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UNUSUAL PROPERTIES OF ADIABATIC INVARIANCE IN A BILLIARD MODEL RELATED TO THE ADIABATIC PISTON PROBLEM

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Abstract. We consider the motion of two massive particles along a straight line. A lighter particle bounces back and forth between a heavier particle and a stationary wall, with all collisions being ideally elastic. This is one of canonical models in the theory of adiabatic invariants. It is known that if the lighter particle moves much faster than the heavier one, and the kinetic energies of the particles are of the same order, then the product of the speed of the lighter particle and the distance between the heavier particle and the wall is an adiabatic invariant: its value remains approximately constant over a long period. We show that the value of this adiabatic invariant, calculated at the collisions of the lighter particle with the wall, is a constant of motion (i.e., *an exact adiabatic invariant*). On the other hand, the value of this adiabatic invariant at the collisions between the particles slowly, linearly in time, decays with each collision.

The model we consider is a highly simplified version of the classical adiabatic piston problem, where the lighter particle represents a gas particle, and the heavier particle represents the piston.

1. Statement of the problem

Consider the motion of two particles along a straight line. Denote m and M masses of these particles. We assume $m < M$. We examine the case where the lighter particle bounces back and forth between a stationary wall and the heavier particle, undergoing elastic collisions with both. We refer to the lighter particle as the "gas particle" and the heavier particle as the "piston", due to the connection of this model to the classical adiabatic piston problem [3] (Fig.1).

Take stationary wall as the origin of the coordinate line. Let the particles move in the positive half-line. Denote x and X coordinates of the gas particle and the piston, respectively. Thus, $0 \leqslant x \leqslant X$, and X is the distance between the stationary wall and the piston. Denote v and V velocities of the gas particle and the piston,

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respectively. We assume that $|v| > |V|$. Thus, the gas particle moves faster than the piston. Denote $I = |v|X$. It is known that if $m \ll M$, $|v| \gg |V|$, and the kinetic energies of the particles are of the same order, then I is an adiabatic invariant, meaning its value remains approximately constant over long time intervals. This is one of canonical examples in the theory of adiabatic invariants (see, e.g., [4], Problem 13.8(a)). Total number of collisions in this problem is calculated in [1,2]. We aim to study the behavior of I in more details.

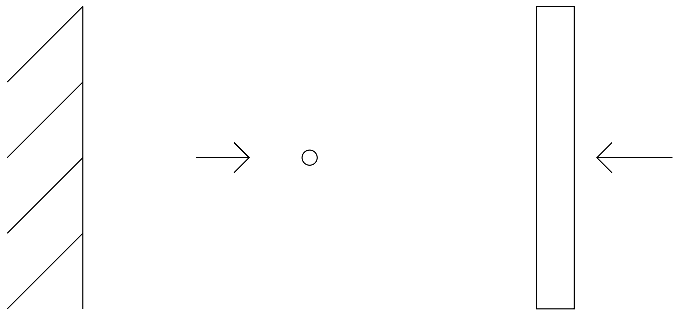

FIGURE 1. Diagram displaying the interaction between the piston and gas particle

2. Values of adiabatic invariant at collisions with the stationary wall

Let the gas particle collide with the stationary wall at a certain moment of time. Denote by v_0 the velocity of the gas particle immediately after this collision, $v_0 > 0$. Denote by V_0, X_0 the piston's velocity and coordinate (i.e. the distance between the stationary wall and the piston) at the time of this collision. The value of the adiabatic invariant I at this collision is $I_0 = v_0 X_0$.

The time interval between this collision and the subsequent collision of the gas particle with the piston is

$$\Delta t_1 = \frac{X_0}{v_0 - V_0}.$$

The distance between the stationary wall and the piston at the time of the collision of the gas particle with the piston is

$$X_1 = X_0 + V_0 \Delta t_1 = \frac{X_0}{v_0 - V_0} v_0.$$

The velocities of the gas particle and the piston before this collision are still v_0, V_0. Denote by v_1, V_1 the velocities of the gas particle and the piston immediately after this collision. According to the standard formulas for ideally elastic collisions:

$$v_1 = \frac{(m - M)v_0 + 2MV_0}{M + m}, \quad V_1 = \frac{(M - m)V_0 + 2mv_0}{M + m}. \tag{2.1}$$

The gas particle will then collide with the stationary wall after the time interval

$$\Delta t_2 = \frac{X_1}{-v_1}.$$

The distance between the stationary wall and the piston at this collision is

$$X_2 = X_1 + V_1 \Delta t_2 = X_1 + \frac{V_1 X_1}{-v_1} = X_1 \frac{v_1 - V_1}{v_1}.$$

The value of the adiabatic invariant I at this collision is $I_2 = -v_1 X_2$. We have

$$I_2 = -v_1 X_2 = -v_1 X_1 \frac{v_1 - V_1}{v_1} = -X_1(v_1 - V_1) = -\frac{X_0}{v_0 - V_0} v_0 (v_1 - V_1) = -I_0 \frac{v_1 - V_1}{v_0 - V_0}.$$

According to (2.1)

$$v_1 - V_1 = \frac{(m-M)v_0 + 2MV_0}{M+m} - \frac{(M-m)V_0 + 2mv_0}{M+m} = -(v_0 - V_0).$$

Thus, $I_2 = I_0$. One can repeat these calculations for any two consecutive collisions with the stationary wall. Therefore, the adiabatic invariant I retains the same value at any collision with the stationary wall. In the terminology of [5], the value of I at collisions with the stationary wall is an exact adiabatic invariant.

3. Values of adiabatic invariant at collisions with the piston

Let the gas particle collide with the piston in some moment of time. Denote by v_1, V_1 velocities of the gas particle and the piston immediately after this collision, $v_1 < 0$. Denote by X_1 the piston's coordinate (i.e., the distance between the stationary wall and the piston) at the time of this collision. The value of the adiabatic invariant I at this collision is $I_1 = -v_1 X_1$.

The gas particle will collide with the piston again after a time Δt_3 which can be found from the equation

$$X_1 + V_1 \Delta t_3 = (-v_1)\Delta t_3 - X_1.$$

Thus,

$$\Delta t_3 = -\frac{2X_1}{V_1 + v_1}.$$

The distance between the stationary wall and the piston at this collision is

$$X_3 = X_1 + V_1 \Delta t_3 = X_1 - V_1 \frac{2X_1}{V_1 + v_1}.$$

The velocity of the gas particle just after this collision is

$$v_3 = \frac{(m-M)(-v_1) + 2MV_1}{M+m}.$$

We assume that $v_3 < 0$. Thus, the particle moves to the stationary wall after this collission. The value of the adiabatic invariant I just after this collision is $I_3 = -v_3 X_3$. We have

$$I_3 = -v_3 X_3 = -\frac{(m-M)(-v_1) + 2MV_1}{M+m} \Big(X_1 - V_1 \frac{2X_1}{V_1 + v_1} \Big).$$

This can be simplified to

$$I_3 = I_1 + \frac{2X_1(MV_1^2 + mv_1^2)}{(M+m)(V_1 + v_1)}.$$

Thus, the change of the value of the adiabatic invariant I calculated at the times of collisions of the gas particle and the piston is

$$\Delta I_3 = \frac{2X_1(MV_1^2 + mv_1^2)}{(M+m)(V_1+v_1)}.$$

Calculate the rate of this change

$$k = \frac{\Delta I_3}{\Delta t_3} = -\frac{(MV_1^2 + mv_1^2)}{(M+m)}$$

But $(MV_1^2 + mv_1^2)/2$ is the kinetic energy of the system. It does not change at collisions. Thus, the value k is the same for all collisions. Therefore, the value of the adiabatic invariant at the collisions between the gas particle and the piston linearly in time decays with each collision. However, in an adiabatic situation, where $|v_1| \gg |V_1|$ and $m \ll M$, this decay occurs very slowly. Indeed, the case of interest is when the kinetic energies of the gas particle and the piston are of the same order and are of order 1: $MV_1^2 \sim mv_1^2 \sim 1$. We can take $v_1 \sim 1, m \sim 1, V_1 \sim \varepsilon \ll 1, M \sim 1/\varepsilon^2, X \sim 1$. Then $I_1 \sim 1$ and $k \sim \varepsilon^2$.

4. Numerical Illustrations

This section is dedicated to numerical illustrations of the dynamics in the considered problem. We assume the masses of the particle and piston to be $m = 1$ and $M = 10000$, respectively. The initial velocities of the particle and piston are $v_0 = 1$ and $V_0 = -0.01$. Thus, the piston initially moves towards the stationary wall. The initial positions of the particle and piston are $x_0 = 0$ and $X_0 = 1$.

Fig. 2 illustrates the main finding of this paper: the values of the adiabatic invariant I at the collisions between the particle and the stationary wall remain constant (represented by the horizontal line in Fig. 2), while the values at the collisions with the piston decrease linearly in time (represented by the inclined line in Fig. 2). As seen in Figs. 3 and 4, the piston initially moves towards the stationary wall. Due to the approximate conservation of the adiabatic invariant, as the distance between the piston and the stationary wall decreases, the particle's velocity increases, resulting in a rise in pressure on the piston. Consequently, the piston eventually comes to a stop and begins moving in the opposite direction, with its velocity increasing after each subsequent collision. Figure 5 describes the evolution of the pistons velocity and position after each collision with the piston. As can also be seen in Figures 3 and 4, the piston travels towards the stationary wall until a turning point is reached, at which the piston halts and travels away from the stationary wall, increasing in speed after further collisions. The piston velocity begins to plateau as the piston moves further away due to the particle having less kinetic energy to transfer to the piston as they collide.

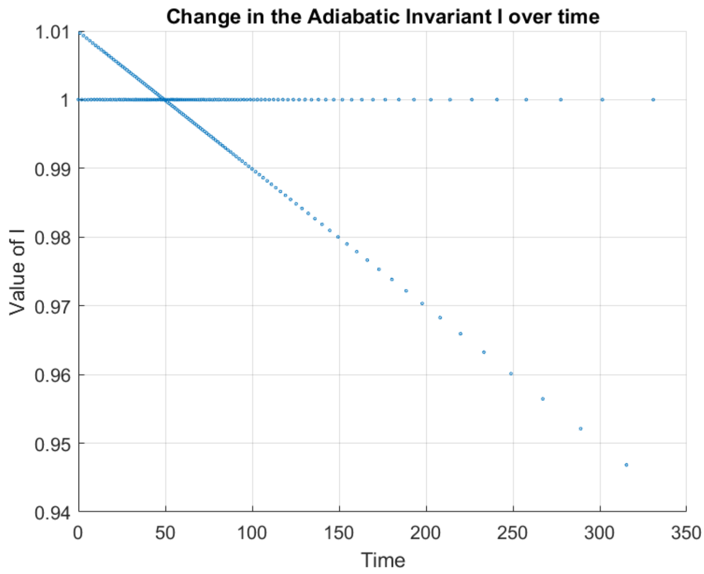

FIGURE 2. Graph depicting the change in the adiabatic invariant I as the system evolves in time

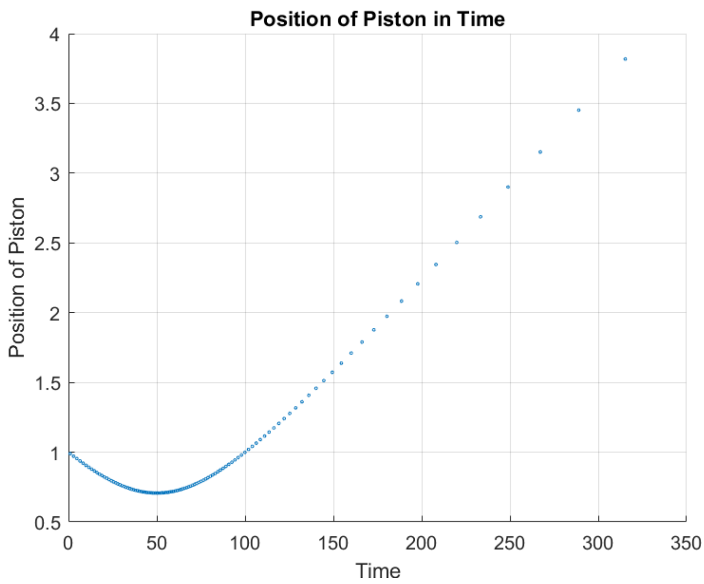

FIGURE 3. Graph depicting the position of the piston as the system evolves in time

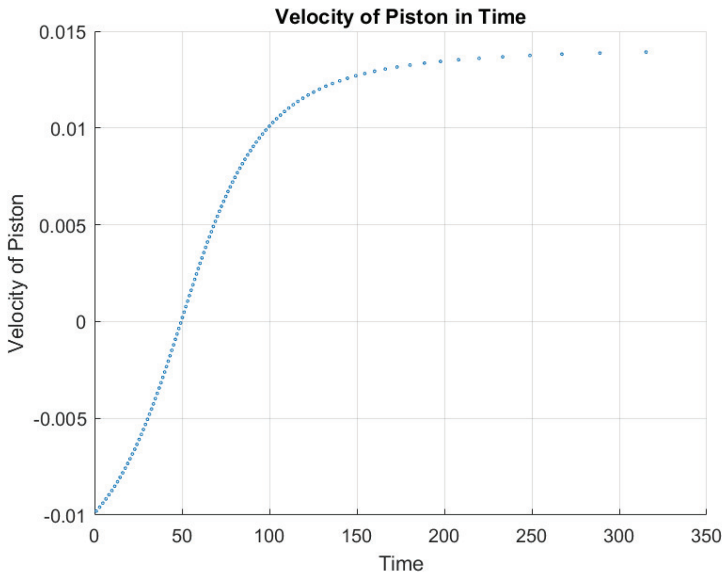

FIGURE 4. Graph depicting the velocity of the piston as the system evolves in time

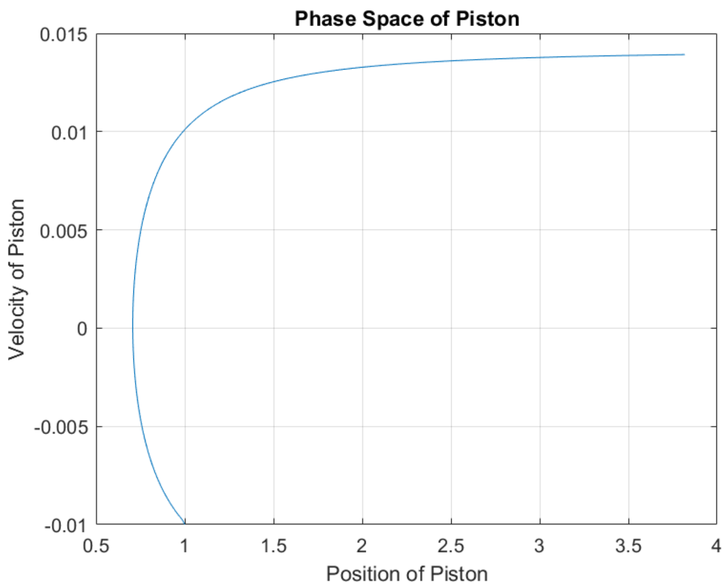

FIGURE 5. Trajectory of the piston in the phase plane

5. Conclusions

The problem considered is a slow-fast Hamiltonian system with two degrees of freedom and elastic collisions. In this system, the fast degree of freedom corresponds to the gas particle, while the slow degree of freedom corresponds to the piston. In such systems, the expected behavior of the adiabatic invariant is that its value oscillates with a small amplitude around some constant value. However, in the problem examined, the behavior of the adiabatic invariant is somewhat unusual. Specifically, its value at the collision of the gas particle with the stationary wall remains constant, while its value at the collision of the gas particle with the piston linearly in time decays from collision to collision.

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НЕУОБИЧАЈЕНЕ ОСОБИНЕ АДИЈАБАТСКЕ ИНВАРИЈАНТЕ У БИЛИЈАРНОМ МОДЕЛУ ПОВЕЗАНОМ СА ПРОБЛЕМОМ АДИЈАБАТСКОГ КЛИПА

Резиме. Разматрамо кретање две материјалне тачке (честице) дуж праве линије. Лакша честица се одбија напред-назад између теже честице и непокретног зида, при чему су сви судари идеално еластични. Ово је један од канонских модела у теорији адијабатских инваријанти. Познато је да ако се лакша честица креће много брже од теже, а кинетичке енергије су им истог реда, онда је производ брзине лакше честице и растојања између теже честице и зида адијабатска инваријанта: њена вредност остаје приближно константна током дужег периода. Показујемо да је вредност ове адијабатске инваријанте, израчуната при сударима лакше честице са зидом, константа кретања (тј. тачна адијабатска инваријанта). С друге стране, вредност ове адијабатске инваријанте при сударима између честица полако, линеарно у времену, опада са сваким сударом.

Модел који разматрамо је веома поједностављена верзија класичног адијабатског проблема клипа, где лакша честица представља честицу гаса, а тежа честица представља клип.

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