THEORETICAL AND APPLIED MECHANICS
Volume 52 (2025) Issue 2, 217–231 DOI: https://doi.org/10.2298/TAM250108014B

ON THE QUASI-DIAGONALIZATION AND UNCOUPLING OF DAMPED CIRCULATORY MULTI-DEGREE-OF-FREEDOM SYSTEMS

Ranislav M. Bulatović and Firdaus E. Udwadia

Abstract. The decomposition of linear multi-degree-of-freedom systems with damping, circulatory, and potential forces is considered through a real linear coordinate transformation generated by an orthogonal matrix. Criteria are derived that establish the conditions under which such a transformation exists, allowing these systems to be decomposed into independent, uncoupled subsystems, each with a maximum dimension of two. These criteria are expressed in terms of the properties of systems' coefficient matrices. Several numerical examples are provided to demonstrate the analytical results.

1. Introduction

An interesting and long-standing topic in the dynamics of linear multi-degree-of-freedom (MDOF) systems is the investigation of the possibility of decomposing them into a series of mutually independent low-dimensional subsystems. It is well known that the normal modes of a linear symmetric undamped MDOF dynamical system constitute a modal matrix, which defines a real congruence transformation (real change of coordinates) that diagonalizes (completely uncouples) the system. Rayleigh [1] extended the use of normal mode analysis to a damped MDOF system, in which the damping matrix is a linear combination of its inertia and potential (stiffness) matrices. Caughey and O'Kelly showed in 1965 that such a symmetric damped system can be uncoupled by modal analysis if and only if its damping and potential matrices commute with respect to the inverse of its inertia matrix [2]. In this case, the system is said to be classically damped, and upon uncoupling, such a system can be treated as a series of independent single-degree-of-freedom subsystems. Rayleigh damping (proportional damping) is just a special case of classical damping. Today, classically damped systems are frequently assumed in the design and modeling of linear MDOF systems.

2020 *Mathematics Subject Classification:* 70J10, 34A30.

Key words and phrases: linear multi-degree-of-freedom dynamical system, potential force, damping force, circulatory force, diagonalization and quasi-diagonalization, real change of coordinates, orthogonal transformation, dynamics, vibrations.

The presence of gyroscopic and/or circulatory forces in an MDOF system makes it asymmetric and the system cannot be completely uncoupled through the use of a real congruence. The best that can be done is to uncouple the system into subsystems each of which has at most two-degrees-of-freedom. This is what has recently been accomplished for several categories of asymmetric MDOF systems [3–6], and necessary and sufficient (n&s) conditions for their uncoupling have been obtained. Because the behavior of a low-dimensional subsystem is considerably easier to understand than that of an MDOF system with numerous degrees of freedom, the decomposition of an MDOF system into such subsystems is useful in providing a better understanding of the MDOF system's behavior as well as in providing more accurate computational methods in the determination of its response to external forces.

Systems of interest in this paper are linear MDOF systems described by the equation

$$\tilde{M}\ddot{q} + \tilde{D}\dot{q} + \tilde{K}q + \tilde{N}q = \tilde{f}(t) \tag{1.1}$$

where $\tilde{M}, \tilde{D}, \tilde{K}$ and $\tilde{N}$ are n by n constant real matrices; $\tilde{M}$ is symmetric and positive definite ($\tilde{M} = \tilde{M}^T > 0$), $\tilde{D}$ and $\tilde{K}$ are symmetric, and $\tilde{N}$ is skew-symmetric ($\tilde{N} = -\tilde{N}^T$). $\tilde{M}$ is the inertia matrix, $\tilde{D}, \tilde{K}$ and $\tilde{N}$ correspond to damping, potential, and circulatory (positional non-conservative) forces, respectively [7]. The n-vector of generalized coordinates is denoted by q, $\tilde{f}(t)$ is the external forcing vector, and the dots indicate differentiation with respect to time, t. The matrices $\tilde{K}$ and $\tilde{N}$ can also be thought of as the symmetric and skew-symmetric additive parts, respectively, of a given stiffness matrix. Since any arbitrary matrix can be uniquely expressed as the sum of a symmetric and a skew-symmetric part, Eq. (1.1) also describes a damped multi-degree-of-freedom system whose stiffness matrix is arbitrary (non-symmetric). Systems modeled by Eq. (1.1) are found in various areas of physics and engineering (elasticity, fluid dynamics, rotor dynamics, controls, plasma physics, etc.).

Our overall goal is to obtain n&s conditions for the system described by (1.1) to be uncoupled so that a real change of coordinates $q = Pp$ where P is a real nonsingular matrix, transforms it into a canonical (simplest) form that is maximally uncoupled. It was previously shown in [3] that gyroscopic conservative systems can be uncoupled into at most two degrees of freedom independent subsystems when the appropriate two n&s conditions are met. Reference [4] extends this result to damped gyroscopic MDOF potential systems wherein the symetric damping matrix has a special form. Ref. [5] deals with developing the n&s conditions for uncoupling MDOF potential systems with arbitrary damping matrices through quasi-diagonalization, yielding independent subsystems with at most two degrees of freedom. MDOF gyroscopic systems with arbitrary stiffness matrices are considered in Ref. [6], and the n&s uncoupling conditions for such systems are obtained. The dynamical system (1.1) considered in this paper does not belong to any of the classes of systems considered in Refs. [3–6]; however, the approach developed in [5] can be applied for uncoupling it, as shown below.

The organization of the paper is as follows. In Section 2 some results in linear algebra are briefly presented, including a recent result on the simultaneous quasi-diagonalization (the term that will be specified later) of a skew-symmetric matrix and two symmetric matrices by means of real orthogonal congruence. This is applied in Section 3 to develop the conditions for the decoupling of MDOF damped circulatory systems by a real change of coordinates. Section 4 gives the conclusions.

2. Algebraic preliminaries

It is well known that for an $n \times n$ real skew-symmetric matrix N there exists an n by n real orthogonal matrix Q such that

$$Q^T N Q = \mathrm{diag}(v_1 J_2, \dots, v_{n/2} J_2) \quad \text{for} \quad n \quad \text{even} \\ = \mathrm{diag}(v_1 J_2, \dots, v_{(n-1)/2} J_2, 0) \quad \text{for} \quad n \quad \text{odd}, \tag{2.1}$$

where J_2 is the two-dimensional skew-symmetric matrix

$$J_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

and some of the real numbers v_j may be zero (see, for example [8, p. 65]). Furthermore, $J_2^2 = -I_2$, where I_2 is the 2 by 2 identity matrix.

The block-diagonal form of matrix (2.1), which we shall refer to as *quasi-diagonal*, is the simplest possible (canonical) form of a skew-symmetric matrix with respect to orthogonal similarities, while the canonical form for a real symmetric matrix is, of course, a diagonal matrix consisting of its eigenvalues along the diagonal. We note that if $\mathrm{Rank}(N) = 2m$ (the rank of a skew-symmetric matrix must be even), then m of the v_j are nonzero. The two-dimensional blocks appearing along the diagonal of matrix (2.1) can then be ordered, with no loss of generality, in such a way that the first m of them are nonzero, i.e., we can put

$$Q^T N Q = \mathrm{N} = \mathrm{diag}(v_1 J_2, \dots, v_m J_2, 0_{n-2m}), \tag{2.2}$$

where the real numbers $v_j \neq 0$, $j = 1, \dots, m$, and 0_{n-2m} is an $(n-2m)$ by $(n-2m)$ zero matrix. The nonzero numbers v_j correspond to (complex) conjugate pairs of purely imaginary eigenvalues of N, namely,$\pm v_j i$, $i = \sqrt{-1}$, with the zero eigenvalue of N having a multiplicity of $(n-2m)$. Note the distinction between N and N in (2.2): N is a real skew-symmetric matrix, while N is a quasi-diagonal, skew-symmetric matrix whose structure is given in (2.2).

The following assertion plays a key role in our further considerations.

THEOREM 2.1. *Let $K = K^T$, $D = D^T$ and $N = -N^T$ be $n \times n$ real matrices, and let be* $\mathrm{Rank}(N) = 2m$. *Necessary and sufficient conditions for a real orthogonal matrix Q to exist such that*

$$Q^T K Q = \Lambda = \mathrm{diag}(\lambda_1, \dots, \lambda_n), \qquad \textit{and} \tag{2.3}$$

$$Q^T D Q = \Delta = \mathrm{diag}(\delta_1, \dots, \delta_n), \qquad \textit{and} \tag{2.4}$$

$$Q^T N Q = \mathrm{N} = \mathrm{diag}(v_1 J_2, \dots, v_m J_2, 0_{n-2m}), \tag{2.5}$$

are that the following seven commutation conditions be met:

$$[K, D] = 0, \tag{2.6}$$

$$[K, N^2] = 0, \quad [K, NKN] = 0, \tag{2.7}$$

$$[D, N^2] = 0, \quad [D, NDN] = 0, \tag{2.8}$$

and

$$[K, NDN] = 0, \quad [D, NKN] = 0, \tag{2.9}$$

where the commutator of any two square matrices A and B is defined as $[A, B] := AB - BA$.

Proof. The proof of this theorem is given in Ref. [5]. □

When the matrices K, D and N can be reduced to the forms (2.3)–(2.5) by a real orthogonal congruence, we shall say that these matrices are *simultaneously quasi-diagonalized by the real orthogonal matrix Q*, or *simultaneously orthogonally quasi-diagonalized*, for short.

Remark 2.1. Conditions in (2.6)–(2.9) are equivalent to the symmetry of the following set of matrices

$$KD, \quad KG^2, \quad DN^2, \quad (KN)^2, \quad (DN)^2, \quad KNDN, \quad DNKN.$$

Remark 2.2. It can be verified by direct computation that the six conditions in (2.7)–(2.9) are satisfied when $n = 2$ and $[K, D] = 0$.

The following statement, which was obtained earlier in [9] (see also [3] and [10]), follows directly from Theorem 2.1.

Corollary 2.1. *Let $K = K^T$ and $N = -N^T$ be n by n real matrices, and let be* Rank$(N) = 2m$. *Necessary and sufficient conditions that there exists a real orthogonal matrix Q such that Q^TKQ and Q^TNQ are such as in* (2.3) *and* (2.5) *respectively, are that*

$$[K, N^2] = 0 \tag{2.10}$$

and

$$[K, NKN] = 0. \tag{2.11}$$

Proof. Application of Theorem 2.1 with $D = 0$ gives the result. □

Remark 2.3. When conditions (2.10) and (2.11) are satisfied, then

$$[K^j, NK^kN] = 0,$$

where j and k are non-negative integers. Indeed, under conditions (2.10) and (2.11), according to Corollary 2.1, there exists a real orthogonal matrix Q such that $K = Q\Lambda Q^T$ and $N = Q\mathrm{N}Q^T$ with Λ and N as in (2.3) and (2.5), respectively. Then $[K^j, NK^kN] = Q[\Lambda^j, \mathrm{N}\Lambda^k\mathrm{N}]Q^T = 0$ because Λ^j and $\mathrm{N}\Lambda^k\mathrm{N}$ are diagonal matrices.

We give the following three lemmas next, which will be used later.

Lemma 2.1. *Let $K = K^T$ and $N = -N^T$ be n by n real matrices. If all the nonzero eigenvalues of the skew-symmetric matrix N are distinct, then the condition*

$$[K, N^2] = 0$$

implies the condition

$$[K, NKN] = 0$$

Proof. See [3]. □

Lemma 2.2. *Let $K = K^T$, $D = D^T$ and $N = -N^T$ be n by n real matrices. If all the nonzero eigenvalues of the skew-symmetric matrix N are distinct, then the three conditions*

$$[K, D] = 0, \quad [K, N^2] = 0, \quad [D, N^2] = 0, \tag{2.12}$$

imply the conditions

$$[K, NKN] = 0, \quad [D, NDN] = 0, \tag{2.13}$$

and

$$[K, NDN] = 0, \quad [D, NKN] = 0. \tag{2.14}$$

Proof. See [5]. □

Lemma 2.3. *Let $K = K^T$, $D = D^T$ and $N = -N^T$ be n by n real matrices. If all eigenvalues of the symmetric matrix K are distinct, then the conditions*

$$[K, D] = 0, \quad [K, N^2] = 0, \quad [K, NKN] = 0, \tag{2.15}$$

imply the conditions

$$[D, N^2] = 0, \quad [D, NKN] = 0, \quad [K, NDN] = 0 \quad [D, NDN] = 0. \tag{2.16}$$

Proof. Since K and D commute and the matrix K has all distinct eigenvalues, the matrix D can be expressed in the polynomial form

$$D = \sum_{j=0}^{n-1} a_j K^j,$$

where a_j are real numbers [11]. Then we obtain

$$[D, N^2] = \sum_{j=0}^{n-1} a_j [K^j, N^2], \quad [D, NKN] = \sum_{j=0}^{n-1} a_j [K^j, NKN],$$

$$[K, NDN] = \sum_{j=0}^{n-1} a_j [K, NK^j N],$$

and

$$[D, NDN] = \sum_{j,k=0}^{n-1} a_j a_k [K^j, NK^k N].$$

From this, in view of Remark 2.3, we get (2.16). □

3. Uncoupling of damped circulatory systems

We begin with the observation that the change of coordinates $q = \tilde{M}^{-1/2}x$, where $\tilde{M}^{-1/2}$ denotes the inverse of the unique positive definite square root of $\tilde{M}$, transforms (1.1) to the simpler form

$$\ddot{x} + D\dot{x} + Kx + Nx = f(t), \tag{3.1}$$

where

$$D = D^T = \tilde{M}^{-1/2}\tilde{D}\tilde{M}^{-1/2}, \tag{3.2}$$

$$K = K^T = \tilde{M}^{-1/2}\tilde{K}\tilde{M}^{-1/2}, \tag{3.3}$$

$$N = -N^T = \tilde{M}^{-1/2}\tilde{N}\tilde{M}^{-1/2}, \tag{3.4}$$

and

$$f(t) = \tilde{M}^{-1/2}\tilde{f}(t). \tag{3.5}$$

The systems described in (1.1) and (3.1) are equivalent and we will focus mainly on (3.1) in the subsequent analysis. We shall refer to the matrices D, K, and N as the damping matrix, the potential matrix, and the circulatory matrix, respectively.

RESULT 3.1. *Consider the system described in Eq.* (3.1) *in which* $\mathrm{Rank}(N) = 2m$. *Then conditions*

$$[K, D] = 0 \tag{3.6}$$

$$[K, N^2] = 0 \quad [K, NKN] = 0, \tag{3.7}$$

$$[D, N^2] = 0 \quad [D, NDN] = 0, \tag{3.8}$$

$$[K, NDN] = 0 \quad [D, NKN] = 0, \tag{3.9}$$

are necessary and sufficient conditions for Eq. (3.1) *to be transformed by an orthogonal change of coordinates* $x = Qp$ *to the equation*

$$\ddot{p} + \Delta\dot{p} + \Lambda p + \mathrm{N}p = Q^T f(x) \tag{3.10}$$

with

$$\Delta = \mathrm{diag}(\delta_1, \dots, \delta_n), \tag{3.11}$$

$$\Lambda = \mathrm{diag}(\lambda_1, \dots, \lambda_n), \tag{3.12}$$

and

$$\mathrm{N} = \mathrm{diag}(v_1 J_2, \dots, v_m J_2, 0_{n-2m}), \tag{3.13}$$

where δ_j, λ_j, *and* v_j *are real numbers with* $v_j \neq 0$, $j = 1, \dots, m$.

PROOF. Using the real orthogonal transformation $x = Qp$ with $Q^TQ = I$, Eq. (3.1) after multiplication from the left by Q^T becomes

$$\ddot{p} + Q^TDQ\dot{p} + Q^TKQp + Q^TNQp = Q^Tf(x)$$

in the new coordinate p. Theorem 2.1 states that an orthogonal matrix Q exists, such that

$$Q^TDQ = \Delta, \quad Q^TKQ = \Lambda, \quad Q^TNQ = \mathrm{N}$$

where Δ, Λ, and N are as in (3.11), (3.12) and (3.13), if and only if conditions (3.6)–(3.9) are satisfied. □

COROLLARY 3.1. *The system described in Eq.* (3.1) *with two degrees-of-freedom can be transformed by an orthogonal congruence transformation to the form* (3.10)–(3.13) *if and only if* $[K, D] = 0$.

PROOF. See Remark 2.2. □

As seen, Eqs. (3.10)–(3.13) describe a set of *independent*, *uncoupled* (real) subsystems, m of which are quasi-diagonalized two-degree-of-freedom and $n - 2m$ of which are, in general, damped single-degree-of-freedom potential subsystems. Recall that $2m$ is the rank of the circulatory matrix N. When the skew-symmetric matrix N is nonsingular and conditions (3.6)–(3.9) are satisfied, then the system uncouples into $m = n/2$ independent two-degree-of-freedom subsystems, each generally functioning as a damped circulatory system. When n is odd, then the uncoupled system has at least one single degree of freedom subsystem which, in general, is a damped potential system.

The uncoupling conditions (3.6)–(3.9) trivially hold (disappear) in the following three cases: $D = N = 0$, $K = N = 0$ and $K = D = 0$. In the first two cases, it is well known that the system can be transformed to the completely uncoupled (diagonal) forms $\ddot{p} + \Lambda p = 0$ and $\ddot{p} + \Delta\dot{p} = 0$, respectively. In the third case the system can be reduced to the quasi-diagonal form $\ddot{p} + \mathrm{N}p = 0$. When $N = 0$, the uncoupling conditions disappear except for $[K, D] = 0$, which is a necessary and sufficient condition for the complete uncoupling (i.e., diagonalization) of damped potential systems (Caughey–O'Kelly [2]). On the other hand, if $D = 0$, conditions (3.6)–(3.9), clearly, become two conditions (3.7) that are necessary and sufficient for the quasi-diagonalization of potential circulatory systems, which were mentioned earlier in the case when the circulatory matrix N is nonsingular [12, 13].

Let us illustrate Result 3.1 with the following example.

EXAMPLE 3.1. Consider a four-degree-of-freedom system, as described above, in which

$$K = \begin{bmatrix} 4 & -2 & 0 & 0 \\ -2 & 4 & 0 & 0 \\ 0 & 0 & 5 & 3 \\ 0 & 0 & 3 & 5 \end{bmatrix}, \quad N = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0.2 & 0.1 & 0 & 0 \\ 0.1 & 0.2 & 0 & 0 \\ 0 & 0 & 0.3 & 0.2 \\ 0 & 0 & 0.2 & 0.3 \end{bmatrix}.$$

Evidently, $[K, N^2] = [D, N^2] = 0$ because $N^2 = -I$.

To determine if the other commutation conditions (3.6)–(3.9) are satisfied, we use Remark 2.1. We obtain:

$$KD = \frac{1}{10}\begin{bmatrix} 6 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 21 & 19 \\ 0 & 0 & 19 & 21 \end{bmatrix} = (KD)^T, \quad \text{i.e.,} \quad [K, D] = 0;$$

$$(KN)^2 = \begin{bmatrix} -14 & -2 & 0 & 0 \\ -2 & -14 & 0 & 0 \\ 0 & 0 & -14 & -2 \\ 0 & 0 & -2 & -14 \end{bmatrix} = (KN)^{2T}, \quad \text{i.e.,} \quad [K, NKN] = 0;$$

$$(DN)^2 = \frac{1}{100}\begin{bmatrix} -8 & -7 & 0 & 0 \\ -7 & -8 & 0 & 0 \\ 0 & 0 & -8 & -7 \\ 0 & 0 & -7 & -8 \end{bmatrix} = (DN)^{2T}, \quad \text{i.e.,} \quad [D, NDN] = 0;$$

$$KNDN = \frac{1}{10}\begin{bmatrix} -8 & -2 & 0 & 0 \\ -2 & -8 & 0 & 0 \\ 0 & 0 & -13 & -11 \\ 0 & 0 & -11 & -13 \end{bmatrix} = (KNDN)^T, \quad \text{i.e.,} \quad [K, NDN] = 0; \text{ and}$$

$$DNKN = \frac{1}{10}\begin{bmatrix} -13 & -11 & 0 & 0 \\ -11 & -13 & 0 & 0 \\ 0 & 0 & -8 & -2 \\ 0 & 0 & -2 & -8 \end{bmatrix} = (DNKN)^T, \quad \text{i.e.,} \quad [D, NKN] = 0.$$

Thus, conditions (3.6)–(3.9) of Result 3.1 are satisfied. Taking into account that $\text{Rank}(N) = 4$, i.e., $m = 2$, the system in this example can be transformed by a real orthogonal transformation into two independent two-dimensional subsystems. Indeed, one easily verifies that the orthogonal change of coordinates $x = Qp$, where

$$Q = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

decomposes the system into two independent, uncoupled two-degree-of-freedom subsystems described by

$$\begin{bmatrix} \ddot{p}_1 \\ \ddot{p}_2 \end{bmatrix} + \begin{bmatrix} 0.1\dot{p}_1 \\ 0.1\dot{p}_2 \end{bmatrix} + \begin{bmatrix} 6p_1 \\ 6p_2 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} f_1(t) - f_2(t) \\ f_3(t) - f_4(t) \end{bmatrix}$$

and

$$\begin{bmatrix} \ddot{p}_3 \\ \ddot{p}_4 \end{bmatrix} + \begin{bmatrix} 0.3\dot{p}_3 \\ 0.5\dot{p}_4 \end{bmatrix} + \begin{bmatrix} 2p_3 \\ 8p_4 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} p_3 \\ p_4 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} f_1(t) + f_2(t) \\ f_3(t) + f_4(t) \end{bmatrix}$$

In the case of proportional damping, the following result is obtained as a consequence of Result 3.1.

Corollary 3.2. *Let* $K = K^T$, $D = aI + bK$, $N = -N^T$ *and* $\text{Rank}(N) = 2m$. *Then conditions* (3.7), *i. e.,*

$$[K, N^2] = 0, \quad [K, NKN] = 0,$$

are necessary and sufficient for Eq. (3.1) *to be transformed by an orthogonal change of coordinates* $x = Qp$ *to the form*

$$\ddot{p} + (aI + b\Lambda)\dot{p} + \Lambda p + \mathrm{N}p = Q^T f(t)$$

with Λ *and* N *as in Eqs.* (3.12) *and* (3.13).

PROOF. When $D = aI + bK$ and two commutation conditions (3.7) are satisfied, we obtain

$$[K, D] = [K, aI + bK] = a[K, I] + b[K, K] = 0,$$
$$[D, N^2] = [aI + bK, N^2] = a[I, N^2] + b[K, N^2] = 0,$$
$$[D, NKN] = [aI + bK, NKN] = a[I, NKN] + b[K, NKN] = 0,$$
$$[K, NDN] = [K, aN^2 + bNKN] = a[K, N^2] + b[K, NKN] = 0, \quad \text{and}$$
$$[D, NDN] = a[I, NDN] + b[K, NDN] = 0$$

Thus, all the commutation conditions of Result 3.1 are satisfied, and the statement follows. It is worth noting that a more direct proof could have been obtained by applying Corollary 2.1. □

REMARK 3.1. When the non-zero eigenvalues of the circulatory matrix N are distinct then, because of Lemma 2.1, the two commutation conditions in Corollary 3.2 reduce to just the single condition $[K, N^2] = 0$.

COROLLARY 3.3. *If the matrices K, D and N commute pairwise, i.e.,*

$$[K, D] = 0, \quad [K, N] = 0, \quad [D, N] = 0, \tag{3.14}$$

and $\mathrm{Rank}(N) = 2m$*, then there exists an orthogonal change of coordinates that transforms Eq.* (3.1) *to the form* (3.10) *with*

$$\Lambda = \mathrm{diag}(\lambda_1 I_2, \ldots, \lambda_m I_2, \lambda_{2m+1}, \ldots, \lambda_n), \tag{3.15}$$

and

$$\Delta = \mathrm{diag}(\delta_1 I_2, \ldots, \delta_m I_2, \delta_{2m+1}, \ldots, \delta_n), \tag{3.16}$$

and N *as in* (3.13).

PROOF. Evidently, if the matrices K, D and N commute pairwise, then conditions (3.6)–(3.9) are satisfied, and, according to Result 3.1, there exists a real orthogonal transformation that transforms Eq. (3.1) to the form (3.10)–(3.13). Moreover, the two last conditions in (3.14) correspond to the conditions $[\Lambda, \mathrm{N}] = 0$ and $[\Delta, \mathrm{N}] = 0$ that require $\lambda_1 = \lambda_2, \lambda_3 = \lambda_4, \ldots, \lambda_{2m-1} = \lambda_{2m}$ and $\delta_1 = \delta_2, \delta_3 = \delta_4, \ldots, \delta_{2m-1} = \delta_{2m}$, because $v_j \neq 0$. Denoting repeated numbers λ_j and δ_j with $\lambda_1, \lambda_2, \ldots, \lambda_m$ and $\delta_1, \delta_2, \ldots, \delta_m$ we get (3.15) and (3.16). □

The pairwise commutation of K, D and N given in conditions (3.14) ensures that the conditions in (3.6)–(3.9) are all satisfied. However, the reverse is not true. This is because the set of matrices $\{K, D, N\}$ that satisfy (3.6)–(3.9) is much "larger" than the set that satisfies (3.14). As a simple example, when $n = 2$ and K is proportional to the identity matrix, all 2 by 2 matrices D and N satisfy conditions (3.6)–(3.9), while the satisfaction of (3.14) restricts the matrix D to being proportional to the identity matrix.

When the matrices K, D and N have certain spectral characteristics that often occur in real-world applications, the number of necessary and sufficient uncoupling conditions in Result 3.1 can be reduced as follows.

Result 3.2. *The system described in Eq.* (3.1) *in which* $\mathrm{Rank}(N) = 2m$ *can be reduced to the form given in Eqs.* (3.10)–(3.13) *using a real orthogonal transformation*

(a) *When the eigenvalues of the potential matrix K are distinct: if and only if*

$$[K, D] = 0, \quad [K, N^2] = 0, \quad [K, NKN] = 0; \tag{3.17}$$

(b) *When the nonzero eigenvalues of the circulatory matrix N are distinct: if and only if*

$$[K, D] = 0, \quad [K, N^2] = 0, \quad [D, N^2] = 0; \tag{3.18}$$

(c) *When the eigenvalues of the potential matrix K are distinct and the nonzero eigenvalues of the circulatory matrix N are distinct: if and only if*

$$[K, D] = 0, \quad [K, N^2] = 0. \tag{3.19}$$

Proof. Parts (a) and (b) follow readily from Result 3.1 and Lemmas 2.3 and 2.2, respectively. Part (c) follows from part (a) and Lemma 2.1. □

Remark 3.2. The roles of K and D can be interchanged in the parts (a) and (c) of the above assertion. Thus, when the eigenvalues of the damping matrix D are distinct then the conditions in (3.17) and (3.19), in which the symbols K and D are interchanged, are necessary and sufficient for (3.1) to be transformed using a real orthogonal transformation to (3.10)–(3.13).

Example 3.2. Consider the four-degree-of-freedom system described by (3.1) in which

$$K = \begin{bmatrix} 7 & 3 & -1 & -1 \\ 3 & 9 & 3 & 0 \\ -1 & 3 & 7 & 1 \\ -1 & 0 & 1 & 7 \end{bmatrix}, \quad N = \begin{bmatrix} 0 & -5 & -2 & -4 \\ 5 & 0 & -5 & -2 \\ 2 & 5 & 0 & -4 \\ 4 & 2 & 4 & 0 \end{bmatrix},$$

$$D = \begin{bmatrix} 1.3 & 0.1 & -0.3 & -0.2 \\ 0.1 & 1.1 & 0.1 & 0 \\ -0.3 & 0.1 & 1.3 & 0.2 \\ -0.2 & 0 & 0.2 & 1.4 \end{bmatrix}.$$

The spectra of N and K are $\{\pm 3\sqrt{2}i, \pm 6\sqrt{2}i\}$ and $\{3, 6, 9, 12\}$ respectively, and we can apply Result 3.2(c). We obtain

$$KD = \begin{bmatrix} 9.9 & 3.9 & -3.3 & -3 \\ 3.9 & 10.5 & 3.9 & 0 \\ -3.3 & 3.9 & 9.9 & 3 \\ -3 & 0 & 3 & 10.2 \end{bmatrix} = (KD)^T, \quad \text{i.e.,} \quad [K, D] = 0,$$

and

$$KN^2 = -18 \begin{bmatrix} 22 & 15 & -4 & -10 \\ 15 & 33 & 15 & 0 \\ -4 & 15 & 22 & 10 \\ -10 & 0 & 10 & 16 \end{bmatrix} = (KN^2)^T, \quad \text{i.e.,} \quad [K, N^2] = 0.$$

Thus all conditions of Result 3.2(c) are satisfied and the system can be transformed by a real orthogonal transformation into two independent two-dimensional subsystems. Indeed, the orthogonal coordinate transformation $x = Qp$, where

$$Q = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & -1/\sqrt{2} & 1 & -1/\sqrt{2} \\ -1 & 0 & 0 & -\sqrt{2} \\ 1 & 1/\sqrt{2} & -1 & -1/\sqrt{2} \\ 0 & -\sqrt{2} & -1 & 0 \end{bmatrix}$$

transforms the system into the following uncoupled form that has two independent two-degree-of-freedom subsystems

$$\begin{bmatrix} \ddot{p}_1 \\ \ddot{p}_2 \end{bmatrix} + \begin{bmatrix} 0.9\dot{p}_1 \\ 1.2\dot{p}_2 \end{bmatrix} + \begin{bmatrix} 3p_1 \\ 6p_2 \end{bmatrix} + 3\sqrt{2} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} g_1(t) \\ g_2(t) \end{bmatrix}$$

$$\begin{bmatrix} \ddot{p}_3 \\ \ddot{p}_4 \end{bmatrix} + \begin{bmatrix} 1.8\dot{p}_3 \\ 1.2\dot{p}_4 \end{bmatrix} + \begin{bmatrix} 9p_3 \\ 12p_4 \end{bmatrix} + 6\sqrt{2} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} p_3 \\ p_4 \end{bmatrix} = \begin{bmatrix} g_3(t) \\ g_4(t) \end{bmatrix}$$

where $[g_1(t)\ g_2(t)\ g_3(t)\ g_4(t)]^T = Q^T f(t)$ with

$$g_1(t) = \frac{1}{\sqrt{3}}(f_1(t) - f_2(t) + f_3(t)), \qquad g_2(t) = \frac{1}{\sqrt{6}}(f_3(t) - f_1(t) - 2f_4(t)),$$

$$g_3(t) = \frac{1}{\sqrt{3}}(f_1(t) - f_3(t) - f_4(t)) \quad \text{and} \quad g_4(t) = -\frac{1}{\sqrt{6}}(f_1(t) + 2f_2(t) + f_3(t)).$$

All of the above results can be translated for the original system described by Eq. (1.1) using Eqs. (3.2)–(3.4). For example, taking into account that the eigenvalues of the matrices D, K and N are the same as those of $\tilde{M}^{-1}\tilde{D}, \tilde{M}^{-1}\tilde{K}$ and $\tilde{M}^{-1}\tilde{N}$ respectively, Result 3.2 can be translated for (1.1) as follows.

Result 3.3. *The system described in Eq.* (1.1) *in which* $\mathrm{Rank}(\tilde{N}) = 2m$ *can be reduced by a real change of coordinates* $q = Pp$ *to the equation*

$$\ddot{p} + \Delta\dot{p} + \Lambda p + \mathrm{N}p = P^T \tilde{f}(t) \tag{3.20}$$

where Δ, Λ *and* N *are as in Eqs.* (3.11)–(3.13)

(a) *When the eigenvalues of the matrix* $\tilde{M}^{-1}\tilde{K}$ *are distinct: if and only if*

$$\tilde{K}\tilde{M}^{-1}\tilde{D} = \tilde{D}\tilde{M}^{-1}\tilde{K}, \quad \tilde{K}\tilde{M}^{-1}\tilde{N}\tilde{M}^{-1}\tilde{N} = \tilde{N}\tilde{M}^{-1}\tilde{N}\tilde{M}^{-1}\tilde{K}, \quad (\tilde{K}\tilde{M}^{-1}\tilde{N}\tilde{M}^{-1})^2 = (\tilde{N}\tilde{M}^{-1}\tilde{K}\tilde{M}^{-1})^2; \tag{3.21}$$

(b) *When the nonzero eigenvalues of the matrix* $\tilde{M}^{-1}\tilde{N}$ *are distinct: if and only if*

$$\tilde{K}\tilde{M}^{-1}\tilde{D} = \tilde{D}\tilde{M}^{-1}\tilde{K}, \quad \tilde{K}\tilde{M}^{-1}\tilde{N}\tilde{M}^{-1}\tilde{N} = \tilde{N}\tilde{M}^{-1}\tilde{N}\tilde{M}^{-1}\tilde{K}, \quad \tilde{D}\tilde{M}^{-1}\tilde{N}\tilde{M}^{-1}\tilde{N} = \tilde{N}\tilde{M}^{-1}\tilde{N}\tilde{M}^{-1}\tilde{D}; \tag{3.22}$$

(c) *When the eigenvalues of the matrix* $\tilde{M}^{-1}\tilde{K}$ *are distinct and the nonzero eigenvalues of the matrix* $\tilde{M}^{-1}\tilde{N}$ *are distinct: if and only if*

$$\tilde{K}\tilde{M}^{-1}\tilde{D} = \tilde{D}\tilde{M}^{-1}\tilde{K}, \tilde{K}\tilde{M}^{-1}\tilde{N}\tilde{M}^{-1}\tilde{N} = \tilde{N}\tilde{M}^{-1}\tilde{N}\tilde{M}^{-1}\tilde{K}. \tag{3.23}$$

Also, in view of Corollary 3.2 and Remark 3.1, we state the following assertion.

Corollary 3.4. *Let $\tilde{D} = a\tilde{M} + b\tilde{K}$, $\mathrm{Rank}(\tilde{N}) = 2m$ and the nonzero eigenvalues of the matrix $\tilde{M}^{-1}\tilde{N}$ are distinct. Then the condition*

$$\tilde{K}\tilde{M}^{-1}\tilde{N}\tilde{M}^{-1}\tilde{N} = \tilde{N}\tilde{M}^{-1}\tilde{N}\tilde{M}^{-1}\tilde{K}$$

is necessary and sufficient for Eq. (1.1) to be transformed by a real change of coordinates $x = Pp$ to the form (3.20) with Λ and N as in Eqs. (3.12) and (3.13).

Example 3.3. Consider the three-degree-of-freedom system described by (1.1) in which

$$\tilde{M} = \begin{bmatrix} 5 & 0 & 4 \\ 0 & 4 & 0 \\ 4 & 0 & 5 \end{bmatrix}, \quad \tilde{K} = \begin{bmatrix} 13 & -3 & 14 \\ -3 & 10 & 3 \\ 14 & 3 & 13 \end{bmatrix}, \quad \tilde{N} = \begin{bmatrix} 0 & 2 & 0 \\ -2 & 0 & 2 \\ 0 & -2 & 0 \end{bmatrix},$$

$$\tilde{D} = \begin{bmatrix} 0.96 & -0.06 & 0.84 \\ -0.06 & 0.76 & 0.06 \\ 0.84 & 0.06 & 0.96 \end{bmatrix}.$$

We first calculate

$$\tilde{M}^{-1} = \frac{1}{9}\begin{bmatrix} 5 & 0 & -4 \\ 0 & 2.25 & 0 \\ -4 & 0 & 5 \end{bmatrix}$$

and then we find that the spectrum of $\tilde{M}^{-1}\tilde{K}$ is $\{-2, 3, 3.5\}$, i.e., the matrix $\tilde{M}^{-1}\tilde{K}$ has distinct eigenvalues. Furthermore, since $\tilde{N}$ is a three-dimensional nonzero skew-symmetric matrix, the matrix $\tilde{M}^{-1}\tilde{N}$ has a conjugate pair of purely imaginary eigenvalues, and therefore Result 3.3(c) can be applied. Thus, we obtain

$$\tilde{K}\tilde{M}^{-1}\tilde{D} = \begin{bmatrix} 2.685 & -0.51 & 2.715 \\ -0.51 & 2.26 & 0.51 \\ 2.715 & 0.51 & 2.685 \end{bmatrix} = (\tilde{K}\tilde{M}^{-1}\tilde{D})^T = \tilde{D}\tilde{M}^{-1}\tilde{K}$$

and

$$\tilde{K}\tilde{M}^{-1}\tilde{N}\tilde{M}^{-1}\tilde{N} = \begin{bmatrix} 1 & 6 & -1 \\ 6 & -20 & -6 \\ -1 & -6 & 1 \end{bmatrix} = (\tilde{K}\tilde{M}^{-1}\tilde{N}\tilde{M}^{-1}\tilde{N})^T = \tilde{N}\tilde{M}^{-1}\tilde{N}\tilde{M}^{-1}\tilde{K}$$

and, according to Result 3.3(c), there exists a change of coordinates $q = Pp$ that decomposes the system into two independent subsystems: one with two degrees of freedom and another with a single degree of freedom. Indeed, the transformation $q = Pp$ with

$$P = \begin{bmatrix} \frac{3}{\sqrt{22}} & \frac{1}{\sqrt{11}} & \frac{1}{3\sqrt{2}} \\ \frac{1}{\sqrt{22}} & -\frac{3}{2\sqrt{11}} & 0 \\ -\frac{3}{\sqrt{22}} & -\frac{1}{\sqrt{11}} & \frac{1}{3\sqrt{2}} \end{bmatrix}$$

reduces the system to the form

$$\begin{bmatrix} \ddot{p}_1 \\ \ddot{p}_2 \end{bmatrix} + \begin{bmatrix} 0.1\dot{p}_1 \\ 0.21\dot{p}_2 \end{bmatrix} + \sqrt{2}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} + \begin{bmatrix} -2p_1 \\ 3.5p_2 \end{bmatrix} = \begin{bmatrix} g_1(t) \\ g_2(t) \end{bmatrix}$$

$$\ddot{p}_3 + 0.2\dot{p} + 3p_3 = g_3(t),$$

where $[g_1(t) \quad g_2(t) \quad g_3(t)]^T = P^T\tilde{f}(t)$ with $g_1(t) = \frac{1}{\sqrt{22}}(3\tilde{f}_1(t) + \tilde{f}_2(t) - 3\tilde{f}_3(t))$, $g_2(t) = \frac{1}{2\sqrt{11}}(2\tilde{f}_1(t) - 3\tilde{f}_2(t) - 2\tilde{f}_3(t))$ amd $g_3(t) = \frac{1}{3\sqrt{2}}(\tilde{f}_1(t) + \tilde{f}_3(t))$.

Note that the matrix $\tilde{D}$ of this example can be expressed as $\tilde{D} = 0.14\tilde{M} + 0.02\tilde{K}$ and therefore Corollary 3.4 can also be applied.

Remark 3.3. The results presented above may be interpreted as counterparts to the uncoupling results obtained via a real coordinate transformation, recently derived for other mathematically and physically distinct classes of linear asymmetric dynamical systems [3–6].

4. Conclusion

This paper investigates linear n-degree-of-freedom (MDOF) systems with potential, damping and circulatory forces, building upon our earlier work in the uncoupling of linear dynamical systems [3–6].

Since the circulatory matrix is skew, it precludes the decomposition of such a system into n uncoupled subsystems through the use of a *real* coordinate change. The best that can be done using a real coordinate change is to uncouple the system into subsystems each of which has at most two-degrees-of-freedom. The conditions for such a decoupling are provided here. The main findings are summarizes below.

In the general case, the MDOF system under consideration, in which the circulatory matrix has rank $2m \leqslant n$, can be decomposed by a suitable real linear change of coordinates into m uncoupled two-degree-of-freedom subsystems and $(n - 2m)$ single-degree of freedom subsystems if and only if the seven conditions obtained in the paper are satisfied. Each of the two-degree-of-freedom subsystems corresponds to a circulatory system in canonical form (with diagonal potential and damping matrices), while each of the single-degree-of-freedom subsystems corresponds to a potential system. Such an uncoupling is useful in providing a deeper physical understanding of the behavior of an MDOF system with numerous degrees of freedom in terms of one- and two-degree-of-freedom subsystems that are much simpler and easier to understand, as well as in providing more accurate computational methods in the determination of its response to external forces.

The seven n&s uncoupling conditions, as expected place restrictions on the system's matrices. It is shown that the incorporation of additional information about the spectra of the matrices, which many real-life systems commonly possess, enables a reduction in the number of these conditions. When the potential matrix K has distinct eigenvalues, the number of n&s uncoupling conditions reduce to three; when the non-zero eigenvalues of the circulatory matrix N are distinct, they reduce also to three; and when the eigenvalues of K are distinct and non-zero eigenvalues of N are distinct, they reduce to two.

Several illustrative examples are considered throughout the paper to give clarity to the analytical results that are obtained.

Lastly, we point out that the results obtained here are also useful in analysing the behavior of MDOF nonlinear dynamical systems in the vicinity of their equilibria, about which they can be linearized.

References

1. J. W. S. Rayleigh, *The Theory of Sound*, Vol. 1, 1894; reprinted by Dover, New York, 1945.
2. T. K. Caughey, M. E. J. O'Kelly, *Classical normal modes in damped linear dynamic systems*, J. Appl. Mech. **32** (1965), 583–588.
3. R. M. Bulatovic, F. E. Udwadia, *Decomposition and uncoupling of multi-degree-of-freedom gyroscopic conservative systems*, J. Appl. Mech. **91**(3) (2024), 031003.
4. F. E. Udwadia, R. M. Bulatovic, *Uncoupling of linear multi-degree-of-freedom damped gyroscopic potential systems*, J. Appl. Mech. **91**(4) (2024), 041007.
5. F. E. Udwadia, R. M. Bulatovic, *Uncoupling of damped linear potential multi-degree-of-freedom structural and mechanical systems*, J. Appl. Mech. **91**(9) (2024), 091004.
6. R. M. Bulatovic, F. E. Udwadia, *On the quasi-diagonalization and uncoupling of gyroscopic circulatory multi-degree-of-freedom systems*, J. Appl. Mech. **92**(2) (2025), 021005.
7. D. R. Merkin, *Introduction to the Theory of Stability*, Springer-Verlag, New York, 1997.
8. R. Bellman, *Introduction to Matrix Analysis*, McGraw-Hill, New York, 1970.
9. R. M. Bulatovich, *Simultaneous reduction of a symmetric matrix and a skew-symmetric one to canonical form*, Math. Montisnigri **8** (1997), 33–36. (in Russian)
10. Kh. D. Ikramov, *On a quasidiagonalizability criterion for real matrices*, Comput. Math. Math. Phys. **40** (2000), 4–17.
11. R. A. Horn, C. R. Johnson, *Matrix Analysis*, Cambridge University Press, Cambridge, 1985.
12. R. M. Bulatovic, *On the stability of potential systems under the action of non-conservative positional forces*, Regul. Chaotic Dyn. **25**(4) (2020), 330–337.
13. R. M. Bulatovic, *On the stability of linear circulatory systems*, Z. Angew. Math. Phys. **50** (1999), 669–674.

О КВАЗИ-ДИЈАГОНАЛИЗАЦИЈИ И РАСПРЕЗАЊУ ПРИГУШЕНИХ ЦИРКУЛАТОРНИХ СИСТЕМА СА КОНАЧНИМ БРОЈЕМ СТЕПЕНИ СЛОБОДЕ

Резиме. Разматра се декомпозиција линеарних пригушених циркулаторних система са коначним бројем степени слободе помоћу реалне линеарне координатне трансформације, генерисане ортогоналном матрицом. Изведени су критеријуми који садрже услове егзистенције координатних трансформација које омогућавају декомпозицију система на независне, међусобно неспрегнуте подсистеме, при чему ниједан од њих нема више од два степена слободе. Критеријуми су изражени кроз карактеристике описних матрица система. Неколико нумеричких примјера дато је ради илустрације аналитичких резултата.

Faculty of Mechanical Engineering
University of Montenegro
Podgorica
Montenegro
ranislav@ucg.ac.me
http://orcid,org/0009-0006-6952-6299

(Received 08.01.2025)
(Revised 02.09.2025)
(Available online 29.09.2025)

Civil and Environmental Engineering, and
Aerospace and Mechanical Engineering
University of Southern California
Los Angeles
CA, USA
feuusc@gmail.com