tau,\cdot)$, $j \in J$, $i \in I$ are convex on $\mathbb{R}^n$ for a.e. $\tau \in \Delta$ and $g_j(\tau,\cdot)$, $j \in J$ is concave on $\mathbb{R}^n$ for a.e. $\tau \in \Delta$. Then for each $j \in J$, the following system*

$$\begin{aligned}
&\int_\Delta (f_l(\tau,z(\tau)) - \hat{w}_l g_l(\tau,z(\tau)))d\tau < 0,\\
&\int_\Delta (f_l(\tau,z(\tau)) - \hat{w}_l g_l(\tau,z(\tau)))d\tau\\
&\qquad + M\int_\Delta (f_j(\tau,z(\tau)) - \hat{w}_j g_j(\tau,z(\tau)))d\tau < 0, \quad j \neq l,\ j \in J,
\end{aligned} \tag{3.2}$$

has no solution in Ω_P, where M is constant of the definition of proper efficient solution.

Proof. Suppose that $\hat{z}(\cdot)$ is a proper efficient solution for (VFCTP). Then from [30], $\hat{z}(\cdot)$ is a proper efficient solution for (VFCTP) if and only if $\hat{z}(\cdot)$ is a proper efficient solution for (VCTP)

$$\text{(VCTP)} \quad \min_{z(\cdot)\in L_\infty(\Delta;\mathbb{R}^n)} \int_\Delta \phi(\tau,z(\tau))d\tau = \left(\int_\Delta \phi_1(\tau,z(\tau))d\tau,\ldots,\int_\Delta \phi_k(\tau,z(\tau))d\tau\right)$$
$$\text{subject to } z(\cdot) \in \Omega_P,$$

where $\phi_j(\tau,z(\tau)) = f_j(\tau,z(\tau)) - \hat{w}_j g_j(\tau,z(\tau))$, $j \in J$ and $\hat{w} = \frac{\int_\Delta f(\tau,\hat{z}(\tau))d\tau}{\int_\Delta g(\tau,\hat{z}(\tau))d\tau}$.
We will suppose that the statement of the Lemma is not true. Let the system (3.2)

be consistent, provided $z(\cdot)$ is a proper efficient solution, it follows

$$\frac{\int_\Delta (f_l(\tau,\hat{z}(\tau)) - \hat{w}_l g_l(\tau,\hat{z}(\tau)))d\tau - \int_\Delta (f_l(\tau,z(\tau)) - \hat{w}_l g_l(\tau,z(\tau)))d\tau}{\int_\Delta (f_j(\tau,z(\tau)) - \hat{w}_j g_j(\tau,z(\tau)))d\tau - \int_\Delta (f_j(\tau,\hat{z}(\tau)) - \hat{w}_j g_j(\tau,\hat{z}(\tau)))d\tau} \leqq M$$

for some j such that

$$\int_\Delta (f_j(\tau,z(\tau)) - \hat{w}_j g_j(\tau,z(\tau)))d\tau > \int_\Delta (f_j(\tau,\hat{z}(\tau)) - \hat{w}_j g_j(\tau,\hat{z}(\tau)))d\tau = 0,$$

i.e.,

$$\frac{-\int_\Delta (f_l(\tau,z(\tau)) - \hat{w}_l g_l(\tau,z(\tau)))d\tau}{\int_\Delta (f_j(\tau,z(\tau)) - \hat{w}_j g_j(\tau,z(\tau)))d\tau} \leqq M,$$

for some j such that

$$\int_\Delta (f_j(\tau,z(\tau)) - \hat{w}_j g_j(\tau,z(\tau)))d\tau > 0.$$

The set

$$\left\{ j : \int_\Delta (f_j(\tau,z(\tau)) - \hat{w}_j g_j(\tau,z(\tau)))d\tau > 0 \right\} \neq \emptyset,$$

because otherwise

$$\int_\Delta (f_j(\tau,z(\tau)) - \hat{w}_j g_j(\tau,z(\tau)))d\tau \leqq 0 \quad \forall j \in J,$$

which would imply that $\hat{z}(\cdot)$ is not an efficient solution, since

$$\int_\Delta (f_l(\tau,z(\tau)) - \hat{w}_l g_l(\tau,z(\tau)))d\tau < 0 \implies \int_\Delta (f(\tau,z(\tau)) - \hat{w} \circ g(\tau,z(\tau)))d\tau \neq 0.$$

Hence,

$$\int_\Delta (f_l(\tau,z(\tau)) - \hat{w}_l g_l(\tau,z(\tau)))d\tau + M \int_\Delta (f_j(\tau,z(\tau)) - \hat{w}_j g_j(\tau,z(\tau)))d\tau \geqq 0,$$

contradicting (3.2). Thus, the proof is complete. □

The transposition theorem plays a key role in deriving optimality conditions for a wide range of extremal problems. In [29], Aryutunov et al. established a transposition theorem for convex systems of inequalities, and we will rely on their result in this section as well. To apply the theorem effectively, a suitable regularity condition must hold. The results that follow then provide necessary and sufficient conditions for saddle-point optimality.

THEOREM 3.1. *Let $\hat{z}(\cdot)$ be a proper efficient solution of* (VFCTP). *Assume that* (SQ) *and* (RC) *are satisfied. Then there exists $(\hat{\lambda}, \hat{v}(\cdot)) \in \Lambda^+ \times V$ such that*

$$\hat{v}'(\tau)h(\tau,\hat{z}(\tau)) = 0 \quad \text{a.e. in } \Delta \tag{3.3}$$

and $(\hat{z}(\cdot), \hat{\lambda}, \hat{v}(\cdot))$ is a KKTSP for (VFCTP).

PROOF. Let $l \in J$ be fixed. Since $\hat{z}$ is a proper efficient solution of (VFCTP), then by Lemma 3.1, we obtain that there is no $z(\cdot) \in L_\infty(\Delta; \mathbb{R}^n)$ such that the following system is consistent

$$\int_\Delta (f_l(\tau, z(\tau)) - \hat{w}_l g_l(\tau, z(\tau)))d\tau < 0,$$
$$\int_\Delta (f_l(\tau, z(\tau)) - \hat{w}_l g_l(\tau, z(\tau)))d\tau + M \int_\Delta (f_j(\tau, z(\tau)) - \hat{w}_j g_j(\tau, z(\tau)))d\tau < 0, j \neq l, j \in J,$$
$$h(\tau, z(\tau)) \leqq 0 \quad \text{a.e. in } \Delta.$$

It is clear that all assumptions of Theorem 1 [29] are satisfied. The solvability condition is implied by (SQ). Therefore, there exists a nonzero function $(\hat{\varphi}^l(\cdot), \hat{\psi}^l(\cdot), \hat{u}^l(\cdot)) \in L_\infty(\Delta; \mathbb{R}) \times L_\infty(\Delta; \mathbb{R}^{k-1}) \times L_\infty(\Delta; \mathbb{R}^m)$, with $\hat{\varphi}^l(\tau) \geqq 0, \hat{\psi}^l(\tau) \geqq 0, \hat{u}^l(\tau) \geqq 0$ a.e. in Δ, and $(\hat{\varphi}^l(\tau), \hat{\psi}^l(\tau)) \ngeqq 0$, such that

$$\hat{\varphi}^l(\tau)\chi_l(\tau, z(\tau)) + \sum_{\substack{j=1 \\ j \neq i}}^{k} \hat{\psi}^l_j(\tau)\chi_j(\tau, z(\tau)) + \sum_{i=1}^{m} \hat{u}^l_i(\tau) h_i(\tau, z(\tau)) \geqq 0,$$

$\forall z(\cdot) \in L_\infty(\Delta; \mathbb{R}^n)$ a.e. in Δ, i.e.,

$$\begin{aligned} \text{(3.4)} \quad & \hat{\varphi}^l(\tau) \int_\Delta (f_l(s, z(s)) - \hat{w}_l g_l(s, z(s)))ds \\ & + \sum_{\substack{j=1 \\ j \neq i}}^{k} \hat{\psi}^l_j(\tau) \Big(\int_\Delta (f_l(s, z(s)) - \hat{w}_l g_l(s, z(s)))ds + M \int_\Delta (f_j(s, z(s)) - \hat{w}_j g_j(s, z(s)))ds \Big) \\ & + \sum_{i=1}^{m} \hat{u}^l_i(\tau) h_i(\tau, z(\tau)) \geqq 0, \forall z(\cdot) \in L_\infty(\Delta; \mathbb{R}^n) \text{ a.e. in } \Delta. \end{aligned}$$

Putting $z(\cdot) = \hat{z}(\cdot)$ we obtain $\sum_{i=1}^m \hat{u}^l_i(\tau) h_i(\tau, \hat{z}(\tau)) \geqq 0$ a.e. in Δ. Since the point $\hat{z}(\cdot) \in \Omega_P$ and $\hat{u}^l(\tau) \geqq 0$ a.e. in Δ, we have that the opposite inequality is also satisfied. From the fact it follows

$$\hat{u}^l(\tau)' h(\tau, \hat{z}(\tau)) = \sum_{i=1}^{m} \hat{u}^l_i(\tau) h_i(\tau, \hat{z}(\tau)) = 0 \quad \text{a.e. in } \Delta.$$

Furthermore, integrating inequality (3.4) on Δ we have

$$\begin{aligned} & \hat{\varphi}^l \int_\Delta (f_l(\tau, z(\tau)) - \hat{w}_l g_l(\tau, z(\tau)))\, d\tau \\ & + \sum_{\substack{j=1 \\ j \neq i}}^{k} \hat{\psi}^l_j \Big(\int_\Delta (f_l(\tau, z(\tau)) - \hat{w}_l g_l(\tau, z(\tau)))d\tau + M \int_\Delta (f_j(\tau, z(\tau)) - \hat{w}_j g_j(\tau, z(\tau)))d\tau \Big) \\ & + \int_\Delta \hat{u}^l(\tau)' h(\tau, z(\tau)) d\tau \geqq 0, \forall z(\cdot) \in L_\infty(\Delta; \mathbb{R}^n), \end{aligned}$$

where $\hat{\psi}^l_j = \int_\Delta \hat{\psi}^l_j(\tau) d\tau$, $\hat{\varphi}^l = \int_\Delta \hat{\varphi}^l(\tau) d\tau$, and according to the assumption $(\hat{\varphi}^l(\tau), \hat{\psi}^j(\tau)) \ngeqq 0$, the value of least one integral above is strictly positive, i.e., $(\hat{\varphi}^l, \hat{\psi}^j) \neq 0$.

Further, we obtain,

$$
\begin{aligned}
(3.5)\quad & \hat{\varphi}^l \left(\int_\Delta f_l(\tau, z(\tau))d\tau - \frac{\int_\Delta f_l(\tau, \hat{z}(\tau))d\tau}{\int_\Delta g_l(\tau, \hat{z}(\tau))d\tau} \int_\Delta g_l(\tau, z(\tau))d\tau \right) \\
& + \sum_{\substack{j=1 \\ j\neq i}}^{k} \hat{\psi}_j^l \Bigg[\left(\int_\Delta f_l(\tau, z(\tau))d\tau - \frac{\int_\Delta f_l(\tau, \hat{z}(\tau))d\tau}{\int_\Delta g_l(\tau, \hat{z}(\tau))d\tau} \int_\Delta g_l(\tau, z(\tau))d\tau \right) \\
& + M \left(\int_\Delta f_j(\tau, z(\tau))d\tau - \frac{\int_\Delta f_j(\tau, \hat{z}(\tau))d\tau}{\int_\Delta g_j(\tau, \hat{z}(\tau))d\tau} \int_\Delta g_j(\tau, z(\tau))d\tau \right) \Bigg] \\
& + \int_\Delta \hat{u}^l(\tau)' h(\tau, z(\tau))d\tau \geqq 0, \ \forall z(\cdot) \in L_\infty(\Delta; \mathbb{R}^n).
\end{aligned}
$$

Setting $\hat{\lambda} = (\hat{\lambda}_1, \hat{\lambda}_2, \dots, \hat{\lambda}_k)$, $\hat{v}(\tau) = (\hat{v}_1(\tau), \dots, \hat{v}_m(\tau))$, where

$$
\hat{\lambda}_l = \frac{\hat{\varphi}^l + \sum_{\substack{j=1 \\ j\neq l}}^{k} \hat{\psi}_l^j}{\sum_{l=1}^{k} \left(\hat{\varphi}^l + \sum_{\substack{j=1 \\ j\neq l}}^{k} \hat{\psi}_l^j \right)} > 0,
$$

$$
\hat{v}_i(\tau) = \frac{\sum_{l=1}^{k} \hat{u}_i^l(\tau)}{\sum_{l=1}^{k} \left(\hat{\varphi}^l + \sum_{\substack{j=1 \\ j\neq l}}^{k} \hat{\psi}_l^j \right)} \geqq 0, \quad l \in J, \ i \in I, \ \text{a.e. in } \Delta
$$

and summing inequality (3.5) for $l \in J$, we have that for such $\hat{\lambda}$ and $\hat{v}(\tau)$ the conditions

$$
\hat{\lambda}' e = 1, \ \hat{\lambda} > 0, \tag{3.6}
$$

$$
\hat{v}(\tau) \geqq 0, \quad \text{a.e. in } \Delta, \tag{3.7}
$$

$$
\hat{v}'(\tau) h(\tau, \hat{z}(\tau)) = 0, \quad \text{a.e. in } \Delta \ \text{ and} \tag{3.8}
$$

$$
\begin{aligned}
(3.9)\quad & \hat{\lambda}' \int_\Delta \left(f(\tau, z(\tau)) - \left(\frac{\int_\Delta f(\tau, z(\tau))d\tau}{\int_\Delta g(\tau, z(\tau))d\tau} \right) \circ g(\tau, z(\tau)) \right) d\tau + \int_\Delta \hat{v}'(\tau) h(\tau, z(\tau))d\tau \\
& \geqq \hat{\lambda}' \int_\Delta \left(f(\tau, \hat{z}(\tau)) - \left(\frac{\int_\Delta f(\tau, z(\tau))d\tau}{\int_\Delta g(\tau, z(\tau))d\tau} \right) \circ g(\tau, \hat{z}(\tau)) \right) d\tau + \int_\Delta \hat{v}'(\tau) h(\tau, \hat{z}(\tau))d\tau
\end{aligned}
$$

are satisfied.

From (3.6)–(3.9) we have

$$
\begin{aligned}
& \hat{\lambda}' \int_\Delta \left(f(\tau, z(\tau)) - \left(\frac{\int_\Delta f(\tau, \hat{z}(\tau))d\tau}{\int_\Delta g(\tau, \hat{z}(\tau))d\tau} \right) \circ g(\tau, z(\tau)) \right) d\tau + \int_\Delta \hat{v}'(\tau) h(\tau, z(\tau))d\tau \\
& \geqq \hat{\lambda}' \int_\Delta \left(f(\tau, \hat{z}(\tau)) - \left(\frac{\int_\Delta f(\tau, \hat{z}(\tau))d\tau}{\int_\Delta g(\tau, \hat{z}(\tau))d\tau} \right) \circ g(\tau, \hat{z}(\tau)) \right) d\tau + \int_\Delta \hat{v}'(\tau) h(\tau, \hat{z}(\tau))d\tau \\
& = 0 \geqq \hat{\lambda}' \int_\Delta \left(f(\tau, \hat{z}(\tau)) - \left(\frac{\int_\Delta f(\tau, \hat{z}(\tau))d\tau}{\int_\Delta g(\tau, \hat{z}(\tau))d\tau} \right) \circ g(\tau, \hat{z}(\tau)) \right) d\tau \\
& + \int_\Delta v'(\tau) h(\tau, \hat{z}(\tau))d\tau, \ \forall z(\cdot) \in L_\infty(\Delta; \mathbb{R}^n), \ \forall v(\cdot) \in V.
\end{aligned}
$$

Therefore, $(\hat{z}(\cdot), \hat{\lambda}, \hat{v}(\cdot)) \in L_\infty(\Delta;\mathbb{R}^n) \times \Lambda^+ \times V$ is a KKTSP of (VFCTP) with complementary slackness condition (3.3). Thus, the proof is complete. □

From the proof of preceding Theorem we obtain the following necessary conditions for (VFCTP).

THEOREM 3.2. *Let $\hat{z}(\cdot)$ be a proper efficient solution of (VFCTP). Assume that (RC) and (SQ) are satisfied. Then, there exist the multipliers $(\hat{\lambda}, \hat{v}(\cdot)) \in \mathbb{R}^k \times L_\infty(\Delta;\mathbb{R}^m)$, such that the following conditions are satisfied:*

(1) $\hat{\lambda} \in \Lambda^+,\ \hat{v}(\cdot) \in V$,

(2) $\hat{v}'(\tau)h(\tau, \hat{z}(\tau)) = 0$ a.e. in Δ,

(3) $$\hat{\lambda}' \int_\Delta \left(f(\tau, z(\tau)) - \left(\frac{\int_\Delta f(\tau, \hat{z}(\tau))d\tau}{\int_\Delta g(\tau, \hat{z}(\tau))d\tau} \right) \circ g(\tau, z(\tau)) \right) d\tau + \int_\Delta \hat{v}'(\tau)h(\tau, z(\tau))d\tau \geqq 0, \quad \forall z(\cdot) \in L_\infty(\Delta;\mathbb{R}^n).$$

The assertions of Theorem 3.1 are also sufficient for the proper efficiency of the function $\hat{z}(\cdot)$.

THEOREM 3.3. *Let $(\hat{z}(\cdot), \hat{\lambda}, \hat{v}(\cdot)) \in L_\infty(\Delta;\mathbb{R}^n) \times \Lambda^+ \times V$ be a KKTSP of (VFCTP). Then $\hat{z}(\cdot)$ is a proper efficient solution of (VFCTP).*

PROOF. If we put $v \equiv 0$ in (3.1), we have

$$\hat{\lambda}' \int_\Delta (f(\tau, z(\tau)) - \hat{w} \circ g(\tau, z(\tau)))d\tau + \int_\Delta \hat{v}'(\tau)h(\tau, z(\tau))d\tau \geqq 0, \quad \forall z(\cdot) \in L_\infty(\Delta;\mathbb{R}^n).$$

Particularly for all $z(\cdot) \in \Omega_P$. In such case,

$$\hat{\lambda}' \int_\Delta (f(\tau, z(\tau)) - \hat{w} \circ g(\tau, z(\tau)))d\tau + \int_\Delta \hat{v}'(\tau)h(\tau, z(\tau))d\tau \leqq \hat{\lambda}' \int_\Delta (f(\tau, z(\tau)) - \hat{w} \circ g(\tau, z(\tau)))d\tau,$$

so that

$$\hat{\lambda}' \int_\Delta (f(\tau, z(\tau)) - \hat{w} \circ g(\tau, z(\tau)))d\tau \geqq 0, \quad z(\cdot) \in \Omega_P. \tag{3.10}$$

In the following, the proof will be carried out in two steps.

Step 1: Now, suppose contrary, that $\hat{z}(\cdot)$ is not an efficient solution for (VFCTP). Then there exists $\bar{z}(\cdot) \in \Omega_P$ such that

$$\hat{\lambda}_j \frac{\int_\Delta f_j(t, \bar{z}(\tau))d\tau}{\int_\Delta g_j(t, \bar{z}(\tau))d\tau} \leqq \hat{\lambda}_j \frac{\int_\Delta f_j(\tau, \hat{z}(\tau))d\tau}{\int_\Delta g_j(\tau, \hat{z}(\tau))d\tau} = \hat{\lambda}_j \hat{w}_j \quad \text{for all } j \in J,$$

and

$$\hat{\lambda}_i \frac{\int_\Delta f_i(t, \bar{z}(\tau))d\tau}{\int_\Delta g_i(t, \bar{z}(\tau))d\tau} < \hat{\lambda}_i \frac{\int_\Delta f_i(\tau, \hat{z}(\tau))d\tau}{\int_\Delta g_i(\tau, \hat{z}(\tau))d\tau} = \hat{\lambda}_i \hat{w}_i \quad \text{for some } i \in J,$$

i.e.,

$$\hat{\lambda}_j \int_\Delta (f_j(\tau, \bar{z}(\tau)) - \hat{w}_j g_j(\tau, \bar{z}(\tau)))d\tau \leqq 0 \quad \text{for all } j \in J, \tag{3.11}$$

and

$$\hat{\lambda}_i \int_\Delta (f_i(\tau, \bar{z}(\tau)) - \hat{w}_i g_i(\tau, \bar{z}(\tau)))d\tau < 0 \quad \text{for some } i \in J. \tag{3.12}$$

Since $\hat{\lambda} \in \Lambda^+$, (3.11) and (3.12) imply

$$\sum_{j \in J} \hat{\lambda}_j \int_\Delta (f_j(\tau, \bar{z}(\tau)) - \hat{w}_j g_j(\tau, \bar{z}(\tau)))d\tau = \hat{\lambda}' \int_\Delta (f(\tau, \bar{z}(\tau)) - \hat{w} \circ g(\tau, \bar{z}(\tau)))d\tau < 0.$$

This inequality contradicts (3.10). Thus, $\hat{z}(\cdot)$ must be an efficient solution for (VFCTP). The proof of Step 1 is complete.

Step 2 : Assume that $\hat{z}(\cdot)$ is no proper efficient solution for (VFCTP). We know that $\hat{z}(\cdot)$ is no proper efficient solution for (VCTP). Let $\mid J \mid = k$ be the cardinality of J. Then we choose

$$M = (k-1) \max_{i,j \in J} \frac{\hat{\lambda}_j}{\hat{\lambda}_i}, \qquad \text{for } k \geqslant 2,$$

and we have for some $i \in J$ and some $z(\cdot) \in \Omega_P$ with

$$\int_\Delta (f_i(\tau, z(\tau)) - \hat{w}_i g_i(\tau, z(\tau)))d\tau < 0,$$

$$\frac{\int_\Delta (f_i(\tau, \hat{z}(\tau)) - \hat{w}_i g_i(\tau, \hat{z}(\tau)))d\tau - \int_\Delta (f_i(\tau, z(\tau)) - \hat{w}_i g_i(\tau, z(\tau)))d\tau}{\int_\Delta (f_j(\tau, z(\tau)) - \hat{w}_j g_j(\tau, z(\tau)))d\tau - \int_\Delta (f_j(\tau, \hat{z}(\tau)) - \hat{w}_j g_j(\tau, \hat{z}(\tau)))d\tau} > M$$

for all $j \in J$, with

$$\int_\Delta (f_j(\tau, z(\tau)) - \hat{w}_j g_j(\tau, z(\tau)))d\tau > \int_\Delta (f_j(\tau, \hat{z}(\tau)) - \hat{w}_j g_j(\tau, \hat{z}(\tau)))d\tau = 0.$$

It follows that

$$\begin{aligned} -\int_\Delta (f_i(\tau, z(\tau)) - \hat{w}_i g_i(\tau, z(\tau)))d\tau &> M \int_\Delta (f_j(\tau, z(\tau)) - \hat{w}_j g_j(\tau, z(\tau)))d\tau \\ &\geqq (k-1) \frac{\hat{\lambda}_j}{\hat{\lambda}_i} \int_\Delta (f_j(\tau, z(\tau)) - \hat{w}_j g_j(\tau, z(\tau)))d\tau, \quad \forall j \neq i \end{aligned} \tag{3.13}$$

i.e.,

$$-\frac{\hat{\lambda}_i}{k-1} \int_\Delta (f_i(\tau, z(\tau)) - \hat{w}_i g_i(\tau, z(\tau)))d\tau > \hat{\lambda}_j \int_\Delta (f_j(\tau, z(\tau)) - \hat{w}_j g_j(\tau, z(\tau)))d\tau, \quad \forall j \neq i.$$

Summing over $j \neq i$, we have

$$-\hat{\lambda}_i \int_\Delta (f_i(\tau, z(\tau)) - \hat{w}_i g_i(\tau, z(\tau)))d\tau > \sum_{\substack{j=1 \\ j \neq i}}^{k} \hat{\lambda}_j \int_\Delta (f_j(\tau, z(\tau)) - \hat{w}_j g_j(\tau, z(\tau)))d\tau,$$

so that

$$\sum_{j\in J}\hat{\lambda}_j\int_{\Delta}(f_j(\tau,z(\tau))-\hat{w}_j g_j(\tau,z(\tau)))d\tau=\hat{\lambda}'\int_{\Delta}(f(\tau,z(\tau))-\hat{w}\circ g(\tau,z(\tau)))d\tau<0,$$

which also contradicts (3.10). Thus, the proof of Step 2 is complete. Therefore, $\hat{z}(\cdot)$ is a proper efficient solution of (VFCTP). □

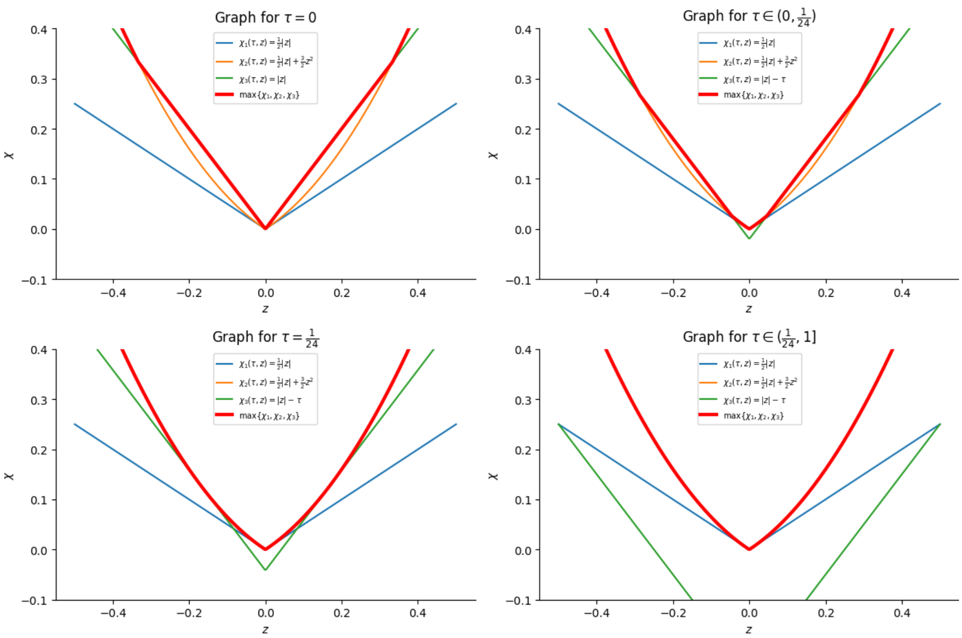
As an illustration, we will consider the following example. Let $\Delta=[0,1]$.

Example 3.1.

$$\text{(VPFCTP)}\qquad \min_{z(\cdot)\in L_\infty([0,1];\mathbb{R})}\left(\frac{\int_0^1 f_1(\tau,z(\tau))d\tau}{\int_0^1 g_1(\tau,z(\tau))d\tau},\frac{\int_0^1 f_2(\tau,z(\tau)),d\tau}{\int_0^1 g_2(\tau,z(\tau))d\tau}\right)$$
$$\text{s.t. } h_1(\tau,z(\tau))\leqq 0\quad \text{a.e. in } [0,1],$$

where $f_1(\tau,z):=\tau|z|$, $f_2(\tau,z):=z^2+2\tau$, $g_1(\tau,z):=\ln(z+2\tau+3)$, $g_2(\tau,z):=4\tau-z^2$, $h_1(\tau,z):=|z|-\tau$. It can be easily verified that $\hat{z}(\tau)=0$ for a.e. $\tau\in[0,1]$ is a proper efficient solution of preceding problem and $\hat{w}=(\hat{w}_1,\hat{w}_2)=\left(0,\frac{1}{2}\right)$. We have that, for almost every $\tau\in[0,1]$, Slater's condition (SQ) is satisfied for $s(\tau)=\tau-\frac{1}{2}$. For $M=1$, it is obvious that $\chi_1(\tau,z)=\frac{1}{2}|z|$, $\chi_2(\tau,z)=\frac{1}{2}|z|+\frac{3}{2}z^2$, $\chi_3(\tau,z)=|z|-\tau$ and

$$\partial_z\chi_1=\begin{cases}\{-\frac{1}{2}\}, & z<0,\\ [-\frac{1}{2},\frac{1}{2}], & z=0,\\ \{\frac{1}{2}\}, & z>0,\end{cases}\quad \partial_z\chi_2=\begin{cases}\{-\frac{1}{2}+3z\}, & z<0,\\ [-\frac{1}{2},\frac{1}{2}], & z=0,\\ \{\frac{1}{2}+3z\}, & z>0,\end{cases}\quad \partial_z\chi_3=\begin{cases}\{-1\}, & z<0,\\ [-1,1], & z=0,\\ \{1\}, & z>0.\end{cases}$$



In the following we will determine the set $\mathcal{I}(\tau,z)=\{i:\chi_i(\tau,z)=\max_{1\leqq j\leqq 3}\chi_j(\tau,z)\}$.

Take $\bar{z}(\tau) \equiv 0$, $R = \alpha = \frac{1}{2}$. For $z \in \mathbb{R} \setminus \left(-\frac{1}{2}, \frac{1}{2}\right)$ we have $\mathcal{I}(\tau, z) = \{2\}$. For almost every $\tau \in [0,1]$, the regularity of the system

$$\chi_1(\tau, z) < 0, \quad \chi_2(\tau, z) < 0, \quad \chi_3(\tau, z) \leqq 0, \qquad z \in \mathbb{R}$$

is verified with $e_1 = e_1(\tau, z) = 1$ for $z \geqq \frac{1}{2}$ and $e_2 = e_2(\tau, z) = -1$ for $z \leqq -\frac{1}{2}$. Indeed, we have

- $\langle \partial_z \chi_2(\tau, z), e_1 \rangle \geqq \alpha$ for $z \geqq \frac{1}{2}$,
- $\langle \partial_z \chi_2(\tau, z), e_2 \rangle \geqq \alpha$ for $z \leqq -\frac{1}{2}$.

Here $J = \{1, 2\}$, $I = \{1\}$. Define the multipliers $\hat{\lambda} = (\hat{\lambda}_1,\ \hat{\lambda}_2) = \left(\frac{1}{2}, \frac{1}{2}\right) \in \Lambda^+$, $\hat{v}(\tau) = 0$ a.e. in $[0,1]$. Then:

(1) This is obvious.
(2) Complementarity slackness condition holds:

$$\hat{v}(\tau)\, h_1(\tau, \hat{z}(\tau)) = 0 \cdot (-\tau) = 0, \quad \text{a.e. in } [0,1].$$

(3) In the sequel, from inequality

$$\begin{aligned}
\mathcal{L}(z(\cdot),&(\hat{\lambda}_1, \hat{\lambda}_2), \hat{v}(\cdot)) \\
&= \int_0^1 \left[\underbrace{\hat{\lambda}_1}_{\frac{1}{2}} \cdot \tau |z(\tau)| + \underbrace{\hat{\lambda}_2}_{\frac{1}{2}} \cdot \left(z^2(\tau) + 2\tau - \underbrace{\hat{w}_2}_{\frac{1}{2}} \cdot (4\tau - z^2(\tau)) \right) \right] d\tau \\
&\quad + \int_0^1 \underbrace{\hat{v}(\tau)}_{0} \cdot (|z(\tau)| - \tau) d\tau = \int_0^1 \left(\frac{1}{2} \tau |(z(\tau)| + \frac{3}{4} z^2(\tau) \right) d\tau \\
&\geqq 0 = \mathcal{L}(\hat{z}(\cdot), (\hat{\lambda}_1, \hat{\lambda}_2), \hat{v}(\cdot)) \quad \forall z(\cdot) \in L_\infty([0,1]; \mathbb{R}),
\end{aligned}$$

it can be easily verified the condition (3) is satisfied.

Theorem 3.4. *Let $\hat{z}(\cdot)$ be a proper efficient solution of $(VFCTP)$ and let*

$$\hat{w} := \frac{\int_\Delta f(\tau, \hat{z}(\tau)) d\tau}{\int_\Delta g(\tau, \hat{z}(\tau)) d\tau} \in \mathbb{R}^k_+.$$

Assume that the mixed affine/convex structure holds, i.e., there exist index sets $J_A \subseteq \{1, \ldots, k\}$ and $I_A \subseteq \{1, \ldots, m\}$ such that, for each $j \in J_A$,

$$f_j(\tau, z) = \langle a_j(\tau), z \rangle + b_j(\tau), \quad g_j(\tau, z) = \langle c_j(\tau), z \rangle + d_j(\tau),$$

and for each $i \in I_A$,

$$h_i(\tau, z) = \langle p_i(\tau), z \rangle + q_i(\tau),$$

for a.e. $\tau \in \Delta$ and all $z \in \mathbb{R}^n$. Let $J_N := \{1, \ldots, k\} \setminus J_A$ and $I_N := \{1, \ldots, m\} \setminus I_A$. Assume that (RC) and (SQ) hold. Then, there exist multipliers $(\hat{\lambda}, \hat{v}(\cdot)) \in \mathbb{R}^k \times L_\infty(\Delta; \mathbb{R}^m)$ such that the following conditions are satisfied:

(1) *$\hat{\lambda} \in \Lambda^+$ and $\hat{v}(\cdot) \in V$ (i.e., $\hat{v}(\tau) \geqq 0$ a.e. in Δ);*
(2) *$\hat{v}(\tau)' h(\tau, \hat{z}(\tau)) = 0$ for a.e. $\tau \in \Delta$;*

(3) *for all* $z(\cdot) \in L_\infty(\Delta; \mathbb{R}^n)$,

$$\int_\Delta \Bigg[\sum_{j\in J_A} \hat{\lambda}_j(\langle a_j(\tau) - \hat{w}_j c_j(\tau),\, z(\tau)\rangle + b_j(\tau) - \hat{w}_j d_j(\tau)) + \sum_{j\in J_N} \hat{\lambda}_j(f_j(\tau, z(\tau)) - \hat{w}_j g_j(\tau, z(\tau))) + \sum_{i\in I_A} \hat{v}_i(\tau)(\langle p_i(\tau), z(\tau)\rangle + q_i(\tau)) + \sum_{i\in I_N} \hat{v}_i(\tau)\, h_i(\tau, z(\tau)) \Bigg] d\tau \geqq 0.$$

3.1. Improved necessary optimality conditions for (VFCTP) under mixed affine structure and novel constraint qualification.

THEOREM 3.5 (Affine objective with mixed phase constraints: necessary conditions under (SC)). *Let* $\hat{z}(\cdot)$ *be a proper efficient solution of* (VFCTP) *and define*

$$\hat{w} := \frac{\int_\Delta f(\tau, \hat{z}(\tau))d\tau}{\int_\Delta g(\tau, \hat{z}(\tau))d\tau} \in \mathbb{R}^k_+.$$

Assume that $J_N = \emptyset$, *i.e., for each* $j \in J = J_A$,

$$f_j(\tau, z) = \langle a_j(\tau), z\rangle + b_j(\tau), \quad g_j(\tau, z) = \langle c_j(\tau), z\rangle + d_j(\tau),$$

and that the phase constraints are split as $I = I_A \cup I_N$, *where for each* $i \in I_A$,

$$h_i(\tau, z) = \langle p_i(\tau), z\rangle + q_i(\tau),$$

while for each $i \in I_N$ *the function* $h_i(\tau, \cdot)$ *is convex (not necessarily affine). Assume that* (RC) *and* (SQ) *hold. Then, there exist multipliers* $(\hat{\lambda}, \hat{v}(\cdot)) \in \mathbb{R}^k \times L_\infty(\Delta; \mathbb{R}^m)$ *such that*

(1) $\hat{\lambda} \in \Lambda_+$ *and* $\hat{v}(\cdot) \in V$;
(2) $\hat{v}(\tau)' h(\tau, \hat{z}(\tau)) = 0$ *for a.e.* $\tau \in \Delta$;
(3) *for all* $z(\cdot) \in L_\infty(\Delta; \mathbb{R}^n)$,

$$\int_\Delta \Bigg[\sum_{j\in J_A} \hat{\lambda}_j(\langle a_j(\tau) - \hat{w}_j c_j(\tau),\, z(\tau)\rangle + b_j(\tau) - \hat{w}_j d_j(\tau)) + \sum_{i\in I_A} \hat{v}_i(\tau)(\langle p_i(\tau), z(\tau)\rangle + q_i(\tau)) + \sum_{i\in I_N} \hat{v}_i(\tau)\, h_i(\tau, z(\tau)) \Bigg] d\tau \geqq 0.$$

Moreover, if the solvability requirement (SC) *holds, then necessarily*

(0) $\hat{v}_N(\cdot) \not\equiv 0$, *where* $\hat{v}_N(\cdot) := (\hat{v}_i(\cdot))_{i\in I_N}$.

PROOF OF CONDITION (0). Assume that (SQ) and (RC) hold and that there exist multipliers $(\hat{\lambda}, \hat{v}(\cdot)) \in \mathbb{R}^k \times L_\infty(\Delta; \mathbb{R}^m)$ satisfying (1)–(3) of Theorem 3.4. We prove that $\hat{v}_N(\cdot) \not\equiv 0$.

Step 0 (Contradiction hypothesis). Suppose to the contrary that

$$\hat{v}_N(\cdot) \equiv 0, \quad \text{i.e.,} \quad \hat{v}_i(\cdot) \equiv 0 \qquad \forall i \in I_N. \tag{3.14}$$

Then, condition (3) reduces to an inequality involving only the affine parts:

$$\int_\Delta \Bigg[\sum_{j \in J_A} \hat{\lambda}_j(\langle a_j(\tau) - \hat{w}_j c_j(\tau), z(\tau)\rangle + b_j(\tau) - \hat{w}_j d_j(\tau)) + \sum_{i \in I_A} \hat{v}_i(\tau)(\langle p_i(\tau), z(\tau)\rangle + q_i(\tau))\Bigg] d\tau \geqq 0, \quad \forall z(\cdot) \in L_\infty(\Delta; \mathbb{R}^n). \tag{3.15}$$

Step 1 (Affine decomposition). Define for a.e. $\tau \in \Delta$ and $z \in \mathbb{R}^n$ the affine integrand

$$\Psi(\tau, z) := \sum_{j \in J_A} \hat{\lambda}_j(\langle a_j(\tau) - \hat{w}_j c_j(\tau), z\rangle + b_j(\tau) - \hat{w}_j d_j(\tau)) + \sum_{i \in I_A} \hat{v}_i(\tau)(\langle p_i(\tau), z\rangle + q_i(\tau)).$$

Then $\Psi(\tau, \cdot)$ is affine in z and can be written as

$$\Psi(\tau, z) = \langle \eta(\tau), z\rangle + \beta(\tau), \tag{3.16}$$

where

$$\eta(\tau) := \sum_{j \in J_A} \hat{\lambda}_j(a_j(\tau) - \hat{w}_j c_j(\tau)) + \sum_{i \in I_A} \hat{v}_i(\tau)\, p_i(\tau),$$
$$\beta(\tau) := \sum_{j \in J_A} \hat{\lambda}_j(b_j(\tau) - \hat{w}_j d_j(\tau)) + \sum_{i \in I_A} \hat{v}_i(\tau)\, q_i(\tau).$$

With this notation, (3.15) becomes

$$\int_\Delta \Psi(\tau, z(\tau)) d\tau \geqq 0 \quad \forall z(\cdot) \in L_\infty(\Delta; \mathbb{R}^n). \tag{3.17}$$

Step 2 (Showing $\eta(\tau) = 0$ a.e. in Δ by the test functions z_t). We prove that

$$\eta(\tau) = 0 \quad \text{for a.e. } \tau \in \Delta. \tag{3.18}$$

Assume the contrary. Then, there exists a measurable set $E \subseteq \Delta$ with $\mu(E) > 0$ such that $\eta(\tau) \neq 0$ for all $\tau \in E$. To avoid any boundedness issue, define for each $M \in \mathbb{N}$ the measurable set $E_M := \{\tau \in E : \ \|\eta(\tau)\| \leqq M\}$. Since $E = \bigcup_{M=1}^\infty E_M$ and $\mu(E) > 0$, there exists $M_0 \in \mathbb{N}$ with $\mu(E_{M_0}) > 0$. Fix such an M_0 and set $F := E_{M_0}$. For $t > 0$, define $z_t(\cdot) \in L_\infty(\Delta; \mathbb{R}^n)$ by

$$z_t(\tau) := \begin{cases} -t\, \eta(\tau), & \tau \in F, \\ 0, & \tau \notin F. \end{cases}$$

Clearly $z_t(\cdot) \in L_\infty(\Delta; \mathbb{R}^n)$ because $\|\eta(\tau)\| \leqq M_0$ on F. Using (3.16) and (3.17) we obtain

$$0 \leqq \int_\Delta \Psi(\tau, z_t(\tau)) d\tau = \int_F \Big(\langle \eta(\tau), -t\, \eta(\tau)\rangle + \beta(\tau)\Big) d\tau + \int_{\Delta \setminus F} \beta(\tau) d\tau$$
$$= -t \int_F \|\eta(\tau)\|^2\, d\tau + \int_\Delta \beta(\tau) d\tau.$$

Since $\mu(F) > 0$ and $\eta(\tau) \neq 0$ on F, we have $\int_F \|\eta(\tau)\|^2 \, d\tau > 0$. Letting $t \to +\infty$ yields a contradiction. Hence (3.18) holds. Consequently, (3.16) reduces to $\Psi(\tau, z) = \beta(\tau)$ a.e., and (3.17) simplifies to

$$\int_\Delta \beta(\tau) d\tau \geqq 0.$$

Step 3 Contradiction with the solvability requirement (SC). Taking $\zeta(\cdot)$ from (SC) and substituting $z(\cdot) = \zeta(\cdot)$ into (3.17), we obtain

$$0 \leqq \sum_{j\in J_A} \hat{\lambda}_j \int_\Delta (f_j(\tau, \zeta(\tau)) - \hat{w}_j g_j(\tau, \zeta(\tau)))d\tau + \int_\Delta \sum_{i\in I_A} \hat{v}_i(\tau)\, h_i(\tau, \zeta(\tau))d\tau.$$

Since $\hat{v}_i(\tau) \geqq 0$ a.e. and $h_i(\tau, \tilde{z}(\tau)) \leqq 0$ a.e. for all $i \in I_A$, the second integral is less than or equal to zero. Therefore,

$$0 \leqq \sum_{j\in J_A} \hat{\lambda}_j \int_\Delta (f_j(\tau, \zeta(\tau)) - \hat{w}_j g_j(\tau, \zeta(\tau)))d\tau. \tag{3.19}$$

On the other hand, (SC) provides an index $\ell \in J = J_A$ such that

$$\int_\Delta (f_\ell(\tau, \zeta(\tau)) - \hat{w}_\ell g_\ell(\tau, \zeta(\tau)))d\tau < 0.$$

Because $\hat{\lambda} \in \Lambda_+$ (hence $\hat{\lambda}_\ell > 0$), the weighted sum in (3.19) contains a strictly negative term with a strictly positive weight, and all terms are finite. Consequently,

$$\sum_{j\in J_A} \hat{\lambda}_j \int_\Delta \big(f_j(\tau, \zeta(\tau)) - \hat{w}_j g_j(\tau, \zeta(\tau))\big)d\tau < 0,$$

which contradicts (3.19). Thus (3.14) is false and $\hat{v}_N(\cdot) \not\equiv 0$. This completes the proof. □

EXAMPLE 3.2. Let $\Delta = [0, 1]$, $n = 1$, $k = 2$, and $m = 2$. We consider the vector fractional continuous-time problem (VFCTP):

$$\min_{z(\cdot)\in L_\infty([0,1];\mathbb{R})} \left(\frac{\int_0^1 f_1(\tau, z(\tau))d\tau}{\int_0^1 g_1(\tau, z(\tau))d\tau}, \frac{\int_0^1 f_2(\tau, z(\tau))d\tau}{\int_0^1 g_2(\tau, z(\tau))d\tau} \right)$$
$$h_1(\tau, z(\tau)) \leqq 0, \qquad \text{a.e. on } [0, 1],$$
$$h_2(\tau, z(\tau)) \leqq 0, \qquad \text{a.e. on } [0, 1],$$

where $h_1(\tau, z) = z - \frac{\tau}{2}$ (affine in z), $h_2(\tau, z) = z^2 - \frac{\tau^2}{16}$ (convex in z), $g_1(\tau, z) = g_2(\tau, z) = 2 + \tau$, $f_1(\tau, z) = (2 + \tau) \cdot 1 + \frac{\tau}{2} z + \frac{\tau^2}{8}$, $f_2(\tau, z) = (2 + \tau) \cdot 2 + \frac{\tau}{2} z + \frac{\tau^2}{8}$ (affine in z and depending on τ). Also, we have $I_A = \{1\}$ and $I_N = \{2\}$. From $h_2(\tau, z) \leqq 0$ we obtain $|z| \leqq \tau/4$, hence the feasible set is

$$\Omega = \Big\{ z(\cdot) \in L_\infty([0, 1]) : -\frac{\tau}{4} \leqq z(\tau) \leqq \frac{\tau}{4} \text{ a.e.} \Big\}.$$

Let $\hat{z}(\tau) := -\frac{\tau}{4}$ for a.e. $\tau \in [0, 1]$. Hence $\hat{w} = (1, 2) \in \mathbb{R}^2_+$. We have that, for almost every $\tau \in [0, 1]$, Slater's condition (SQ) is satisfied for $s(\tau) = 0$. Recall that the nonaffine constraint is $h_2(\tau, z) = \|z\|^2 - \tau^2/16$, $z \in \mathbb{R}^2$, which is convex in z

and has quadratic growth. In particular, for every $\tau \in [0,1]$ we have $\partial_z h_2(\tau, z) = \{2z\}$. It is easy to check that condition (RC) holds. Recall that (SC) requires a function $\zeta(\cdot)$ satisfying only the affine constraints (i.e., for $i \in I_A$) and such that $\int_0^1 \big(f_\ell - \hat{w}_\ell g_\ell\big) < 0$ for some ℓ. Choose $\zeta(\tau) = -\tau$. Then $h_1(\tau, \zeta(\tau)) = -3\tau/2 \leqq 0$ a.e., and

$$\int_0^1 (f_1(\tau, \zeta(\tau)) - \hat{w}_1 g_1(\tau, \zeta(\tau)))d\tau = \int_0^1 \Big(\frac{\tau}{2}(-\tau) + \frac{\tau^2}{8}\Big)d\tau = \int_0^1 \Big(-\frac{3}{8}\tau^2\Big)d\tau < 0.$$

Therefore, (SC) holds (with $\ell = 1$). Define $\hat{\lambda} = \big(\frac{1}{2}, \frac{1}{2}\big) \in \Lambda_+$, $\hat{v}_1(\tau) \equiv 0$, $\hat{v}_2(\tau) \equiv 1$. Then, (1) and (2) hold by construction. For any $z \in \mathbb{R}$, we have $f_1(\tau, z) - \hat{w}_1 g_1(\tau, z) = \frac{\tau}{2}z + \frac{\tau^2}{8}$, $f_2(\tau, z) - \hat{w}_2 g_2(\tau, z) = \frac{\tau}{2}z + \frac{\tau^2}{8}$, so the integrand in (3) becomes nonnegative. Integrating over Δ yields (3). Moreover, $\hat{v}_N(\cdot) = \hat{v}_2(\cdot) \not\equiv 0$.

Example 3.3. Let $\Delta = [0,1]$, $n = 2$, $k = 2$, and $m = 2$. Let $z = (z_1, z_2) \in \mathbb{R}^2$. We impose the phase constraints (a.e. in Δ)

$$h_1(\tau, z) = z_1 - \frac{\tau}{2} \leqq 0 \quad (I_A = \{1\}), \qquad h_2(\tau, z) = \|z\|^2 - \frac{\tau^2}{16} \leqq 0 \quad (I_N = \{2\}).$$

Define common denominators $g_1(\tau, z) = g_2(\tau, z) = 2 + \tau$ and fix a parameter $\beta > 0$. Define $f_1(\tau, z) = (2+\tau)\cdot 1 + \frac{\tau}{2}z_1 + \frac{\tau^2}{8} + \beta\frac{\tau}{4}z_2$ and $f_2(\tau, z) = (2+\tau)\cdot 2 + \frac{\tau}{2}z_1 + \frac{\tau^2}{8} - \beta\frac{\tau}{4}z_2$. Thus, both objective components are affine in z (so $J_N = \emptyset$). It can be easily verified that $\hat{z}(\tau) = \big(-\frac{\tau}{4}, 0\big)$ for a.e. $\tau \in [0,1]$ is a proper efficient solution and $\hat{w}_1 = 1$, $\hat{w}_2 = 2$. Take $s(\tau) \equiv (0,0)$. Then $h_1(\tau, s(\tau)) < 0$ and $h_2(\tau, s(\tau)) < 0$ for $\tau \in (0,1]$, hence (SQ) holds. The nonaffine function $h_2(\tau, z) = \|z\|^2 - \tau^2/16$ is convex in z. Similarly, as in the previous example (RC) is obviously valid. Let $\zeta(\tau) = (-\tau, 0)$. We have $h_1(\tau, \zeta(\tau)) \leqq 0$ a.e. in $[0,1]$. Moreover, (SC) also holds (with $\ell = 1$). Let $\hat{\lambda} = \big(\frac{1}{2}, \frac{1}{2}\big)$, $\hat{v}_1(\tau) \equiv 0$ and $\hat{v}_2(\tau) \equiv 1$. Then (1) holds. Condition (2) holds since $h_2(\tau, \hat{z}(\tau)) = 0$ a.e. and $\hat{v}_1 \equiv 0$. Also, condition (3) obviously holds and $\hat{v}_N = \hat{v}_2 \not\equiv 0$.

Theorem 3.6 (Affine objective with mixed phase constraints: necessary conditions under (SD)). *Let $\hat{z}(\cdot)$ be a proper efficient solution of $(VFCTP)$ and define*

$$\hat{w} := \frac{\int_\Delta f(\tau, \hat{z}(\tau))d\tau}{\int_\Delta g(\tau, \hat{z}(\tau))d\tau} \in \mathbb{R}^k_+.$$

Assume that $J_N = \emptyset$, i.e., for each $j \in J = J_A$,

$$f_j(\tau, z) = \langle a_j(\tau), z\rangle + b_j(\tau), \quad g_j(\tau, z) = \langle c_j(\tau), z\rangle + d_j(\tau),$$

and that the phase constraints are split as $I = I_A \cup I_N$, where for each $i \in I_A$,

$$h_i(\tau, z) = \langle p_i(\tau), z\rangle + q_i(\tau),$$

while for each $i \in I_N$ the function $h_i(\tau, \cdot)$ is convex (not necessarily affine). Assume that (RC) and (SQ) hold. Then, there exist multipliers $(\hat{\lambda}, \hat{v}(\cdot)) \in \mathbb{R}^k \times L_\infty(\Delta; \mathbb{R}^m)$ such that

(1) $\hat{\lambda} \in \Lambda_+$ *and* $\hat{v}(\cdot) \in V$;
(2) $\hat{v}(\tau)' h(\tau, \hat{z}(\tau)) = 0$ *for a.e.* $\tau \in \Delta$;

(3) $\int_\Delta \big[\sum_{j\in J_A} \hat{\lambda}_j(\langle a_j(\tau) - \hat{w}_j c_j(\tau), z(\tau)\rangle + b_j(\tau) - \hat{w}_j d_j(\tau))$
$+ \sum_{i\in I_A} \hat{v}_i(\tau)(\langle p_i(\tau), z(\tau)\rangle + q_i(\tau)) + \sum_{i\in I_N} \hat{v}_i(\tau)\, h_i(\tau, z(\tau))\big]\, d\tau \geqq 0,$
for all $z(\cdot) \in L_\infty(\Delta;\mathbb{R}^n)$.

Moreover, if (SD) *holds, then necessarily*

(0) $\hat{v}_N(\cdot) \not\equiv 0,$ *where* $\hat{v}_N(\cdot) := (\hat{v}_i(\cdot))_{i\in I_N}$.

Proof. Assume that (SQ) and (RC) hold and that there exist multipliers $(\hat{\lambda}, \hat{v}(\cdot)) \in \mathbb{R}^k \times L_\infty(\Delta;\mathbb{R}^m)$ satisfying (1)–(3) of Theorem 3.4. We prove that $\hat{v}_N(\cdot) \not\equiv 0$.

Step 0 (Contradiction hypothesis). Suppose to the contrary that

$$\hat{v}_N(\cdot) \equiv 0, \quad \text{i.e.,} \quad \hat{v}_i(\cdot) \equiv 0 \qquad \forall i \in I_N.$$

Then condition (3) reduces to an inequality involving only the affine parts:

$$\int_\Delta \Bigg[\sum_{j\in J_A} \hat{\lambda}_j(\langle a_j(\tau) - \hat{w}_j c_j(\tau), z(\tau)\rangle + b_j(\tau) - \hat{w}_j d_j(\tau)) + \sum_{i\in I_A} \hat{v}_i(\tau)(\langle p_i(\tau), z(\tau)\rangle + q_i(\tau)) \Bigg] d\tau \geqq 0, \quad \forall z(\cdot) \in L_\infty(\Delta;\mathbb{R}^n).$$

Then the KKT-type inequality of Theorem 3.6 reduces to an affine-only inequality: for all $z(\cdot) \in L_\infty(\Delta;\mathbb{R}^n)$,

$$\int_\Delta \big(\langle \eta(\tau), z(\tau)\rangle + \beta(\tau)\big) d\tau \geqq 0, \tag{3.20}$$

where

$$\eta(\tau) := \sum_{j\in J} \hat{\lambda}_j(a_j(\tau) - \hat{w}_j c_j(\tau)) + \sum_{i\in I_A} \hat{v}_i(\tau)\, p_i(\tau) \quad \text{a.e. in } \Delta,$$

$$\beta(\tau) := \sum_{j\in J} \hat{\lambda}_j(b_j(\tau) - \hat{w}_j d_j(\tau)) + \sum_{i\in I_A} \hat{v}_i(\tau)\, q_i(\tau) \quad \text{a.e. in } \Delta.$$

By the standard z_t test-function argument (as in Step 2 of the previous Theorem), (3.20) implies

$$\eta(\tau) = 0 \quad \text{for a.e. } \tau \in \Delta. \tag{3.21}$$

Step 1 (Contradiction using (SD)). We have

$$0 = \langle \eta(\tau), d(\tau)\rangle = \sum_{j\in J} \hat{\lambda}_j \langle a_j(\tau) - \hat{w}_j c_j(\tau), d(\tau)\rangle + \sum_{i\in I_A} \hat{v}_i(\tau)\langle p_i(\tau), d(\tau)\rangle \text{ a.e. in } \Delta.$$

By (SD), each term $\langle a_j(\tau) - \hat{w}_j c_j(\tau), d(\tau)\rangle$ is strictly negative, and since $\hat{\lambda} \in \Lambda_+$ we have $\hat{\lambda}_j > 0$ for all $j \in J$. Therefore,

$$\sum_{j\in J} \hat{\lambda}_j \langle a_j(\tau) - \hat{w}_j c_j(\tau), d(\tau)\rangle < 0 \quad \text{for a.e. in } \Delta.$$

Also, (SD) gives $\langle p_i(\tau), d(\tau)\rangle \leqq 0$ and $\hat{v}_i(\tau) \geqq 0$ a.e. in Δ, hence

$$\sum_{i\in I_A} \hat{v}_i(\tau)\langle p_i(\tau), d(\tau)\rangle \leqq 0 \quad \text{for a.e. in } \Delta.$$

Adding the two inequalities yields $\langle \eta(\tau), d(\tau)\rangle < 0$ a.e. in Δ contradicting $\langle \eta(\tau), d(\tau)\rangle = 0$ from (3.21). Hence $\hat{v}_N(\cdot) \not\equiv 0$. $\square$

EXAMPLE 3.4. Let $\Delta = [0,1]$, $n = 2$, $k = 2$, $m = 2$, and keep the same constraints as in Example 3.3. Let $h_1(\tau, z) = z_1 - \frac{\tau}{2} \leqq 0 \quad (I_A = \{1\})$, $h_2(\tau, z) = \|z\|^2 - \frac{\tau^2}{16} \leqq 0 \quad (I_N = \{2\})$. Let $g_1(\tau, z) = g_2(\tau, z) = 3 + \tau - \frac{1}{8} z_1$. On the feasible set $\|z(\tau)\| \leqq \tau/4$ we have $z_1(\tau) \leqq \tau/4$, hence $g_j(\tau, z) > 0$. Fix $\beta > 0$ and define $f_1(\tau, z) = 1 \cdot g_1(\tau, z) + \frac{\tau}{2} z_1 + \frac{\tau^2}{8} + \beta \frac{\tau}{4} z_2$ and $f_2(\tau, z) = 2 \cdot g_2(\tau, z) + \frac{\tau}{2} z_1 + \frac{\tau^2}{8} - \beta \frac{\tau}{4} z_2$. Thus both objective components are affine in z (so $J_N = \emptyset$). Let $\hat{z}(\tau) = \left(-\frac{\tau}{4}, 0\right)$ a.e. in $[0,1]$. Hence, $\hat{w}_1 = 1$, $\hat{w}_2 = 2$. (SQ) holds with $s(\tau) \equiv (0,0)$. (RC) holds exactly as in previous example, since $h_2(\tau, z)$ is convex and has quadratic growth with $\partial_z h_2(\tau, z) = 2z$. Let $d(\tau) \equiv (-1, 0)$. Then, for $\tau > 0$, $\langle a_1(\tau) - \hat{w}_1 c(\tau), d(\tau)\rangle < 0$, $\langle a_2(\tau) - \hat{w}_2 c(\tau), d(\tau)\rangle < 0$. Moreover, $h_1(\tau, z) = \langle p_1, z\rangle + q_1(\tau)$ with $p_1 = (1, 0)$, so $\langle p_1, d(\tau)\rangle \leqq 0$. Hence, (SD) holds. Let $\hat{\lambda} = \left(\frac{1}{2}, \frac{1}{2}\right)$, $\hat{v}_1(\tau) \equiv 0$ and $\hat{v}_2(\tau) \equiv 1$. Conditions (1), (2) and (3) are obviously satisfied. Also

$$\hat{v}_N = \hat{v}_2 \not\equiv 0.$$

THEOREM 3.7 (Fully affine case). *Let $\hat{z}(\cdot) \in L_\infty(\Delta; \mathbb{R}^n)$ be a proper efficient solution of $(VFCTP)$ and define*

$$\hat{w} := \frac{\int_\Delta f(\tau, \hat{z}(\tau)) d\tau}{\int_\Delta g(\tau, \hat{z}(\tau)) d\tau} \in \mathbb{R}^k_+.$$

Assume that all objective components are affine in the second argument, i.e., for each $j \in J = \{1, \ldots, k\}$,

$$f_j(\tau, z) = \langle a_j(\tau), z\rangle + b_j(\tau), \quad g_j(\tau, z) = \langle c_j(\tau), z\rangle + d_j(\tau),$$

and that all phase constraints are affine in the second argument, i.e., for each $i \in I = \{1, \ldots, m\}$,

$$h_i(\tau, z) = \langle p_i(\tau), z\rangle + q_i(\tau),$$

for a.e. $\tau \in \Delta$ and all $z \in \mathbb{R}^n$. Assume that (RC) and (SQ) hold. Then, there exist multipliers $(\hat{\lambda}, \hat{v}(\cdot)) \in \mathbb{R}^k \times L_\infty(\Delta; \mathbb{R}^m)$ such that

(1) *$\hat{\lambda} \in \Lambda_+$ and $\hat{v}(\cdot) := (\hat{v}_i(\cdot))_{i \in I} \in V$ (i.e. $\hat{v}(\tau) \geqq 0$ for a.e. $\tau \in \Delta$);*

(2) *$\hat{v}_i(\tau)(\langle p_i(\tau), z(\tau)\rangle + q_i(\tau)) = 0$, $i \in I$ for a.e. $\tau \in \Delta$;*

(3.1) *$\sum_{j \in J} \hat{\lambda}_j (a_j(\tau) - \hat{w}_j c_j(\tau)) + \sum_{i \in I} \hat{v}_i(\tau) p_i(\tau) = 0$ a.e. in Δ;*

(3.2) *$\int_\Delta \left(\sum_{j \in J} \hat{\lambda}_j (b_j(\tau) - \hat{w}_j d_j(\tau)) + \sum_{i \in I} \hat{v}_i(\tau) q_i(\tau)\right) d\tau \geqq 0$.*

PROOF. **Step 0**. Under (RC) and (SQ), Theorem 3.6 provides multipliers $(\hat{\lambda}, \hat{v}(\cdot)) \in \mathbb{R}^k \times L_\infty(\Delta; \mathbb{R}^m)$ such that $\hat{\lambda} \in \Lambda_+$, $\hat{v}(\cdot) \in V$, complementary slackness holds, and the basic KKT-type inequality of Theorem 3.6 is satisfied for all $z(\cdot) \in L_\infty(\Delta; \mathbb{R}^n)$. Using the fully affine structure of f, g, h, this inequality becomes exactly

$$\int_{\Delta}\left[\sum_{j\in J_A}\hat{\lambda}_j(\langle a_j(\tau)-\hat{w}_j c_j(\tau), z(\tau)\rangle + b_j(\tau)-\hat{w}_j d_j(\tau)) + \sum_{i\in I_A}\hat{v}_i(\tau)(\langle p_i(\tau), z(\tau)\rangle + q_i(\tau))\right]d\tau \geqq 0.$$

Step 1 The proof of this step is the same as the proof of Step 2 in Theorem 3.5. Hence, with $\eta(\tau)=0$ a.e. in Δ, condition (3) reduces to $\int_{\Delta}\beta(\tau)d\tau \geqq 0$, which completes the proof. □

Solution of Example: Time-dependent surface-grinding model. In the notation of (VFCTP), we have $k=2$, $m=2$, and $f_1(\tau,p)=1+p-\tau$, $f_2(\tau,p)=2(p-\tau)-1$, $g_1(\tau,p)=g_2(\tau,p)=p-\tau$, $h_1(\tau,p)=1+\tau-p$, $h_2(\tau,p)=p-\tau-2$. The feasible set is $\Omega_P=\{p(\cdot)\in L_\infty([0,1];\mathbb{R}) : 1\leqslant p(\tau)-\tau\leqslant 2$ a.e. on $[0,1]\}$. Obviously, $\widehat{p}(\tau)=\tau+\frac{3}{2}$ is a proper efficient solution of problem (G) with $M=1$. For every $\tau\in[0,1]$, the functions $f_1(\tau,p)$, $f_2(\tau,p)$ are affine in p, hence convex. Also, $g_1(\tau,p)=g_2(\tau,p)=p-\tau$ are affine in p, hence concave as well as convex. Therefore, $-g_1(\tau,p)$, $-g_2(\tau,p)=\tau-p$ are convex. Finally, $h_1(\tau,p)=1+\tau-p$, $h_2(\tau,p)=p-\tau-2$ are affine in p, hence convex. All functions are measurable in τ, continuous in p, and bounded on bounded subsets of $\mathbb{R}$. Moreover, for every feasible $p(\cdot)$, $p(\tau)-\tau\geqslant 1$ a.e. $[0,1]$, so $\int_0^1 g_j(\tau,p(\tau))d\tau>0$, $j=1,2$. Hence, the problem belongs to the mixed-affine VFCTP framework. Set $A:=\int_0^1\big(p(\tau)-\tau\big)d\tau$. Since $1\leqslant p(\tau)-\tau\leqslant 2$ a.e. on $[0,1]$, we have $1\leqslant A\leqslant 2$. Thus $(J_1(p(\cdot)),J_2(p(\cdot)))=(1+\frac{1}{A}, 2-\frac{1}{A})$, $A\in[1,2]$. For $\widehat{p}(\tau)=\tau+\frac{3}{2}$, we obtain $\widehat{w}=\big(\frac{5}{3},\frac{4}{3}\big)$. We have that, for almost every $\tau\in[0,1]$, Slater's condition (SQ) is satisfied for $s(\tau)=\tau+\frac{5}{4}$. The problem is fully affine. We have $J_A=\{1,2\}$, $J_N=\emptyset$, $I_A=\{1,2\}$, $I_N=\emptyset$. Therefore, the mixed-affine structure required in Theorem 3.3 holds. Also, it is clear that the (RC) condition is satisfied. We now construct multipliers satisfying the conclusion of Theorem 3.3. Let $\widehat{\lambda}=\big(\frac{1}{2},\frac{1}{2}\big)$, $\widehat{v}_1(\tau)=0$ and $\widehat{v}_2(\tau)=0$, a.e. in $[0,1]$. Conditions (3.1) and (3.2) hold as the two weighted objective components are in exact equilibrium under the chosen multipliers. Since the phase-constraint multipliers are zero, no additional contribution arises from the constraints. Therefore, both the pointwise balance condition and the global integral condition are trivially satisfied.

Conclusion. In this example, the properly efficient pressure profile is $\widehat{p}(\tau)=\tau+\frac{3}{2}$. Mechanically, this means that the commanded contact pressure increases along the grinding pass in order to compensate for the position-dependent reduction encoded by τ. The effective grinding intensity $\widehat{p}(\tau)-\tau=\frac{3}{2}$ remains constant. This gives a balanced operating regime. Increasing the effective intensity would reduce the specific grinding energy, but it would increase wheel wear per removed material volume, while decreasing it would reduce wheel wear but increase the specific energy.

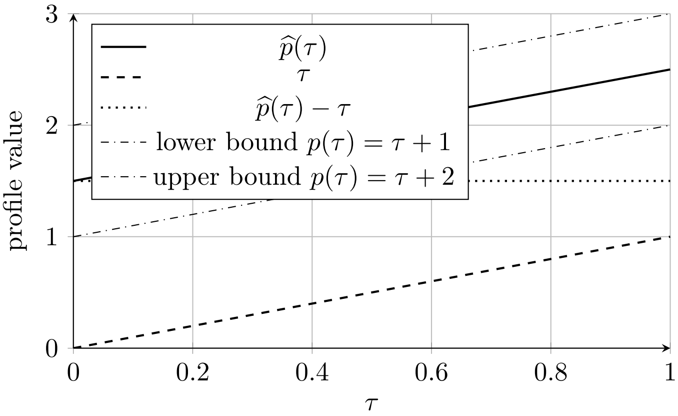


FIGURE 2. Optimal pressure profile and admissible bounds. The profile $\widehat{p}(\tau)=\tau+\frac{3}{2}$ remains strictly inside the feasible band $\tau+1\leqslant p(\tau)\leqslant\tau+2$, while the effective intensity $\widehat{p}(\tau)-\tau$ is constant.

4. Duality

Generally, the dual of a vector programming problem is also a vector problem. For instance, nonlinear vector programming problems in finite-dimensional spaces are discussed in [30, 31]. In the following, we examine a parameter-free efficiency result of the saddle-point type, expressed in terms of a vector-valued Lagrangian function $\mathcal{L}: L_\infty(\Delta;\mathbb{R}^n)\times L_\infty(\Delta;\mathbb{R}^m)\to\mathbb{R}^k$, as $\mathcal{L}(z(\cdot),v(\cdot))=(\mathcal{L}_1(z(\cdot),v(\cdot)),\dots,\mathcal{L}_k(z(\cdot),v(\cdot)))'$ with components $\mathcal{L}_j$, $j\in J$, defined by

$$\mathcal{L}_j(z(\cdot),v(\cdot))=\frac{\int_\Delta (f_j(\tau,z(\tau))+v'(\tau)h(\tau,z(\tau)))d\tau}{\int_\Delta g_j(\tau,z(\tau))d\tau}.$$

We shall say that a pair $(\hat{z}(\cdot),\hat{v}(\cdot))$ is a solution of vector saddle point continuous-time problem (VSPCTP) if

$$\mathcal{L}(\hat{z}(\cdot),v(\cdot))\ngeq\mathcal{L}(\hat{z}(\cdot),\hat{v}(\cdot)) \tag{4.1}$$

$$\mathcal{L}(\hat{z}(\cdot),\hat{v}(\cdot))\ngeq\mathcal{L}(z(\cdot),\hat{v}(\cdot)) \tag{4.2}$$

for all $z(\cdot)\in L_\infty(\Delta;\mathbb{R}^n)$ and all $v(\cdot)\in V$.

We formulate a Lagrangian-type dual problem for (VFCTP) and establish both weak and strong duality theorems. The structure of this dual problem is based on Theorem 3.1. Let

$$\Omega_D=\Big\{(\lambda,\eta,v(\cdot))\in\Lambda^+\times\mathbb{R}^k_+\times V\ :\ \lambda'\eta=\inf_{z(\cdot)\in L_\infty(\Delta;\mathbb{R}^n)}\lambda'\mathcal{L}(z(\cdot),v(\cdot))\Big\}.$$

We consider the vector dual (VDFCTP) of the primal problem (VFCTP):

$$\text{(VDFCTP)}\qquad \begin{array}{c}\max\ \eta\\ \text{subject to } (\lambda,\eta,v(\cdot))\in\Omega_D.\end{array}$$

In the following, the first result, called the weak duality theorem, is a simple consequence of the definition of the dual problem.

THEOREM 4.1 (Weak duality theorem). *Let* $z(\cdot) \in L_\infty(\Delta;\mathbb{R}^n)$ *and* $(\lambda, \eta, v(\cdot)) \in \Omega_D$ *be feasible solutions of* (VFCTP) *and* (VDFCTP)*, respectively. Then*

$$\eta \not\geqq \frac{\int_\Delta f(\tau, z(\tau))d\tau}{\int_\Delta g(\tau, z(\tau))d\tau}.$$

PROOF. For some $\lambda \in \Lambda^+$, $v(\cdot) \in V$, $\lambda'\eta = \inf_{z(\cdot)\in L_\infty(\Delta;\mathbb{R}^n)} \lambda'\mathcal{L}(z(\cdot), v(\cdot))$. Since $z(\cdot) \in \Omega_P$, we have

$$\begin{aligned}\lambda'\eta = \inf_{z(\cdot)\in L_\infty(\Delta;\mathbb{R}^n)} \lambda'\mathcal{L}(z(\cdot), v(\cdot)) &\leqq \lambda'\frac{\int_\Delta f(\tau, z(\tau))d\tau}{\int_\Delta g(\tau, z(\tau))d\tau} + \sum_{j\in J} \frac{\lambda_j \int_\Delta v'(\tau)h(\tau, z(\tau))d\tau}{\int_\Delta g_j(\tau, z(\tau))d\tau}\\ &\leqq \lambda'\frac{\int_\Delta f(\tau, z(\tau))d\tau}{\int_\Delta g(\tau, z(\tau))d\tau}.\end{aligned}$$

Hence,

$$\lambda'\left(\frac{\int_\Delta f(\tau, z(\tau))d\tau}{\int_\Delta g(\tau, z(\tau))d\tau} - \eta\right) \geqq 0 \implies \eta \not\geqq \frac{\int_\Delta f(\tau, z(\tau))d\tau}{\int_\Delta g(\tau, z(\tau))d\tau}.$$

Thus, the proof is complete. □

The following lemma plays a key role in proving the strong duality result in this section.

LEMMA 4.1. *Let* $\hat{z}(\cdot)$ *be a proper efficient solution of* (VFCTP) *and assume that* (RC) *and* (SQ) *are satisfied. Then, there exists* $\hat{v}(\cdot) \in V$ *such that* $(\hat{z}(\cdot), \hat{v}(\cdot))$ *is a solution of the saddle point problem* (VSPCTP) *and slackness condition* (3.3) *holds.*

PROOF. From Theorem 3.1 we obtain that there exists $(\hat{\lambda}, \hat{v}(\cdot)) \in \Lambda^+ \times V$, such that

$$\hat{v}'(\tau)h(\tau, \hat{z}(\tau)) = 0 \quad \text{a.e. in } \Delta \tag{4.3}$$

and $(\hat{z}(\cdot), \hat{\lambda}, \hat{v}(\cdot))$ is a KKTSP for (VFCTP), i.e.,

$$\mathcal{L}_{\hat{w}}(\hat{z}(\cdot), \hat{\lambda}, v(\cdot)) \leqq \mathcal{L}_{\hat{w}}(\hat{z}(\cdot), \hat{\lambda}, \hat{v}(\cdot)) \leqq \mathcal{L}_{\hat{w}}(z(\cdot), \hat{\lambda}, \hat{v}(\cdot)), \tag{4.4}$$

for all $z(\cdot) \in L_\infty(\Delta;\mathbb{R}^n)$ and all $v(\cdot) \in V$. Now, we shall prove that $(\hat{z}(\cdot), \hat{v}(\cdot))$ is a solution of the vector saddle point continuous-time problem (VSPCTP). We will suppose that (4.1) is not true. Then, for some $l \in J$, $\bar{v}(\cdot) \in V$, we have

$$\frac{\int_\Delta (f_l(\tau, \hat{z}(\tau)) + \hat{v}'(\tau)h(\tau, \hat{z}(\tau)))d\tau}{\int_\Delta g_l(\tau, \hat{z}(\tau))d\tau} < \frac{\int_\Delta (f_l(\tau, \hat{z}(\tau)) + \bar{v}(\tau)'h(\tau, \hat{z}(\tau)))d\tau}{\int_\Delta g_l(\tau, \hat{z}(\tau))d\tau} \tag{4.5}$$

and

$$\frac{\int_\Delta (f_j(\tau, \hat{z}(\tau)) + \hat{v}'(\tau)h(\tau, \hat{z}(\tau)))d\tau}{\int_\Delta g_j(\tau, \hat{z}(\tau))d\tau} \leqq \frac{\int_\Delta (f_j(\tau, \hat{z}(\tau)) + \bar{v}'(\tau)h(\tau, \hat{z}(\tau)))d\tau}{\int_\Delta g_j(\tau, \hat{z}(\tau))d\tau} \tag{4.6}$$

for all $j \neq l$, $j \in J$.
Multiplying (4.5) by $\hat{\lambda}_l \int_\Delta g_l(\tau, \hat{z}(\tau))d\tau$ and (4.6) by $\hat{\lambda}_j \int_\Delta g_j(\tau, \hat{z}(\tau))d\tau$, $j \neq l$ and summing, we obtain

$$\begin{aligned}(4.7)\quad \hat{\lambda}' \int_\Delta f(\tau, \hat{z}(\tau))d\tau + \sum_{j\in J} \hat{\lambda}_j \int_\Delta \hat{v}'(\tau)h(\tau, \hat{z}(\tau))d\tau \\ < \hat{\lambda}' \int_\Delta f(\tau, \hat{z}(\tau))d\tau + \sum_{j\in J} \hat{\lambda}_j \int_\Delta \bar{v}'(\tau)h(\tau, \hat{z}(\tau))d\tau.\end{aligned}$$

Since $\sum_{j\in J} \hat{\lambda}_j = 1$, adding the term $-\hat{\lambda}' \int_\Delta \hat{w} \circ g(\tau, \hat{z}(\tau))d\tau$ to both sides of inequality (4.7), we obtain a contradiction to the inequality (4.4). Likewise, we will suppose that (4.2) is not true. Then, for some $l \in J$, $\bar{z}(\cdot) \in L_\infty(\Delta; \mathbb{R}^n)$, we have

$$(4.8)\quad \frac{\int_\Delta (f_l(\tau, \bar{z}(\tau)) + \hat{v}'(\tau)h(\tau, \bar{z}(\tau)))d\tau}{\int_\Delta g_l(\tau, \bar{z}(\tau))d\tau} < \frac{\int_\Delta (f_l(\tau, \hat{z}(\tau)) + \hat{v}'(\tau)h(\tau, \hat{z}(\tau)))d\tau}{\int_\Delta g_l(\tau, \hat{z}(\tau))d\tau}$$

and

$$(4.9)\quad \frac{\int_\Delta (f_j(\tau, \bar{z}(\tau)) + \hat{v}'(\tau)h(\tau, \bar{z}(\tau)))d\tau}{\int_\Delta g_j(\tau, \bar{z}(\tau))d\tau} \leqq \frac{\int_\Delta (f_j(\tau, \hat{z}(\tau)) + \hat{v}'(\tau)h(\tau, \hat{z}(\tau)))d\tau}{\int_\Delta g_j(\tau, \hat{z}(\tau))d\tau}$$

for all $j \neq l$, $j \in J$. Multiplying (4.8) by $\hat{\lambda}_l \int_\Delta g_l(\tau, \bar{z}(\tau))d\tau$ and (4.9) by $\hat{\lambda}_j \int_\Delta g_j(\tau, \bar{z}(\tau))d\tau$, $j \neq l$ and summing, we obtain

$$\begin{aligned}(4.10)\quad \hat{\lambda}' \int_\Delta f(\tau, \bar{z}(\tau))d\tau + \sum_{j\in J} \hat{\lambda}_j \int_\Delta \hat{v}(\tau)'h(\tau, \bar{z}(\tau))d\tau \\ < \sum_{j\in J} \frac{\hat{\lambda}_j \int_\Delta g_j(\tau, \bar{z}(\tau))d\tau \int_\Delta f_j(\tau, \hat{z}(\tau))d\tau}{\int_\Delta g_j(\tau, \hat{z}(\tau))d\tau} + \sum_{j\in J} \hat{\lambda}_j \int_\Delta \hat{v}(\tau)'h(\tau, \hat{z}(\tau))d\tau.\end{aligned}$$

Since $\sum_{j\in J} \hat{\lambda}_j = 1$, (4.3) holds, $\hat{w}_j = \frac{\int_\Delta f_j(\tau, \hat{z}(\tau))d\tau}{\int_\Delta g_j(\tau, \hat{z}(\tau))d\tau}$, if we add the term

$$-\sum_{j\in J} \frac{\hat{\lambda}_j \int_\Delta g_j(\tau, \bar{z}(\tau))d\tau \int_\Delta f_j(\tau, \hat{z}(\tau))d\tau}{\int_\Delta g_j(\tau, \hat{z}(\tau))d\tau}$$

to both sides of inequality (4.10) we obtain

$$\hat{\lambda}' \int_\Delta (f(\tau, \bar{z}(\tau)) - \hat{w} \circ g(\tau, \bar{z}(\tau)))d\tau + \int_\Delta \hat{v}(\tau)'h(\tau, \bar{z}(\tau))d\tau < 0,$$

also contradicting the inequality (4.4). Therefore, (4.1) and (4.2) hold and $(\hat{z}(\cdot), \hat{v}(\cdot))$ is a solution of (VSPCTP). Thus, the proof is complete. □

The following result, known as the strong duality theorem, shows that, under convexity assumptions, suitable regularity condition and Slater's constraint qualification, there is no duality gap between primal (VFCTP) and (VDFCTP).

THEOREM 4.2 (Strong duality theorem). *Let $\hat{z}(\cdot)$ be a proper efficient solution of* (VFCTP)*. Assume that* (RC) *and* (SQ) *are satisfied. Then, there exists* $(\hat{\lambda}, \hat{\eta}, \hat{v}(\cdot)) \in \Lambda^+ \times \mathbb{R}^k \times V$ *such that $\hat{\eta}$ is a proper efficient solution of* (VDFCTP) *and*

$$\frac{\int_\Delta f(\tau, \hat{z}(\tau))d\tau}{\int_\Delta g(\tau, \hat{z}(\tau))d\tau} = \hat{\eta}.$$

PROOF. Step 1: Since $\hat{z}(\cdot)$ is an efficient solution of (VFCTP), by Lemma 4.1, there exist $\hat{v}(\cdot) \in V$ such that $(\hat{z}(\cdot), \hat{v}(\cdot))$ is a solution of the saddle point problem (VSPCTP) and

$$\hat{v}'(\tau)h(\tau, \hat{z}(\tau)) = 0 \quad \text{a.e. in } \Delta. \tag{4.11}$$

Hence, for $\hat{\lambda} \in \Lambda^+$ we have

$$\hat{\lambda}' \frac{\int_\Delta f(\tau, \hat{z}(\tau))d\tau}{\int_\Delta g(\tau, \hat{z}(\tau))d\tau} \leqq \hat{\lambda}' \frac{\int_\Delta (f(\tau, z(\tau)) + e\hat{v}'(\tau)h(\tau, z(\tau)))d\tau}{\int_\Delta g(\tau, z(\tau))d\tau} \quad \forall z(\cdot) \in L_\infty(\Delta; \mathbb{R}^n).$$

Therefore,

$$\hat{\lambda}' \frac{\int_\Delta f(\tau, \hat{z}(\tau))d\tau}{\int_\Delta g(\tau, \hat{z}(\tau))d\tau} \leqq \inf_{z(\cdot) \in L_\infty(\Delta; \mathbb{R}^n)} \hat{\lambda}' \mathcal{L}(z(\cdot), \hat{v}(\cdot)). \tag{4.12}$$

On the other hand, from (4.11) we obtain

$$\inf_{z(\cdot) \in L_\infty(\Delta; \mathbb{R}^n)} \hat{\lambda}' \mathcal{L}(z(\cdot), \hat{v}(\cdot)) \leqq \hat{\lambda}' \mathcal{L}(\hat{z}(\cdot), \hat{v}(\cdot)) = \hat{\lambda}' \frac{\int_\Delta f(\tau, \hat{z}(\tau))d\tau}{\int_\Delta g(\tau, \hat{z}(\tau))d\tau}. \tag{4.13}$$

Setting $\hat{\eta} = \frac{\int_\Delta f(\tau, \hat{z}(\tau))d\tau}{\int_\Delta g(\tau, \hat{z}(\tau))d\tau} = \hat{w}$, from (4.12) and (4.13) we obtain

$$\hat{\lambda}' \hat{\eta} = \inf_{z(\cdot) \in L_\infty(\Delta; \mathbb{R}^n)} \hat{\lambda}' \mathcal{L}(z(\cdot), \hat{v}(\cdot)).$$

Therefore, the solution is feasible.

Step 2: It is obvious that $(\hat{\lambda}, \hat{\eta}, \hat{v}(\cdot))$ is an efficient solution. If this solution is not a proper efficient solution, there will be another feasible solution $(\lambda^0, \eta^0, v^0(\cdot))$, such that for some $j \in J$, inequality $\eta_j^0 - \hat{\eta}_j > M(\hat{\eta}_t - \eta_t^0)$ holds for all $t \in J$ where $\hat{\eta}_t - \eta_t^0 > 0$. Hence, $\hat{\eta}_j - \eta_j^0 < 0$ i.e. $\hat{w}_j < \eta_j^0$, which contradicts inequality $\eta \ngeq \hat{w}$. Thus, the proof is complete. □

EXAMPLE 4.1. Consider the following primal problem from Example 3.1 and the corresponding dual problem (VDFCTP)

$$\begin{aligned} \max \eta =& (\eta_1, \eta_2) \\ \text{subject to } \lambda_1\eta_1 + \lambda_2\eta_2 =& \inf_{z(\cdot) \in L_\infty(\Delta; \mathbb{R}^n)} (\lambda_1 \mathcal{L}_1(z(\cdot), v(\cdot)) + \lambda_2 \mathcal{L}_2(z(\cdot), v(\cdot))), \\ & (\lambda, \eta, v(\cdot)) \in \Lambda^+ \times \mathbb{R}_+^2 \times V, \end{aligned} \tag{VDFCTP}$$

where $k = 2$, $m = 1$ and

$$\begin{aligned} &\mathcal{L}(z(\cdot), v(\cdot)) = (\mathcal{L}_1(z(\cdot), v(\cdot)), \mathcal{L}_2(z(\cdot), v(\cdot)))' \\ &= \left(\frac{\int_0^1 (\tau|z(\tau)| + v(\tau)(|z(\tau)| - \tau))d\tau}{\int_0^1 \ln(z(\tau) + 2\tau + 3)d\tau}, \frac{\int_0^1 (z^2(\tau) + 2\tau + v(\tau)(|z(\tau)| - \tau))d\tau}{\int_0^1 (4\tau - z^2(\tau))d\tau} \right). \end{aligned}$$

However, from Example 3.1 we have that for a.e. τ in $[0,1]$, $\hat{z}(\tau)=0$ is a proper efficient optimal solution of (VPFCTP), $\hat{w}=(\hat{w}_1,\hat{w}_2)=\left(0,\frac{1}{2}\right)$ and (SQ), (RC) are satisfied. Further, there exists $(\hat{\lambda},\hat{\eta},\hat{v}(\cdot))=\left(\left(\frac{1}{2},\frac{1}{2}\right),\left(0,\frac{1}{2}\right),0\right)\in\Lambda^{+}\times\mathbb{R}^2_{+}\times V$ such that $\hat{\eta}$ is an efficient solution of (VDFCTP). Indeed, for $\hat{z}(\tau)=0$ and $\hat{v}(\tau)=0$ a.e. in $[0,1]$ problem (VDFCTP) becomes

$$\text{(VDFCTP)}\qquad \begin{aligned} &\max \eta=(\eta_1,\eta_2)\\ &\text{subject to } \frac{1}{2}\eta_1+\frac{1}{2}\eta_2=\frac{1}{4},\\ &\eta_1\geqq 0,\eta_2\geqq 0, \end{aligned}$$

with the obvious proper efficient solution $\hat{\eta}=\left(0,\frac{1}{2}\right)$. It is obvious that optimal values of (VPFCTP) and (VDFCTP) are equal.

5. Conclusion

This paper investigated a nonsmooth vector fractional continuous-time programming problem with inequality-type phase constraints in the essentially bounded space $L_\infty(\Delta;\mathbb{R}^n)$. The main outcomes can be summarized as follows:

- A continuous-time saddle-point framework was developed for properly efficient solutions by introducing an appropriate KKT saddle-point concept and a Lagrange-type functional tailored to the fractional structure.
- Under a continuous-time Slater-type feasibility condition and a regularity requirement for convex inequality systems, we established KKT-type *necessary* optimality conditions and also provided a *sufficient* saddle-point criterion for proper efficiency.
- For models exhibiting mixed affine/convex structure, we derived refined optimality conditions that separate affine and genuinely nonaffine components. In addition, we introduced a new relaxed Slater-type condition for mixed-affine settings that yields strengthened, parameter-free statements and enforces the nontriviality of multipliers associated with the nonaffine constraint block.
- A vector-valued Lagrangian was proposed and used to formulate a vector dual problem. For this dual model, we proved weak duality and, under the same assumptions (and their mixed-affine relaxed variant), a strong duality theorem ensuring the absence of a duality gap.
- Several examples were provided to illustrate how the assumptions can be checked and how multipliers can be constructed explicitly in representative situations.

Potential extensions include: (i) further weakening of regularity/constraint qualification requirements while preserving multiplier nontriviality; (ii) treatment of additional constraint classes (e.g., equality-type phase constraints or integral-type constraints); (iii) generalized convexity frameworks; and (iv) discretization and numerical schemes that exploit the derived saddle-point and duality structures for computation in practical applications.

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О ОПТИМАЛНОСТИ И ДУАЛНОСТИ У НЕГЛАТКОМ ВЕКТОРСКОМ РАЗЛОМЉЕНОМ ПРОГРАМИРАЊУ СА НЕПРЕКИДНИМ ВРЕМЕНОМ: ПОЈАЧАНИ УСЛОВИ ЗА МЕШОВИТО-АФИНЕ МОДЕЛЕ

Резиме. У овом раду разматрамо проблем неглатког векторског разломљеног програмирања са непрекидним временом са фазним ограничењима неједнакости. Мотивација потиче из модела примењене механике у којима се перформансе описују разломљеним индексима акумулираним током времена, при чему оперативна ограничења морају бити задовољена у готово сваком тренутку посматраног временског интервала. Такви модели обухватају, између осталог, абразивну обраду и брушење, квазистационарну ефикасност крстарења авиона, као и енергетски ефикасно планирање хода код робота. Изводимо неопходне Каруш-Кун-Такерове услове оптималности за правилно ефикасна решења комбиновањем Слејтеровог услова по непрекидном времену и услова регуларности за конвексне системе неједнакости. Такође успостављамо довољан услов оптималности који важи под претпоставкама конвексности.

За моделе са мешовитом афином структуром, јачамо теорију изван класичних оквира заснованих на Слејтеровом услову. Уводимо две додатне проверљиве хипотезе, услов решивости и услов правца раздвајања. Ове претпоставке дају оштрије закључке о мултипликаторима, укључујући нетривијалност мултипликатора повезаних са неафиним ограничењима и воде до побољшања услова оптималности без помоћних параметара. Формулисана је и доказана кључна лема која пружа главни алат који лежи у основи ових резултата. Затим уводимо векторски Лагранжијан и формулишемо одговарајући векторски дуални модел. За тако добијени дуални модел доказујемо теореме слабе и јаке дуалности, при чему теорема јаке дуалности обезбеђује одсуство дуалног јаза. На крају, неколико примера показује како се претпоставке могу проверити у конкретним ситуацијама и како се добијени теоријски резултати примењују помоћу експлицитно конструисаних множилаца.

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