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VORTICES FOR LAKE EQUATIONS
(review with questions and speculations)

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To Darryl Holm, as a token of my esteem.

Abstract. The 'lake equation' on a planar domain D with bathymetry $b(x,y)$ is given by $\partial_t u + (u \cdot \operatorname{grad})u = -\operatorname{grad} p$, $\operatorname{div}(bu) = 0$, with $u \parallel \partial D$. It is well posed as a PDE, but when $b \neq \text{const}$, justifying point vortex models requires the analyst's attention. We focus on Geometric Mechanics aspects, glossing over hard analysis issues. The motivating example is a 'rip current' produced by vortex pairs near a beach shore. For a beach with uniform slope, there is a perfect analogy with Thomson's vortex rings. The stream function produced by a vortex is defined as the Green function of the operator $-\operatorname{div}(\operatorname{grad}\psi/b)$ with Dirichlet boundary conditions. As in elasticity, the lake equations give rise to pseudo-analytical functions and quasi-conformal mappings. Uniformly elliptic equations on close Riemann surfaces could be called 'planet equations'.

1. Introduction

In 1996, Camassa, Holm and Levermore [1,2] introduced the 'small' lake equation. It is a PDE for a planar vector field $u(x,y)$ on a domain D,

$$\partial_t u + (u \cdot \operatorname{grad})u = -\operatorname{grad} p, \quad \operatorname{div}(bu) = 0, \quad u \parallel \partial D$$

where $b(x,y)$ is the bathymetry. It represents a mean field limit of shallow waters when a small vertical component is averaged out. Well-posedness as a PDE was shown in [3]. Vertical columns of fluid supposedly move horizontally in unison, so one adjusts the 2d area form to make the fluid 3d-incompressible: div(bu) = 0. Thus the velocity fields $u \in \mathrm{sDiff}_{\tilde{\mu}}(D)$, with $\tilde{\mu} = b dx \wedge dy$.

Upon D. Holm's suggestion, G. Richardson [4] studied the small lake equation. Applying matching asymptotic expansions around a vortex patch of size $\sim \epsilon$ (for better approximations, one needs information about the inner vortex structure), he showed that the patch has a self-velocity, given in leading order by

$$\frac{\Gamma}{4\pi}\log\left(\frac{1}{\epsilon}\right)\nabla^{\perp}\log b \quad (\nabla^{\perp} = -i\nabla = (\partial/\partial y, -\partial/\partial x)). \tag{1.1}$$

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Thus if $b \neq const.$ a point vortex ($\epsilon = 0$) moves with infinite self-velocity.

This contrasts with the consensus for point vortices when $b \equiv 1$. In that case, the self-velocity is finite and governed by the Robin function R of the Green function G_Δ for the Laplacian, with Dirichlet boundary conditions

$$R(z) = \lim_{\zeta \to z} G_\Delta(z, \zeta) + 1/2\pi \log |z - \zeta| \quad \text{(see eg. [5])}.$$

Indeed, the use of point vortices for the traditional Euler equations in planar domains is classic [6]. Since the 1980's with Marchioro and Pulvirenti [7], its validity has been considered common knowledge. For recent work, see e.g. [8, 9].

Following the core energy method in [10], it was proposed in [11] that in closed Riemann surfaces with a metric, desingularization could be done adding $1/2\pi\, d(p, q)$ to the Green function of the metric Laplacian. This still lacks a real proof although it produces an expected result: close by vortex pairs mimic a geodesic. Once again, the self-motion would be governed by the Robin function. C. Ragazzo studied the Robin function of Bolza's surface [12, 13].

However, B. Gustafsson [14] pointed out that when the surface genus is $\geqslant 1$, the Hamiltonian in [11] is incomplete. One needs also to consider the potential flows and he showed how to describe their interaction with the vortices. Together with Gustafsson and Ragazzo, we developed this theme in [15] *that contains the background for this paper.* A special case is a multiply connected planar domain, that becomes a "pancake" surface via its Schottky double.

The aim here is to outline how to extend the geometric description in [15] *to the lake equations.* In section 2, we describe the analogy of the classic Thomson's vortex rings with return ('rip') currents. The established Hamiltonian description for the motion of vortex rings (see [16, 17] for history) suggests an analogue for the lake equations, presented in section 3. In section 4, the stream function of a vortex is introduced: it is the Green function of $-\operatorname{div}(\operatorname{grad} \psi / b)$. Approximations for the Green function are discussed and a simple toy model for a rip current is presented. Sections 11 and 12 treat multiply connected domains[1]. Extension to closed Riemann surfaces is outlined in section 13, and and we finish with two short comments in section 15.

Studies about the limits of validity of point vortex approximations were done for the lake equation [18–23]. They are not discussed here.

2. Correspondence of vortex rings and rip currents

The infinite self-velocity for a 'pure' point vortex ($\epsilon = 0$) in the lake equation is not that surprising if one is familiar with William Thomson's (later Lord Kelvin) torus vortex [24–28].

The self-velocity along the axis is given by

$$\Gamma[\log(8r/a) - C)/4\pi r$$

where a is the inner radius of the torus, and r is its radius, so it diverges as $a \to 0$. The inner vorticity structure determines the value of constant C.

[1] Our results essentially coincide with the description in Dekeyser and Schaftingen [20].

Recent experimental studies are [29–35]. For a mathematical proof of vortex rings see [36]. We draw attention to studies [37–40]. What is the connection with lake equations?

PROPOSITION 2.1. *Euler's equation for axisymmetric flows without swirl in cylindrical coordinates* (x, r, ϕ) *coincides with the lake equations on* $(x, y), y \geqslant 0$ *with* $b(x, y) = y$*.* (Make $r \leftrightarrow y$ and ignore ϕ.)

'Morally' a (from the torus ring) $\leftrightarrow \epsilon$ (the vortex core) (see [41]), so there is an analogy of rip currents (informations below) with the collision of two equal vortex rings, [33, 42]. The later is a detailed analysis of the turbulent collision.

See Fig. 1 and Fig. 2. Imagine two opposite vortices at the same distance from the shore on a sloping beach, with the positive vortex positioned to the right of the negative vortex. As they move towards the ocean, they approach each other.

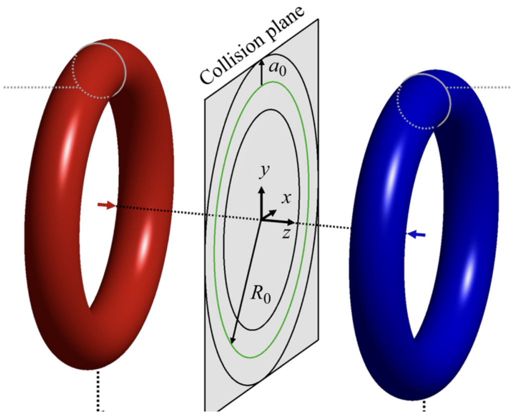

FIGURE 1. Collision of two vortex rings. An important parameter is the thickness. Adapted from [42]. Recent videos: [43, 44].

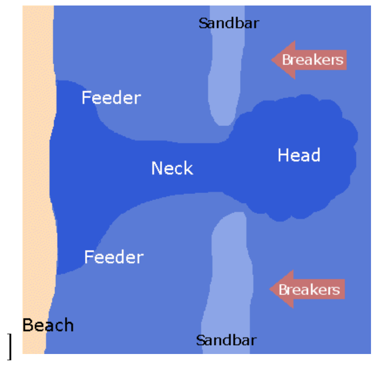

FIGURE 2. Source: wikipedia (public domain). The wide head of the current indicates the merge of vortex couples.

A sequence of such vortex pairs produces the rip current. The current dissolves at the 'head'. This is similar to the mutual destruction of the colliding rings that produces subsidiary small rings all over, but represents a different physical mechanism. Videos: [43, 44].

As Richard Feynman used to say, "same equations have same solutions". Thus, for the lake equations on a uniformly sloping beach, one can use *ipsis verbis* the quite extensive literature on coaxial vortex rings.

3. Geometric Mechanics of lake equations. Green function

In Arnold-Khesin approach [45, 46] for Euler equations, the energy functional

$$\frac{1}{2}\int_M g(u,u)\tilde{\mu}, \quad u \in \mathrm{sDiff}_{\tilde{\mu}}$$

is considered on a Riemannian manifold (M, g) with a volume form $\tilde{\mu}$ *that can be unrelated* to the metric g in M.

THEOREM 3.1. *Euler's equation becomes*

$$u_t + (v, \nabla)u = -\operatorname{grad} p, \quad \operatorname{div}_{\tilde{\mu}} u = 0 \qquad (\nabla \textit{ is the covariant differential of } g).$$

Geometrically, it is better treated dualizing to $\mathrm{sDiff}^*_{\tilde{\mu}}$ *via the musical isomorphim* relative to g, $\nu = u^\flat$, $\nu^\sharp = u$ *and Euler equations rewritten as*

$$\nu_t + L_u\nu = \text{exact} \quad (u = \nu^\sharp \in \mathrm{sDiff}^*_{\tilde{\mu}}).$$

DEFINITION 3.1. The vorticity is defined as a 2-form $\omega = d\nu$.

PROPOSITION 3.1. *Helmholtz's transport formula*

$$\omega_t + L_u\omega = 0 \quad (L_u \textit{ is the Lie derivative})$$

holds exactly the same way as in the usual case in which $\tilde{\mu}$ *is the volume form of* g.

One important consequence is the concept of *isovorticity*:

COROLLARY 3.1. *The vorticity at any moment is transported to the vorticity at any other moment of time by a diffeomorphism preserving the volume element.*

An important information is that, in 2d, isovorticity is equivalent to being in the same coadjoint orbit [47, 48]. In 3d, the situation is more complicated. What does isovorticity imply for 2d point vortex models? This is clear: the ambient *must have constant vorticity*, which is usually zero on a bounded domain, or a constant counter-value on a closed Riemann surface.

For the lake equations, a stream function $\psi(x, y)$ gives rise to the vector field

$$u = (1/b)\nabla^\perp\psi,$$

so that $\operatorname{div}(bu) = 0$. The vorticity vector field is

$$\omega = \operatorname{curl}(\nabla^\perp\psi/b) = -\operatorname{div}(\operatorname{grad}\psi/b)\,\hat{k}.$$

which leads us to consider the elliptic operator, *with Dirichlet boundary conditions*,

$$L_b\psi = -\operatorname{div}\left(\frac{1}{b}\operatorname{grad}\psi\right). \tag{3.1}$$

The stream function of a (bound) point vortex is taken *by definition* to be the Green function $G_{L_b}(z\,;z_o)$ of L_b *vanishing at the boundaries* that we denote G_b for short. The stream function ψ corresponding to a distributed vorticity ω is recovered via

$$\psi(z) = \int_D G_b(z,\zeta)\omega(\zeta)dx\,dy. \tag{3.2}$$

Dualizing to $\nu = u^\flat$ via the Euclian metric, incompressibility entails

$$d(b \star \nu) = 0$$

In the Appendix A of [15] we summarize the properties of the Hodge star $\star$, a conformal invariant object.

Remark 3.1. It may be premature to draw conclusions, but when the domain D is multiply connected the kernel of L_b is non-empty the pair of equations

$$d\eta = 0 \text{ (being irrotational)}, \quad d(b \star \eta) = 0 \text{ (being incompressible)} \tag{3.3}$$

defines the pseudo-harmonic 1-forms. For $b \equiv 1$ these are the harmonic forms.

On closed Riemann surfaces those forms belong to a $2g$ dimensional space, but on a planar domain only half of those forms on the Schottky double are used, i.e., those that are dual to the inner boundaries.

Hamiltonian for vortices on lake equations. The literature on coaxial vortex rings suggests a natural proposal. Let us assume, for the moment, that the domain is simply connected. The velocity $u(x, y)$ for a marker (particle $z = x + iy$ with its $\Gamma \to 0$) is

$$\begin{aligned} u(x,y) &= \frac{1}{b(x,y)}\nabla^\perp\psi, \\ \psi(x,y;z_1,\dots,z_N) &= \sum_{k=1}^N \Gamma_k G_{L_b}(z,z_k). \end{aligned} \tag{3.4}$$

using the Green function G_{L_b} of the elliptic operator (3.1) (more next).

The Hamiltonian system will be

$$\begin{aligned} H &= \sum_{j<k}\Gamma_i\Gamma_k G_{L_b}(z_j,z_k) + \frac{1}{2}\sum_j \Gamma_j^2\,\mathrm{Rich}_b(z_j) \\ \Omega &= \sum_j \Gamma_j b(z_j) dx_j \wedge dy_j. \end{aligned} \tag{3.5}$$

where the vortex self-velocity comes from Richardson's (phenomenological in ϵ_j):

$$\mathrm{Rich}_b(z_j) = \frac{1}{2\pi}\log\left(\frac{1}{\epsilon_j}\right)\log b \quad \text{(see 1.1)}.$$

Remark 3.2. This Hamiltonian system suffices for simply connected domains or for a closed Riemann surface of genus zero. However, the same way as shown in [14], it is *incomplete* for multiply connected domains or closed surfaces of genus $\geqslant 1$. We outline the ideas for the interplay of vortices + pseudoharmonic flows in section 13. It should mimic [15].

We end this section with a query.

QUESTION. For consistency, it is important that, when b approaches a constant value, then $\mathrm{Rich}_b(z)$ becomes the Robin function of the domain. What would be correct to do: (i) establish an interpolation between the expressions, or (ii) add the Robin function to the Hamiltonian (3.5)?

4. Elliptic operators in 2d. The Green functions G_b and $\Delta_{1/b}$

Elliptic operators in $\Re^n$ form a noble area in mathematical physics [49, 50]. In divergence form, operators with variable coefficients appear as

$$L_A\psi = -\operatorname{div}(A(x)\nabla\psi), \quad x \in D \subset \Re^n \tag{4.1}$$

where $A(x)$ (*conductivity matrix*) is positive and satisfies an uniformity condition. In the now classical treatises by Bergman–Schiffer [51] and Vekua [52], several physics and engineering problems (mainly in elasticity) are taken for motivation.

For $n = 2$, together with Lipman Bers, these authors established the theory of pseudoanalytic functions, intrinsically connected with quasi-conformal mappings. Lipman Bers legacy stands high. His 1977 review [53] remains particularly illuminating. There is also a review by Henrici [54] of Vekua's work. Some references: From late 40's throughout 50's, [55–61]. From the 60's to early 80's, [62–65]. A book by Rodin [66] was perhaps the first with the Riemann surfaces viewpoint.

From the 1980's to the early 2000's, the interest somewhat subsided; however, there is a renewed interest, perhaps motivated by AI and image processing. Three recent treatises are available: [67–69]. Specially exciting is the statement in the latter: "any solution becomes harmonic after a quasi-conformal change of coordinates".

5. Green functions near the diagonal. An open question

All operators act on the first slot.

DEFINITION 5.1. The Green function for (4.1) is defined by

$$\begin{cases} L_A G_A(x,y) = \delta_y(x) & x, y \in D \subset \Re^n \\ G_A(x,y) = 0 & \text{when } y \text{ is in the boundary and } G_A(x,y) = G_A(y,x). \end{cases}$$

For lake equations ($n = 2$), one needs information about the Green function (that we denote G_b or G_{L_b}) for the elliptic operator L_A with isotropic conductivity $A = \operatorname{diag}\{1/b\}$.

If the vortices are sufficiently far from boundaries and the bathymetry changes smoothly, with smaller variations compared with their mutual distances, it seems reasonable to use the dominant term of the Green function near the diagonal.

For $b \equiv 1$, it is common knowledge that $G_D(z,\zeta) \sim \Phi_0(z-\zeta)$ with

$$\Phi_0(z) = -\frac{1}{2\pi}\log|z|. \tag{5.1}$$

For the general (4.1), the authors of [70, 71] propose the singular+regular decomposition

$$G_A(x,y) = \frac{\det(A(x)^{-1/2} + \det(A(y)^{-1/2}}{2}\Phi_0\Big(\frac{T_x+T_y}{2}(x-y)\Big) + S_A(x,y),$$

where $T_x^{-1}(T_x^{-1})^\dagger = A(x)$. They claim that although the regular part $S_A(x,y)$ is only continuous at the diagonal, $S_A(x,x)$ is C^∞ smooth.

For $A = \mathrm{diag}\{1/b\}$ this becomes

$$G_b(x,y) \sim \frac{b(x)+b(y)}{2}\Phi_0\Big(\frac{b(x)^{1/2}+b(y)^{1/2}}{2}(x-y)\Big) \tag{5.2}$$

In [20], an alternative expression for the singular part is provided (Proposition 3.1):

$$G_b(x,y) \sim \sqrt{b(x)b(y)}\,G_D(x,y) \tag{5.3}$$

where G_D is the Green function for the usual Laplacian ($b \equiv 1$) with Dirichlet boundary conditions. These expressions are similar, but they are not identical. Which one to use?

6. Changing the operator L_b to $\Delta_{1/b}$

It seems to us that the results in [72] favor the choice (5.3) more than (5.2). The authors consider the operator

$$\begin{aligned} \Delta_a\psi &= -\frac{1}{a(x)}\,\mathrm{div}(a(x)\nabla\psi) \\ &= -\Delta\psi - \nabla\log a\cdot\nabla\psi. \end{aligned} \tag{6.1}$$

on a bounded domain $D \subset \Re^n$, $n \geqslant 2$ with Dirichlet boundary conditions, where $a(x)$ is strictly positive and smooth. Δ is the Euclidean Laplacian and Δ_a can be seen as a perturbation of Δ (look for more below).

By definition, the Green function $\tilde{G} = \tilde{G}_a$ for Δ_a satisfies

$$\begin{cases} \Delta_a\tilde{G}(x,y) = \delta_y(x) & x,y\in D \\ \tilde{G}(x,y) = 0 & \text{when } y \text{ is in the boundary.} \end{cases}$$

and the authors state that the symmetry condition becomes

$$a(y)\tilde{G}(x,y) = a(x)\tilde{G}(y,x) =: \mathscr{G}(x,y) = \mathscr{G}(y,x). \tag{6.2}$$

Let Φ_o be the fundamental solution of the Euclidean Laplacian in $\Re^n$. For $n = 2$ is given by (5.1). They provide a recipe for a recursive expansion in x for the regular part $H(x,y)$ defined by

$$H(x,y) = \tilde{G}(x,y) - \Phi_o(x-y) \quad \text{(symmetry does not hold for } H).$$

Here, we do not need the details of the expansion. H satisfies

$$-\Delta_a H(x,y) = \nabla\log a(x)\cdot\nabla\Phi_o(x-y) = -\frac{1}{2\pi}\nabla\log a(x)\cdot\frac{x-y}{|x-y|^2}$$

Although, in general, $H \notin C^1(D\times D)$, results about elliptic regularity imply that it is at least C^o. Remarkably, the Robin function $H(x,x)$ is C^∞.

Remark 6.1. Inverting an elliptic operator has a smoothing effect and solution of a Poisson equation is two derivatives more regular than the source [73]. When reviewing the literature for lake equations, one must first determine which specific equation is being considered, (3.1) or (6.1).

For our purposes, it is enough to posit that near the diagonal, no matter what anisotropy function $a(x)$ is present, one can take the approximation

$$\tilde{G}_a(x,y) \sim \Phi_o(x-y). \tag{6.3}$$

7. Speculations

We take $a = 1/b$ in (6.1). Since $\Delta_{1/b} = bL_b$ with L_b given by (3.1), we get the important information that

$$L_b \tilde{G}_{1/b}(x,y) = \frac{\delta_y(x)}{b(x)}.$$

To recover the stream of a vorticity $\omega(x)$, one uses the area form $dA = b(x)dx\,dy$:

$$\psi_\omega(z) = \iint_D \tilde{G}_{1/b}(z,\zeta)\omega(\zeta)dA.$$

This implies, as it seems reasonable to us, that a unit vortex stream function could be defined alternatively to G_{L_b} of (3.1), (3.2) taking

$$\psi(x) = b(x_o)\tilde{G}_{1/b}(x,x_0).$$

Let us discuss the underlying reasons. We are in two dimensions so we compute the circulation of the vector field $\nabla^\perp\psi/b$ around a closed curve γ bounding a small region R containing x_o:

$$\begin{aligned}\oint_\gamma \nabla^\perp\psi/b\cdot d\vec{\ell} &= b(x_o)\oint_\gamma \nabla^\perp\tilde{G}_{1/b}(x,x_o)/b(x)\cdot d\vec{\ell}\\ &= b(x_o)\iint_R (-\operatorname{div}_x \nabla\tilde{G}_{1/b}(x,x_o)/b(x))dx\,dy\\ &= b(x_o)\iint_R (L_b)_x\tilde{G}_{1/b}(x,x_o)dx\,dy\\ &= b(x_o)\iint_R \frac{1}{b(x)}\delta_{x_o}(x)dx\,dy = b(x_o)\frac{1}{b(x_o)} = 1\end{aligned}$$

This means that, as the vortices $(x_1,\ldots,x_N)$ are allowed to move around the domain by elements of sDiff, the family of stream functions

$$\psi(x;x_1\ldots,x_N) = \sum_{j=1}^N (\Gamma_j b_j(x_j))\tilde{G}_{1/b}(x,x_j)$$

is *isovorticed with zero background vorticity*. This *ansatz* is close enough to (5.3).

An alternative Hamiltonian description to (3.5), now using $\tilde{G}_{1/b}$, should be not too difficult to produce. For this, the modified symmetry relation (6.2) would be instrumental.

Proposed gross approximation. In view of (6.3), we suggest the approximation

$$u(x) \sim \frac{1}{b(x)}\sum_{j=1}^N (\Gamma_j b_j(x_j))\frac{-i(x-x_j)}{|x-x_j|^2} \tag{7.1}$$

for the fluid flow produced by nearby vortices $x_1, \dots, x_N$. For the vortex dynamics, one would take the interaction terms plus each of Richardson's self velocities.

We hope the above considerations makes sense.

8. Toy example: Vortex pair on a sloping beach

In the so-called *rigid lid model*, it is tacitly assumed that the momentum of the onshore flow by the surface waves [74,75] is averaged and transferred to alongshore currents, which then recirculate via the rip currents.

Let us consider two opposite vortices on a sloping beach $b(y) = \alpha y$ initially at $(\pm x_o, y_o)$. The motion will keep symmetry with respect to y-axis, so we can focus on the right vortex with negative vorticity. The pair will move offshore as they approach. Then (7.1) gives

$$\dot{x} = -p\frac{\Gamma/2}{y}, \quad \dot{y} = \frac{\Gamma/2}{x}$$

where $p = |\log \epsilon|/2\pi$ is a phenomenological parameter. We posit $p = p(\alpha)$ in a decreasing fashion, so that $p \to 0$ as $\alpha \to 0$.

It is readily seen that, in finite time, $x(t)$ attains zero while $y(t) \to \infty$. Thus rip currents can go very far.

A interesting (and more sophisticated) problem could be *vortex sheets* on the lake equations. Possibly [76] and [77] about Birkhoff–Rott may help.

For amusement, we give information on rip currents. Not quite an amusement: rip currents cause most deaths on beaches. There is a large oceanography literature [78–83] and some modeling by fluid mechanicists with various degrees of sophistication, [84–89].

> "Peregrine [85] presents evidence for vortex structures arising from along-shore currents and argues that, in particular, rip currents arise from the pairing of opposite signed vortices in the form of a propagating dipole. ... for typical parameter values, vortices evolve with length scales of O(100 m) and time scales of O(100s)." (from [86]).

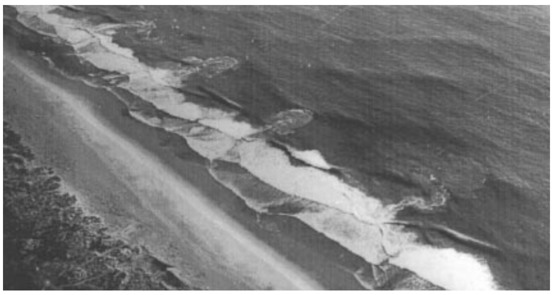

FIGURE 3. Rip currents periodic pattern [85]. Clearly visible are the necks and heads. The wide head of the currents indicates the merge of eddy couples.

9. Three examples of Green functions in (more or less) closed form

Linear profile. In the introduction, we discussed the analogy with vortex rings. Among the references, we highlighted the surveys [16, 17] and the research articles [37–40]. From the latter, only changing notation,

$$\Gamma_j y_j \dot{x_j} = \frac{\partial H}{\partial y_j}, \quad \Gamma_j y_j \dot{y_j} = -\frac{\partial H}{\partial x_j}$$

$$H = \sum_{j=1}^{N} \frac{\Gamma_j^2}{4\pi} y_i \Big[\log \frac{8 y_j}{\epsilon_j} - \frac{7}{4} \Big] + U(x_1, y_1, \dots, x_N, y_N).$$

For the lake equations, one should only replace the self term by the expression given by Richardson. The interaction energy is $U = \frac{1}{2\pi}\sum_{j\neq i} \Gamma_i \Gamma_j G(x_i, y_i, x_j, y_j)$ with

$$G(x, y, x', y') = \frac{y y'}{4\pi} \int_0^{2\pi} \frac{\cos\theta \, d\theta}{\sqrt{(x - x')^2 + y^2 + y'^2 - 2 y y' \cos\theta}}.$$

G can be traced back to Dyson 1893 [26]. See Lamb's Hydrodynamics [27].

The authors of [39] mention the work of Vasilev in 1913 [90] as the first with the Hamiltonian description. The integral can be evaluated explicitly in terms of elliptic functions, equations (1.2, 1.3), page 36. The expression they use for the self-velocity, equation (1.7), page 37, is the one given by Saffman [30].

Remark 9.1. For axisymmetric generalizations in higher dimensions, see [91] and [92]. For the Geometric Mechanics viewpoint, see [93, 94].

Exponential profile. These are sometimes used in physical oceanography [95]. Inverting the notation of [96], let x be alongshore ($-\infty < x < \infty$) and y pointing offshore ($0 < y < \infty$). The profile was given by

$$b(y) = \begin{cases} b_1 \exp[s(y - \ell)], & 0 < y < y_o \\ b_1 \exp[s(y_o - \ell)], & y > y_o. \end{cases}$$

The Green function G was first evaluated by applying a Fourier transform in y. The transformed $\hat{G}$ could be found explicitly. However, certain approximations were required for the Fourier inversion.

Composite materials conductivity. Suppose the function $a(x)$ in (6.1) is formed by a number of piecewise constant indicator functions in D,

$$a(x) = \sum_{\ell=1}^{N} \kappa_\ell \chi_{B_\ell} + \chi_{B_o}, \quad \kappa_\ell > 0,$$

where the B_ℓ are disjoint regions inside D and

$$B_o = D - \cup_{j=1}^{N} B_j$$

In [97], the problem with two balls is examined. The Green function is (more or less) explicitly constructed (Proposition 2.3) and derivative estimates are provided.

10. Numerical methods for elliptic PDEs in inhomogeneous media

Numerical analysis of elliptic PDEs is a much developed area, see e.g. [98]. Here, we just take from the lecture notes from a class at MIT by S. Johnson, [99]. He refers to the Euclidian Laplacian as governing "empty space" on a domain D, and assumes that its Green function G_o can be constructed.

One considers the equation $L_A\psi = f$ with L_A given by (4.1) in the isotropic situation $A = \operatorname{diag} a$. For us, $a = 1/b$. In electrostatics, $\sqrt{a}$ is proportional to the refractive index; in a stretched drum, a is proportional elasticity; a could be a diffusion coefficient or a thermal conductivity.

His approach is to make the problem look as an empty space one, rewriting it as

$$-\Delta\psi = \frac{f}{a} + \nabla \log a \cdot \nabla\psi$$

Formally

$$\psi(x) = \int_D G_o(x,\xi)\Big[\frac{f(\xi)}{a(\xi)} + \nabla'(\log a(\xi)) \cdot \nabla'\psi(\xi)\Big]\, d\,\mathrm{vol}(\xi)$$

This is a volume integral equation for ψ. One may rewrite the VIE as

$$\psi = \phi_o + B\psi, \quad \phi_o = \int_D G_o(x,\xi)\frac{f(\xi)}{a(\xi)}\, d\,\mathrm{vol}(\xi),$$

$$B\psi = \int_D G_o \nabla' \log a \cdot \nabla'\psi\, d\,\mathrm{vol}\,.$$

One can think of the inhomogeneous solution as the sum of "homogeneous" solutions using G_o with right-hand-sides $f(\xi)/a(\xi)$ plus a "scattered" solution due to inhomogeneities of a. There are well-established numerical methods for solving "VIE" problems. There are situations in which the problem simplifies. For instance:

Piecewise homogeneous media. This is the third example in the previous section. Suppose $a \equiv a_1$ in a subdomain $\Omega \subset D$ and $a \equiv a_2$ outside. Then ∇a is a delta function at the interface, multiplied by $\log(a_2/a_1)$. The VIE becomes a SIE which can be handled numerically more easily.

This seems promising for the lake equations, taking a number of level curves of the bathymetry, and assuming constant values between them.

Born–Dyson approximation. When the operator B has some norm < 1, the functional equation $(I - B)\psi = \phi_o$ can be solved via

$$(I - B)^{-1} = \sum_{k=0}^{\infty} B^k.$$

This trick is so common in mathematics that there is no name for it. In physics it is called the Born–Dyson expansion. For nearly homogeneous a, one can take

$$\psi \sim \psi_o + B\psi_o \quad \text{(this is another way to justify (6.3))}.$$

Question. Taking for f a delta function, so $\psi_o = G_o$, perhaps one could produce useful approximations for the Green function of Δ_a.

11. Orthogonal (Hodge) decomposition for multiply connected domains

When the domain has islands, the stream function (3.4) is incomplete. In order to enforce prescribed boundary circulations, another term is needed

$$\psi(z,t) = \sum_{k=1}^{N} \Gamma_k G_b(z, z_k) + \psi_{\rm circ}(z; p_1, \dots, p_g) \tag{11.1}$$

This was called "outside agency" by C. C. Lin [6], but it is a *bona fide* internal entity of the flow. The vortices are driven by $\psi_{\rm circ}$, but in feedback they change $\psi_{\rm circ}$ dynamically. The process is described in [15] when $b \equiv 1$. We now emulate these results for the lake equations. We aim to show that the two terms in (11.1) form an orthogonal decomposition with respect to the area form $dA = b\,dx\,dy$.

The first term: ψ vanishing in all boundaries. Let L_b act on C^∞ functions with constant (perhaps all different) values on the boundaries. We use Green's first identity

$$\int_D \mathrm{div}(\psi X) dx\,dy = \int_D (\psi\,\mathrm{div}\,X + X \cdot \nabla\psi) dx\,dy = \oint_{\partial D} \psi (X \cdot \hat{n}) d\ell$$

Consider two functions ϕ, ψ that *vanish on all boundaries* (or at infinity). Let $X_\phi = \nabla\phi/b$. Interchanging the roles of ψ and ϕ, it follows that

$$\int_D \psi\,\mathrm{div}(\nabla\phi/b) dA - \int_D \phi\,\mathrm{div}(\nabla\psi/b) dA$$
$$= \oint_{\partial D} \not{\psi} (\nabla\phi/b \cdot \hat{n}) d\ell - \oint_{\partial D} \not{\phi} (\nabla\psi/b \cdot \hat{n}) d\ell = 0.$$

So L_b is symmetric at least for functions ψ, ϕ that vanish on all boundaries.

Now take only ψ vanishing at all boundaries. Green gives for $X_\phi = \nabla\phi/b$ that

$$-\int_D \psi\,\mathrm{div}\left(\frac{\nabla\phi}{b}\right) dx\,dy = \int_D \frac{\nabla\phi}{b} \cdot \nabla\psi\,dx\,dy = \int_D \frac{\nabla\phi}{b} \cdot \frac{\nabla\psi}{b} b\,dx\,dy$$

(in the inner products we can replace ∇ by $\nabla^\perp$ and the formulas remain valid).

$$\int_D \psi \Delta^b \phi\,dx\,dy = \int_D \phi \Delta^b \psi\,dx\,dy = \int_D \frac{\nabla^\perp\phi}{b} \cdot \frac{\nabla^\perp\psi}{b} b\,dx\,dy$$

This is the inner product of b-divergence free vector fields, when the stream function of at least one of them (say, ψ) vanishes on all the boundaries ∂D. With ψ vanishing on all the boundaries

$$\langle L_b\psi, \psi\rangle =: \int_D \psi L_b \psi\,dx\,dy = \int_D |\nabla\psi/b|^2 b\,dx\,dy$$

which is twice is the kinetic energy of the vector field $\nabla^\perp\psi/b$. Notice the presence of the area form $dA = b\,dx\,dy$.

Remark 11.1. Any ψ vanishing on all boundaries can be described with the Dirichlet Green function G_b from its vorticity ω, (3.2). The first term in (11.1) corresponds to this situation, but with concentrated vorticities. Lacking a better name, we call these functions *pure vorticity* type.

The second term $\psi_{\mathrm{circ}}(z)$: They are the b-harmonic functions. Let D be bounded with internal curves $\gamma_i, 1 \leqslant i \leqslant g$ and external γ_o. The kernel of L_b has dimension g. This follows from Fredholm theory[2] or invoking a continuation starting from $b \equiv 1$.

DEFINITION 11.1. A function ϕ is b-harmonic when both $d\phi$ (obviously, being exact) and $\star\,, d\phi/b$ are closed 1-forms (see (3.3)). The b-harmonic functions extend to the complex domain as pseudoanalytic functions.

One can take for basis the *b-harmonic measures* $m_k, 1 \leqslant k \leqslant g$, the functions in the kernel of L_b that value 1 in each one of the inner boundaries and zero on the others. As in the Euclidean Laplacian, $0 \leqslant m_\ell < 1$ in the interior of D, and

$$m_o + m_1 + \cdots + m_g \equiv 1.$$

For any $\phi \in \mathrm{Ker}(L_b)$ we call $X = \frac{1}{b}\nabla^\perp\phi$ an *harmonic flow.*

PROPOSITION 11.1. *Relative to $dA = b\,dx\,dy$, the pure vorticity and the b-harmonic functions are orthogonal.*

Recall : $\psi_{\partial D} \equiv 0 \quad \Rightarrow \quad \int_D(\psi \,\mathrm{div}\, Y + Y \cdot \nabla\psi)dx\,dy = \oint_{\partial D} \psi(Y \cdot \hat{n})d\ell = 0.$
Let ψ vanish in all boundaries and $Y = \frac{1}{b}\nabla\phi$, with ϕ b-harmonic: $\mathrm{div}(Y) = 0$.
It follows that

$$\int_D Y \cdot \Big(\frac{1}{b}\nabla\psi\Big) b\,dx\,dy = 0$$

and we can replace ∇ by $\nabla^\perp$.

DEFINITION 11.2. Electrostatic capacity matrix P^b of the b-harmonic measures:

$$P^b_{\kappa\ell} = \int_\Sigma \frac{\nabla m_\kappa}{b} \cdot \frac{\nabla m_\ell}{b} b\,dx\,dy \quad (1 \leqslant \kappa,\ \ell \leqslant g;\ \text{The } \perp\text{'s can be omitted}). \tag{11.2}$$

They are the coefficients of the kinetic energy of the harmonic part,

$$\psi_{\mathrm{circ}}(z) = \sum_{\kappa=1}^{g} C_\kappa m_\kappa(z), \quad T_{\mathrm{har}} = \frac{1}{2} C P^b C^\dagger.$$

12. Reduction

The $C_\kappa = C_\kappa(t)$ evolve in time coupled with the N vortices dynamics $z_j(t)$. Making use of the Helmholtz conservation of circulations on the g inner boundaries, it should be possible to eliminate the C_j and obtain a dynamics just for the vortices. One would amend the Hamiltonian (3.5) with the capacities (11.2).

$$H = \sum_{j<k} \Gamma_i\Gamma_k G_{L_b}(z_j, z_k) + \frac{1}{2}\sum_j \Gamma_j^2\,\mathrm{Rich}_b(z_j) + T_{\mathrm{har}}, \quad T_{\mathrm{har}} = \frac{1}{2} C P^b C^\dagger$$

$$\Omega = \sum_j \Gamma_j b(z_j) dx_j \wedge dy_j.$$

provided we could determine the row vector $C = C(z_1, \ldots, z_N)$ in terms of the boundary circulations, which are constants of motion.

[2] V. Guillemin, Elliptic operators https://math.mit.edu/~vwg/classnotes-spring05.pdf.

Determination of the amended term. The following mimics [15] for the usual situation $b \equiv 1$ and is congenial to the results in [20, eqs. (2.2) to (2.7)].

(i) Consider the pseudoharmonic closed 1-forms $\beta_i = \star\, d\phi_i/b$, where

$$\phi_i = \sum_{j=1}^{g} (P^b)^{-1}_{ij} m_j. \tag{12.1}$$

Claim: $\{\beta_1, \ldots, \beta_g\}$ are *dual* to the inner boundary curves $\{\gamma_1, \ldots, \gamma_g\}$.

$$\oint_{\gamma_i} \beta_j = \delta_{ij}.$$

We put the proof on hold for a moment.

(ii) Let us rewrite the decomposition pure vorticity+b-harmonic (11.1) as

$$\psi(z,t) = \sum_{k=1}^{N} \Gamma_k G_b(z, z_k) + \sum_{\ell=1}^{g} B_\ell \beta_\ell$$

and compute the circulations. For a single unit vortex z_o, the circulation around the boundary γ_ℓ is

$$p_\ell = -\oint_{\gamma_\ell} \frac{\star\, d\psi}{b} = \underbrace{-\oint_{\gamma_\ell} \star\, dG_b(z, z_o)/b}_{m_\ell(z_o)} + B_\ell \quad \Rightarrow \quad B_\ell = p_\ell - m_\ell(z_o)$$

For $b \equiv 1$, the underbraced equality is well known by experts and explained in [15]. It should be valid in general:

$$-\oint_{\gamma_\ell} \star\, dG_b(z, z_o)/b \underset{?}{=} m_\ell(z_o)$$

For N vortices, each with its corresponding strength Γ_ℓ:

$$B_\ell = p_\ell - \sum_{j=1}^{N} \Gamma_j m_\ell(z_j)$$

To finish, one relates the vectors B and C. This is elementary linear algebra. Let $m = (m_1, \ldots, m_g)$ and $\phi = (\phi_1, \ldots, \phi_g)$ in (12.1) be seen as column vectors of functions: $m = P^b \phi$. Since $\psi_{\rm circ} = Cm = B\phi$, it follows that $CP = B$, with C and B regarded as row vectors. Thus

Proposition 12.1.

$$T_{\rm har} = \frac{1}{2} C P^b C^\dagger = \frac{1}{2} B Q^b B^\dagger, \quad Q^b = (P^b)^{-1}.$$

Definition 12.1. We call $Q^b = (P^b)^{-1}$ the hydrodynamical capacity matrix with respect to the dual 1-forms $\beta = \{\beta_1, \ldots, \beta_g\}$ to the curves $\{\gamma_1, \ldots, \gamma_g\}$.

In order to verify (i), we invoke Green's first identity in the third line:

$$\oint_{\gamma_j} \beta_i = \sum_{\ell=1}^{g} P_{i\ell}^{-1} \oint_{\gamma_j} \frac{1}{b}\frac{\partial m_\ell}{\partial n} = \sum_{\ell=1}^{g} P_{i\ell}^{-1} \oint_{\partial D} \frac{m_j}{b}\frac{\partial m_\ell}{\partial n}$$
$$= \sum_{\ell=1}^{g} P_{i\ell}^{-1} \int_D \frac{\nabla m_j \cdot \nabla m_\ell}{b} dx\, dy = \sum_{\ell=1}^{g} P_{i\ell}^{-1} \int_D \frac{\nabla m_j}{b} \cdot \frac{\nabla m_\ell}{b} b\, dx\, dy$$
$$= \sum_{\ell=1}^{g} P_{i\ell}^{-1} P_{\ell j} = \delta_{ij} \quad \text{(for clarity, the superscript } b \text{ in } P^b \text{ has been removed)}$$

13. "Planet equations". Pseudo-harmonic forms on Riemann surfaces

In [15], the study of point vortices on multiply connected planar domains was included in the general setting of Riemann surfaces. If the domain has g internal boundaries and one external, one takes the mirror image to form the Schottky double, a closed Riemann surface of genus g.

In so doing, the classical results of C. C. Lin [6] were reinterpreted in terms of Hamiltonian reduction, à la Marsden and Weinstein. The harmonic part can be incorporated to the Dirichlet Green function, yielding the *hydrodynamic* Green function as in [10].

We now introduce a proposal to extend the lake equations to a closed Riemann surface Σ with a metric in its conformal class, given a bathymetry b. Since the letter g will be used to denote the underlying metric, the surface genus will be denoted κ. We plan to develop this program in a future study. This project may look outrageous for a true fluid-dynamicist, since many factors are disregarded. For instance, we are making the rigid lid assumption (no surface waves) and taking constant density (no gravity effects and inhomogeneities), which was already the case in the lake equations.

However, there is also a mathematical neglect, even worse perhaps. "Tube" effects [100] are ignored in the equations that will live in the rigid lid Σ, a curved manifold bounding an ambient of one more dimension (the depth).

At any rate, we are sure of the mathematical interest, since the main object is the the elliptic operator on functions in the Riemann surface

$$L_b\psi = \operatorname{div}\left(\frac{1}{b}\operatorname{grad}\psi\right) \quad \text{(where div and grad pertain to the metric } g\text{).}$$

Let us denote $\tilde{g} = bg$. As before, the relevant area form is $\tilde{\mu} = b\mu$, where μ is the area form of g. Our vector fields will belong to $\mathrm{sDiff}_{\tilde{g}}$. The Euler equation is written with $\nabla_v v$, the covariant derivative of g.

Given any function ϕ, the vector field $\frac{1}{b}\operatorname{grad}^{\perp}\phi \in \mathrm{sDiff}_{\tilde{g}}$. The operation $\perp$ is to rotate-90 degrees in the tangent plane, which is well defined by the complex structure. However, recall that in order to define the 'curl' of a vectorfield v, it is necessary to take its musical $\nu = v^\flat$ (always with respect to g) and define the vorticity as the two form $\omega = d\nu$.

Remark 13.1. Divergence of a vector field with respect to a measure $\tilde{\mu}$.

$$L_v\tilde{\mu} := [\mathrm{div}_{\tilde{\mu}}\, v]\tilde{\mu}.$$

In dimension 2, $L_v\tilde{\mu} = d(i_v\tilde{\mu}) + i_v \not{d}\not{\tilde{\mu}} = d(i_v\tilde{\mu})$.

Pure vorticity flows.

Definition 13.1. The Green function $G = G_b$ for L_b satisfies

$$\begin{aligned} L_b G_b(s,r)\mu(s) &=: -d_s\Big(\star \frac{d_s G_b(s,r)}{b}\Big) \\ &= \Big(\delta_r(s) - \frac{1}{V}\Big)\mu(s), \quad V = \int_\Sigma \mu. \quad (\star \text{ is the Hodge star}) \end{aligned}$$

A "pure vorticity" stream function is recovered with

$$\psi_\omega(s) = \int_\Sigma G_b(s,r)\omega(r)\mu(r) \quad \text{(note that we use the area form of } g)$$

$$L_b\psi_\omega(s) = \omega(s) - \bar{\omega}, \quad \bar{\omega} = \frac{1}{V}\int_\Sigma \omega(s)\mu(s).$$

"Pure vorticity" (PV) vector fields are constructed with stream functions

$$v_\omega(s) = \frac{1}{b(s)}\,\mathrm{grad}_g^\perp\,\psi_\omega(s) \in PV \subset \mathrm{sDiff}_{\tilde{\mu}}\,.$$

and

$$d\Big(-\star\frac{d\psi}{b}\Big) = -\,\mathrm{div}_g(\mathrm{grad}_g\,\psi/b)\mu_g = -\Delta^\rho\psi\mu_g.$$

However, they are not enough to represent all elements of $\mathrm{sDiff}_{\tilde{\mu}}$.

Pseudo-harmonic 1-forms and pseudo-potential flows. The operator L_b in functions can be extended to act on differential forms, as in Hodge theory for the ordinary Laplacian $\Delta = -\,\mathrm{div}\,\mathrm{grad}$. Its kernel has dimension 2κ. Such forms $\eta \in \ker L_b$ are characterized by $d\eta = d(\star\eta/b) = 0$.As already pointed out by Lipman Bers in the 1950's, these conditions are preserved when Σ is conformally changed.

To any canonical homology basis, there corresponds a dual cohomology basis $\{\alpha\,\beta\}$ of 2κ pseudoharmonic forms. They correspond to the *pseudopotential flows*

$$\frac{1}{b}(\star\alpha)^\sharp, \quad \frac{1}{b}(\star\beta)^\sharp$$

that interact dynamically with the pure vorticity flows.

14. Orthogonality between pure vorticity and pseudo-potential flows

Here, we present one preliminary result in this direction. We claim that the L_2^b-orthogonal complement with respect to the area form $\tilde{\mu}$ inside $\mathrm{sDiff}_{\tilde{g}}$ of the pure vorticity vector fields is given precisely by the pseudo-potential flows. Flats and sharps will be always for the g-metric; we omit the subscripts.

Let us try to characterize the vector fields $v \in \mathrm{sDiff}_{\tilde{\mu}}$ such that

$$\int_\Sigma \Big\langle \frac{1}{\not{b}}\,\mathrm{grad}_g^\perp\,\psi_\omega(s),\; v\Big\rangle_g (\not{b}\mu) = 0, \quad \forall\,\omega \in C^\infty(\Sigma).$$

This is the same as

$$\int_\Sigma \langle \mathrm{grad}_g\, \psi_\omega(s),\, v^\perp\rangle_g \mu = \int_\Sigma d\psi_\omega(v^\perp)\mu = 0$$

We need this Lemma: $d\psi_\omega(v^\perp)\mu = d\psi_\omega \wedge v^\flat$.

PROOF. $g = a(x,y)(dx^2+dy^2)$. If $v = v_1\partial_x + v_2\partial y$ then $v^\flat = a(v_1 dx + v_2 dy)$. □

Thus the condition is $\int_\Sigma d\psi_\omega \wedge v^\flat = 0$, $\forall \omega \in C^\infty(\Sigma)$. Now,

$$d\psi_\omega \wedge v^\flat = d(\psi_\omega v^\flat) - \psi_\omega dv^\flat$$

Since Σ does not have a boundary, the condition rewrites as

$$\int_\Sigma \psi_\omega dv^\flat = 0, \quad \forall\, \omega \in C^\infty(\Sigma) \;\Rightarrow\; dv^\flat = 0 \quad (\text{and } \eta = v^\flat).$$

We also need to impose the $\tilde{g}$ incompressibility of $v = \eta^\sharp$.
We use $i_v\mu = \star v^\flat$ (see Appendix A of [15, eq. (A.4)]). Then for $v = \eta^\sharp$:

$$0 = L_v\tilde{\mu} \;\Rightarrow\; di_v(b\mu) = d(bi_v\mu) = d(b \star v^\flat) = d(\star b\eta) = 0$$

Summarizing, the decomposition is of the form

$$\nu = -\star \frac{d\psi_\omega}{b} \oplus \eta \quad \text{with} \quad d\eta = d(\star b\eta) = 0.$$

Alternatively, we introduce

$$\tilde{\eta} = \star b\eta \quad \text{so that} \quad \eta = -\star \tilde{\eta}/b.$$

The orthogonal decomposition rewrites as

$$\nu = [-\star \frac{d\psi_\omega}{b}] \oplus [-\star \frac{\tilde{\eta}}{b}] \quad \text{with} \quad d\tilde{\eta} = d(\star\tilde{\eta}/b) = 0.$$

and $d\nu = \omega$ since the pseudoharmonic part drops out.

The task. In order to emulate the results in [15], one needs to extend the Riemann relations for a canonical homology basis, but now using the dual basis of pseudoharmonic forms, and then compute the circulations of the flow on the homology generators. With this in hand, it should be a royal road to get the coupled dynamics between the vortices and the pseudopotential flows.

15. Final comment: A desideratum

We hope that the extension of [15] to the lake equations and its generalization to the 'planet' equations, as outlined above, could proceed uneventfully.

We also aimed to point out the direct connection of lake equations to the theory of pseudo-analytical functions and quasi-conformal mappings. Perhaps a good way to conclude is by quoting Lipman Bers [59]:

> "Riemann surfaces were introduced by Riemann as a tool in the investigation of multiple-valued analytic functions. The ideas and methods of Riemann's function theory can also be used in studying multiple-valued solutions of linear partial differential equations of elliptic type."

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ВРТЛОЗИ ЈЕДНАЧИНА ЈЕЗЕРА
(преглед са питањима и спекулацијама)

Резиме. "Језерска једначина" у дводимензионалној области D са функцијом дубине $b(x,y)$ дата је са $\partial_t u + (u \cdot \mathrm{grad})u = -\mathrm{grad}\, p$, $\mathrm{div}(bu) = 0$, где је u $u \parallel \partial D$. Једначина је добро постављена као парцијална диференцијална једначина, али када је $b \neq \mathrm{const}$, оправдавање модела тачкастих вртлога захтева додатну дискусију. Ми се фокусирамо се на аспекте геометријске механике. Мотивирајући пример је "струја кидања" коју производе парови вртлога близу обале. За обалу са једноликим нагибом постоји савршена аналогија са Томсоновим вртложним прстеновима. Функција струјања коју производи вртлог дефинисана је као Гринова функција оператора $-\mathrm{div}(\mathrm{grad}\, \psi/b)$ са Дирихлеовим граничним условима. Као и у еластичности, језерске једначине доводе до псеудоаналитичких функција и квазиконформних пресликавања. Униформно елиптичне једначине на блиским Римановим површима могле би се назвати "планетарне једначине".

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