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# NIPPING ANALYSIS OF A TWO-LEAF SPRING STRENGTHENED BY AN ADDITIONAL FULL-LENGTH LEAF

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Abstract. Nipping analysis of a rectangular two-leaf spring strengthened by an additional full-length leaf is presented. Closed form expressions are derived for the initial gaps between the leaves which make the maximum stresses in all the leaves within the clamped cross-section of the loaded spring equal to each other. The derived expressions are general in the sense that they apply for any values of the introduced leaf-length and thickness parameters. Two initial gaps between the pairs of consecutive leaves are needed to achieve the desired stress reduction. The required gaps can be either positive or negative, depending on the values of introduced spring parameters. For some combination of length and thickness parameters, nipping is not an effective means of stress reduction. There is a particular combination of parameters for which the maximum stresses in all critical cross-sections of leaves become equal to each other. The presented analysis and obtained results may be useful for multi-leaf spring design and related optimization studies.

## 1. Introduction

Leaf springs are used in the vehicle industry to provide support for heavy loads and absorb vibrations due to road irregularities [1–3]. They are made of graduated-length leaves placed one below the other and kept together by clips and bolts (Fig. 1). Leaf springs are initially arc-shaped in such a way that they straighten out under the maximum expected load. Since the length of a leaf spring is much greater than its width and thickness, a loaded spring can be considered as a simply supported mildly-curved beam under the central load. Each half of such a beam acts as a cantilever beam carrying one-half of the total load. Since initial curvatures of leaves are sufficiently small (their radii of curvature being much greater than the leaf lengths), the stress and deflection analysis can be performed by considering the spring configuration with straight leaves and by using the Euler–Bernoulli beam theory. From such an analysis it follows that for leaf springs with a

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uniform width and equal thickness of all leaves, the contact forces between the leaves are different, with the greatest contact force occurring between the two bottom leaves. The bottom-most leaf has the maximum bending moment and thus the greatest bending stress.

To increase the strength and the load capacity of the spring, additional full-length leaves are commonly placed atop the master leaf of the graduated-length portion of the spring (Fig. 1). For a uniform width and equal thickness of all leaves, the maximum stress in such a strengthened spring is still in the bottom-most leaf, while the maximum stress in the added full-length leaves is substantially higher than the maximum stress in the master leaf of the graduated-length portion of the spring. For example, if there are two added full-length leaves atop six graduated-length leaves of length $L$ and the length decrement $L/6$, the maximum stress in the added full-length leaves is 40% higher than the maximum stress in the master leaf [4], assuming the same width and thickness of all leaves. At the same time, it is about 5% smaller than the maximum stress in the bottom-most leaf. From the design point of view, it is of interest to explore the means to decrease the maximum stress in the bottom-most leaf and the added full-length leaves. This can be accomplished by an appropriate choice of the leaf-thicknesses and the length decrements of the graduated-length portion of the spring, or by using leaves with different initial curvatures. In the latter case, upon closing initial gaps between leaves by fastening them with bolts and clamps (referred to as nipping), the leaves become pre-stressed, and, if initial gaps are properly selected, this pre-stress reduces the maximum total stress in the loaded spring during its service. In this paper, we present such a nipping analysis by considering a two-leaf spring stiffened by an added full-length leaf, with all leaves having the same uniform rectangular cross-section. The lengths of two bottom leaves are $L_1$ and $L_2 = \lambda L_1$, and their thicknesses are $h_1$ and $h_2 = \gamma h_1$. The length of the added leaf is $L_1$ and its thickness is $h_a = \alpha h_1$. The values of $0 < \lambda < 1$ and $(\alpha, \gamma) > 0$ are restricted so that the Euler-Bernoulli beam theory can be applied to each considered segment of the spring. The same material is used for all leaves. Two initial gaps are needed to achieve the desired stress reduction, a gap between the two bottom leaves and a gap between the added leaf and the master leaf of the graduated-length portion of the spring. These gaps are determined from the nipping condition that the maximum bending stresses at the clamped ends of all leaves are equal to each other. The closed-form expressions derived for stresses in terms of $(\lambda, \alpha, \gamma)$ may be valuable for the optimization study; for example, if the minimum weight of the spring is an objective function, under the constraint of a prescribed strength or stiffness of the spring [5–7].

In the available literature, we did not find an analysis of nipping-induced stress reduction in leaf springs with rectangular leaves having uniform cross-section throughout their lengths, in contrast to the well-known nipping analysis in the case of leaf springs with triangular endings of leaves (uniform-strength leaves) [8, 9]. For such springs, and for uniform length-decrement and the same leaf-thickness, the maximum stress in the added full-length leaves is 50% higher than in the graduated-length leaves below. Only one initial gap is needed to achieve the stress reduction: a gap between the added full-length leaves and the master leaf of the

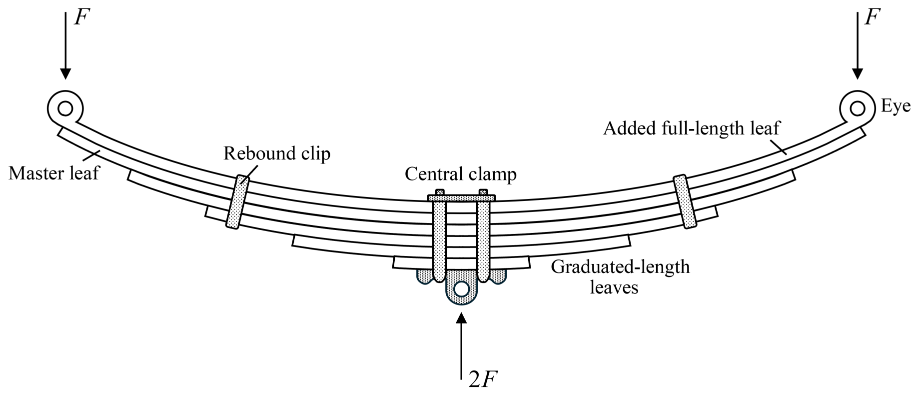



FIGURE 1. A semi-elliptical leaf spring consisting of the master leaf and four graduated-length leaves beneath. An additional full-length leaf is placed atop the master leaf. The depth of the semi-elliptical shape in the preloaded state is such that the spring becomes nearly straight under maximum applied load. The center of the spring is attached to the rear axle of the vehicle. The eyes of the spring are attached to the frame of the vehicle; each eye takes the load $F$ transmitted to the spring by the frame of the vehicle. The upward reactive force from the axle is $2F$.

graduated-length leaves below. Such one-gap nipping, however, does not in general lead to a desired stress reduction for springs with leaves of uniform rectangular cross-section throughout their lengths, which has motivated our double-nipping analysis presented in this paper.

## 2. A two-leaf cantilever spring

Figure 2a shows a two-leaf cantilever spring. Both leaves are made of the same material with the modulus of elasticity $E$, and both have the rectangular cross section of the same width $b$. The length of the upper leaf is $L_1$ and its thickness is $h_1$. The length of the lower leaf is $L_2 = \lambda L_1$ and its thickness is $h_2 = \gamma h_1$, where $0 < \lambda < 1$ and $\gamma > 0$. If the applied end load on the upper leaf is $F$, the force $F_C$ develops between the two leaves at the point of their contact (Fig. 2b,c). The contact force $F_C$ is determined from the condition of equal deflections of two cantilever beams at point $C$,

$$v_C = \frac{FL_2^2}{6EI_1}(3L_1 - L_2) - \frac{F_C L_2^3}{3EI_1} = \frac{F_C L_2^3}{3EI_2},$$

with $I_1 = bh_1^3/12$ and $I_2 = bh_2^3/12$ as the cross-sectional moments of two leaves. This gives

$$F_C = \frac{1}{\lambda} f(\lambda, \gamma) F, \tag{2.1}$$

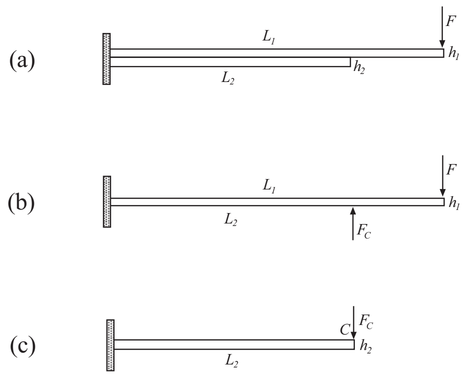



FIGURE 2. (a) A two-leaf cantilever spring made of the upper leaf of length $L_1$ and thickness $h_1$, and the lower leaf of length $L_2$ and thickness $h_2$. The applied force at the end of the upper leaf is $F$. (b) and (c) The free body diagrams of two leaves with the indicated contact force $F_C$.

where

$$f(\lambda,\gamma)=\frac{1}{2}\,(3-\lambda)\,\frac{\gamma^3}{1+\gamma^3} \tag{2.2}$$

is a conveniently introduced geometric parameter, which is always positive ($f>0$). The normalized contact force $F_C/F$ has a monotonically increasing dependence on $\gamma$, and a monotonically decreasing dependence on $\lambda$. For example, if $\lambda=1/2$ and $\gamma=1$, (2.1) gives $F_C=5F/4=1.25F$, while for $\gamma=\lambda=1$, (2.1) gives $F_C=F/2$.

The maximum deflection is at the end of the upper leaf,

$$v_{\max}=\frac{FL_1^3}{3EI_1}-\frac{F_CL_2^3}{3EI_1}-\frac{F_CL_2^2}{2EI_1}\,(L_1-L_2)=\Big[1-\frac{1}{2}\,\lambda(3-\lambda)f\Big]v_1,\quad v_1=\frac{FL_1^3}{3EI_1}.$$

The scaling parameter $v_1$ is the end deflection of a single cantilever leaf of length $L_1$ under the end force $F$. The spring constant $k_2$ of the two-leaf spring is defined by

$$k_2=\frac{F}{v_{\max}}=\frac{k_1}{1-\frac{1}{2}\,\lambda(3-\lambda)f(\lambda,\gamma)},\quad k_1=\frac{F}{v_1}=\frac{3EI_1}{L_1^3}, \tag{2.3}$$

where $k_1$ is the spring constant of a single cantilever leaf of length $L_1$. For example, if $\lambda=1/2$ and $\gamma=1$, (2.3) gives $k_2=(64/39)k_1=1.641k_1$, while for $\lambda=1/2$ and $\gamma=2$, the spring stiffness is approximately double i.e., $k_2=(36/11)k_1=3.273k_1$. For $\gamma=\lambda=1$, (2.3) gives $k_2=2k_1$.

**2.1. Maximum bending stresses.** The (downward) bending moments at the clamped ends of the upper and lower leaves are

$$M_1=FL_1-F_CL_2=[1-f(\lambda,\gamma)]M_*,$$
$$M_2=F_CL_2=f(\lambda,\gamma)M_*,$$

where $M_* = FL_1$ is a scaling factor. The bending moment in the cross-section of the upper leaf above the contact point $C$ is

$$M_C = F(L_1 - L_2) = (1 - \lambda)M_*,$$

independently of $\gamma$. The corresponding maximum bending stresses (being tensile at the upper-most points of the cross-sections) are

$$\sigma_1 = \frac{M_1 h_1}{2I_1} = [1 - f(\lambda, \gamma)]\sigma_*, \quad \sigma_2 = \frac{M_2 h_2}{2I_2} = \frac{1}{\gamma^2} f(\lambda, \gamma)\sigma_*, \tag{2.4}$$

and

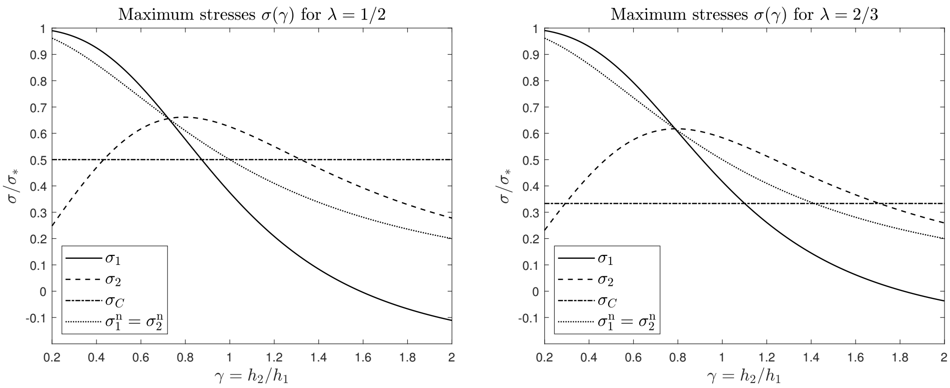
$$\sigma_C = \frac{M_C h_1}{2I_1} = (1 - \lambda)\sigma_*. \tag{2.5}$$

The introduced normalizing stress factor

$$\sigma_* = \frac{M_* h_1}{2I_1} = \frac{FL_1 h_1}{2I_1} \tag{2.6}$$

represents the maximum stress in a single cantilever leaf of length $L_1$ and thickness $h_1$, loaded at its end by the force $F$.

The variations of the maximum bending stresses $\sigma_1$, $\sigma_2$, and $\sigma_C$ versus $\gamma$ for $\lambda = 1/2$ and $\lambda = 2/3$ are shown in Fig. 3. While $\sigma_1$ is monotonically decreasing with $\gamma$ at any $\lambda$, the stress $\sigma_2$ reaches the maximum at $\gamma = \sqrt[3]{3/4} = 0.909$, with its value being approximately equal to $0.26(3 - \lambda)\sigma_*$. Beyond $\gamma = 0.909$, both $\sigma_1$ and $\sigma_2$ decrease with $\gamma$. Such an increase of the thickness of the lower leaf is beneficial




(a) (b)

FIGURE 3. The variations of stresses $\sigma_1$, $\sigma_2$, and $\sigma_C$ with $\gamma$ for: (a) $\lambda = 1/2$, and (b) $\lambda = 2/3$. The normalizing stress is $\sigma_* = FL_1h_1/2I_1$. The variations in maximum stresses at the clamped ends are also presented $\sigma_1^{\text{nip}} = \sigma_2^{\text{nip}} = \sigma_*/(1 + \gamma^2)$ in the case of nipping (dotted curve), considered in section 3. The dotted curve intersects the horizontal line $\sigma_C/\sigma_* = 1 - \lambda$ at $\gamma = [\lambda/(1-\lambda)]^{1/2}$, at which point all three maximum stresses become equal to each other.

for the strength of a two-leaf spring because it decreases its maximum stresses. However, it also increases the stiffness and the weight of the spring, which may not be desirable for damping of vibrations. The variations of $\sigma_1$, $\sigma_2$, and $\sigma_C$ versus $\lambda$, at a given $\gamma$, are linear, as can be recognized from the structure of expressions (2.4) and (2.5).

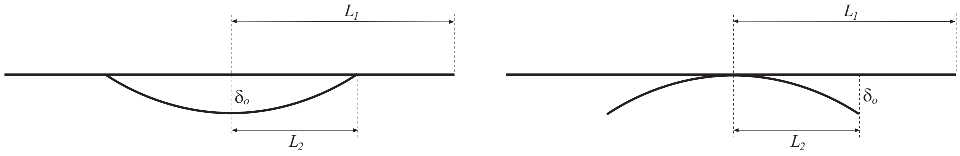
**2.2. Location of the maximum stress.** While $\sigma_2 > 0$ for all $(\lambda, \gamma)$, the stress $\sigma_1$ can be either positive, zero, or negative. It is positive semi-definite ($\sigma_1 \geqslant 0$) if $f \leqslant 1$, i.e., from (2.2), if $(1-\lambda)\gamma^3 \leqslant 2$. For example, for $\lambda = 1/2$, $\sigma_1$ is positive if $\gamma < 1.587$. Furthermore, the magnitude of the maximum stress in the upper leaf can be either in the cross-section of the clamped end, or above the contact point $C$. If $\sigma_1 > 0$, the condition $\sigma_1 > \sigma_C$ gives $f < \lambda$, i.e., $3(1-\lambda)\gamma^3 < 2\lambda$. Thus, if $\sigma_1 > 0$, the maximum stress in the upper leaf is $\sigma_1$ if $3(1-\lambda)\gamma^3 < 2\lambda$, or $\sigma_C$ if $3(1-\lambda)\gamma^3 > 2\lambda$. On the other hand, if $\sigma_1 < 0$, the condition $-\sigma_1 > \sigma_C$ gives $f > 2 - \lambda$, i.e., $(1-\lambda)\gamma^3 < 2(\lambda - 2)$, which is never satisfied for $\lambda < 1$. Therefore, if $\sigma_1 < 0$, the maximum stress in the upper leaf is always $\sigma_C = (1-\lambda)\sigma_*$. Recall that the condition for $\sigma_1 < 0$ is $(1-\lambda)\gamma^3 > 2$.

We next discuss the conditions for the magnitude of the maximum stress in the upper leaf to be greater than in the lower leaf. If $\sigma_1 > 0$, i.e., $(1-\lambda)\gamma^3 < 2$, the maximum stress in the upper leaf is $\sigma_1$, provided that $3(1-\lambda)\gamma^3 < 2\lambda$, and the condition $\sigma_1 > \sigma_2$ is fulfilled if $(1-\lambda)\gamma^3 + (3-\lambda)\gamma - 2 < 0$. For example, if $\lambda = 1/2$, the above inequalities are all satisfied for $\gamma < 0.724$. On the other hand, if $\sigma_1 > 0$, but the maximum stress in the upper leaf is $\sigma_C$, which is the case for $3(1-\lambda)\gamma^3 > 2\lambda$, the condition $\sigma_C > \sigma_2$ is fulfilled if $2(1-\lambda)\gamma^3 - (3-\lambda)\gamma + 2(1-\lambda) > 0$. For example, if $\lambda = 1/2$, the above inequalities are all satisfied for $1.32 < \gamma < 1.587$. Finally, if $\sigma_1 < 0$, i.e., $(1-\lambda)\gamma^3 > 2$, the maximum stress in the upper leaf is $\sigma_C$, and the condition $\sigma_C > \sigma_2$ is fulfilled if $2(1-\lambda)\gamma^3 - (3-\lambda)\gamma + 2(1-\lambda) > 0$. In the case where $\lambda = 1/2$, this implies that $\gamma > 1.587$. In summary, for $\lambda = 1/2$ the maximum stress is in the lower leaf, being equal to $\sigma_2$, if $0.724 < \gamma < 1.32$; otherwise, it is in the upper leaf, being equal to either $\sigma_1$ or $\sigma_C$, as previously specified.

Examples. For $\lambda = 1/2$ and $\gamma = 1/2$, the stresses are $\sigma_1 = 0.861\sigma_*$, $\sigma_C = 0.5\sigma_*$, and $\sigma_2 = 0.556\sigma_*$, i.e., the maximum stress ($\sigma_1$) in the upper leaf is about 55% greater than that in the lower leaf. For $\lambda = 1/2$ and $\gamma = 3/2$, the stresses are $\sigma_1 = 0.036\sigma_*$, $\sigma_C = 0.5\sigma_*$, and $\sigma_2 = 0.429\sigma_*$, i.e., the maximum stress ($\sigma_C$) in the upper leaf is about 16.6% greater than that in the lower leaf. On the other hand, for $\lambda = 1/2$ and $\gamma = 1$, the stresses are $\sigma_1 = 0.375\sigma_*$, $\sigma_C = 0.5\sigma_*$, and $\sigma_2 = 0.625\sigma_*$, i.e., the maximum stress ($\sigma_C$) in the upper leaf is 25% smaller than that in the lower leaf. It also follows that for equal thicknesses of two leaves ($\gamma = 1$), the maximum stress for any $\lambda < 1$ is always in the bottom leaf.

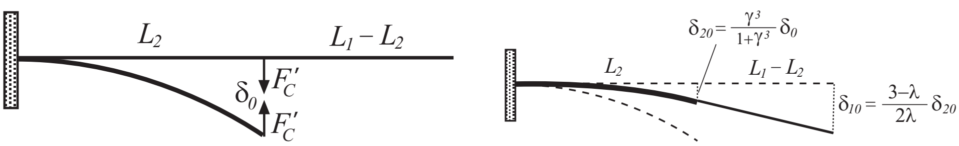
## 3. Nipping analysis of a two-leaf spring

The objective in this section is to determine the decrease of the maximum stress in a two-leaf spring produced by nipping if the leaves are made with different initial curvatures before their fastening into a leaf spring (Fig. 4). For simplicity, but without loss of generality, we consider that the upper leaf is initially straight.




(a) (b)

FIGURE 4. The initial shape of the lower leaf with (a) a central gap $\delta_0$, and (b) two side gaps $\delta_0$, relative to the upper leaf. Both leaves are of uniform width $b$. The length of the upper leaf is $L_1$ and its thickness is $h_1$; the length of the lower leaf is $L_2 = \lambda L_1$ and its thickness is $h_2 = \gamma h_1$. The gap $\delta_0 \ll L_2$ and the width and heights are sufficiently small relative to the lengths of the leaves that the Euler-Bernoulli beam theory can be used in the stress and deflection analysis of the spring.




(a) (b)

FIGURE 5. (a) If there is an initial gap $\delta_0$ between two leaves, the internal force $F'_C$ is required to close the gap. (b) The deformed prestressed spring configuration with a closed gap $\delta_0$. The residual deflection left upon clamping two leaves from part (a) at the contact point $x = L_2$ is $\delta_{20} = \gamma^3\delta_0/(1+\gamma^3)$, while $\delta_{10} = \delta_{20}(3-\lambda)/(2\lambda) \equiv (f/\lambda)\delta_0$ at the end $x = L_1$.

The shape of the lower leaf in the case shown in Fig. 4a may be specified by $v_0(x) = \delta_0[1-(x/L_2)^2]$, where $\delta_0 \ll L_2$ is the initial central gap between two leaves, which ensures that the two leaves do not overlap upon their fastening into a leaf-spring.

The prestresses in two leaves produced by closing the gap $\delta_0$ using the central and lateral clamps have opposite signs in cases (a) and (b) of Fig. 4. Therefore, it is sufficient to consider only one of these two cases and we conveniently choose case (b), with two side gaps $\delta_0$ (Fig. 4b).

Upon closing these gaps, the internal forces develop as indicated in Fig. 5a. The shown cantilever spring represents the right-half of the symmetric spring from Fig. 4b. The internal force $F'_C$ is determined from the closing-of-gap condition,

$$\frac{F'_C L_2^3}{3EI_1} + \frac{F'_C L_2^3}{3EI_2} = \delta_0,$$

which gives

$$F'_C = \frac{3EI_1}{L_2^3}\,\frac{\gamma^3}{1+\gamma^3}\,\delta_0. \tag{3.1}$$

For the latter purposes, (3.1) can be conveniently rewritten as

$$F'_C = F\,\frac{1}{\lambda^3}\,\frac{\gamma^3}{1+\gamma^3}\,\frac{\delta_0}{\delta_1}, \quad \delta_1 = \frac{FL_1^3}{3EI_1}, \tag{3.2}$$

for any value of external force $F$, to be applied upon assembling the spring.

The bending moments at the clamped ends of two leaves, due to $F'_C$, are

$$M'_1 = -M'_2 = F'_C L_2 = \frac{Eh_1\delta_0}{L_1^2}\,\frac{1}{\lambda^2}\,\frac{\gamma^3}{1+\gamma^3} = M_*\,\frac{1}{\lambda^2}\,\frac{\gamma^3}{1+\gamma^3}\,\frac{\delta_0}{\delta_1}.$$

The downward moment is considered positive, to be consistent with the convention used in section 2. The corresponding maximum stresses are

$$\begin{aligned} \sigma'_1 &= \frac{3}{2}\,\frac{Eh_1\delta_0}{L_1^2}\,\frac{1}{\lambda^2}\,\frac{\gamma^3}{1+\gamma^3}, \\ \sigma'_2 &= -\frac{3}{2}\,\frac{Eh_1\delta_0}{L_1^2}\,\frac{1}{\lambda^2}\,\frac{\gamma}{1+\gamma^3}, \end{aligned} \tag{3.3}$$

the negative sign indicating a compressive stress. Equivalently, (3.3) can be expressed as

$$\begin{aligned} \sigma'_1 &= \sigma_*\,\frac{1}{\lambda^2}\,\frac{\gamma^3}{1+\gamma^3}\,\frac{\delta_0}{\delta_1}, \\ \sigma'_2 &= -\sigma_*\,\frac{1}{\lambda^2}\,\frac{\gamma}{1+\gamma^3}\,\frac{\delta_0}{\delta_1}. \end{aligned} \tag{3.4}$$

**3.1. Total stresses.** If the assembled spring is loaded by two downward end forces $F$, balanced by an upward force $2F$ in the middle of the spring, the total stresses are the sum of the stresses (2.4) due to external load $F$, and the prestresses (3.4). Thus, the maximum stresses at the uppermost points of the clamped ends are

$$\begin{aligned} \sigma_1^T &= \sigma_1^F + \sigma'_1 = \sigma_*\Big[1 - f(\lambda,\gamma) + \frac{1}{\lambda^2}\,\frac{\gamma^3}{1+\gamma^3}\,\frac{\delta_0}{\delta_1}\Big], \\ \sigma_2^T &= \sigma_2^F + \sigma'_2 = \sigma_*\Big[\frac{1}{\gamma^2}\,f(\lambda,\gamma) - \frac{1}{\lambda^2}\,\frac{\gamma}{1+\gamma^3}\,\frac{\delta_0}{\delta_1}\Big]. \end{aligned} \tag{3.5}$$

The normal stress in the cross-section above the contact point is unaffected by nipping, i.e., $\sigma_C^T = \sigma_C^F = (1-\lambda)\sigma_*$, as given by (2.5).

The total contact force between the upper (master) leaf and the shorter leaf below, due to both nipping and external load, is

$$F_C^T = F_C^F - F'_C = \frac{fF}{\lambda} - \frac{F}{\lambda^3}\,\frac{\gamma^3}{1+\gamma^3}\,\frac{\delta_0}{\delta_1},$$

where expression (2.1) is used for $F_C^F$, and (3.2) for $F'_C$.

**3.2. Nipping condition.** The initial gap $\delta_0$ can be specified from the nipping condition that the maximum bending stresses at the clamped end of two leaves are equal to each other,

$$|\sigma_1^T| = |\sigma_2^T|. \tag{3.6}$$

There are two cases when (3.6) is satisfied. The first one is $\sigma_1^T = \sigma_2^T$. Upon using (3.5) we obtain

$$\frac{\delta_0}{\delta_1} = \lambda^2 \Big[\frac{3-\lambda}{2} - \frac{1+\gamma^3}{\gamma(1+\gamma^2)}\Big], \quad \delta_1 = \frac{FL_1^3}{3EI_1}. \tag{3.7}$$

The corresponding bending stress is obtained by substituting (3.7) into (3.5),

$$\sigma_1^{\text{nip}} = \sigma_2^{\text{nip}} = \frac{\sigma_*}{1+\gamma^2}. \tag{3.8}$$

The second case is $\sigma_1^T = -\sigma_2^T$, which can occur if and only if $\gamma < 1$. This gives

$$\frac{\delta_0}{\delta_1} = \lambda^2 \Big[\frac{3-\lambda}{2} + \frac{1+\gamma^3}{\gamma(1-\gamma^2)}\Big],$$

with the corresponding bending stress

$$\sigma_1^{\text{nip}} = -\sigma_2^{\text{nip}} = \frac{\sigma_*}{1-\gamma^2}.$$

However, the magnitude of this total stress is greater than that in (3.8), because $\gamma > 0$, and the second nipping condition is of no practical interest. Therefore, (3.7) and (3.8) specify the initial gap $\delta_0$ and the corresponding maximum stresses at the clamped ends of two leaves.

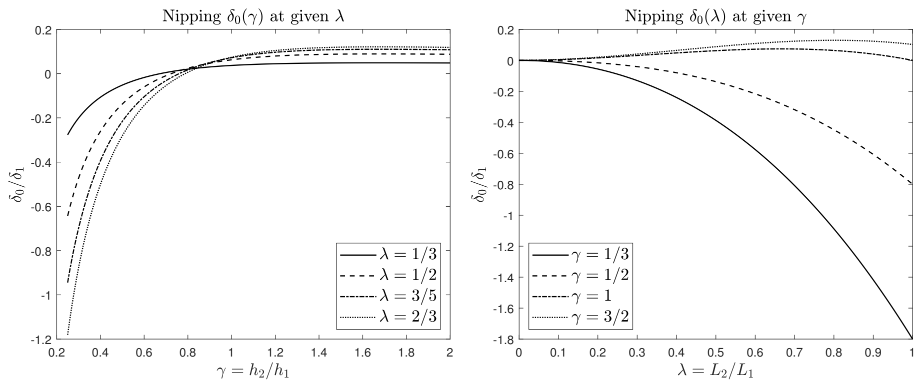
3.2.1. *Making maximum stresses equal to each other.* There is a unique relationship between $\lambda$ and $\gamma$ for which all three maximum stresses in the loaded pre-nipped configuration of the spring are equal to each other. From (2.5) and (3.8), this is

$$\sigma_1^{\text{nip}} = \sigma_2^{\text{nip}} = \sigma_C^{\text{nip}} = (1-\lambda)\sigma_* \quad \Leftrightarrow \quad \gamma^2 = \frac{\lambda}{1-\lambda}. \tag{3.9}$$

For example, for $\lambda = (1/3, 1, 2/3)$, (3.9) gives $\gamma = (\sqrt{2}/2, 1, \sqrt{2})$. The pairs of values $(\lambda, \gamma)$ satisfying the condition $\gamma^2 = \lambda/(1-\lambda)$ are optimal regarding the strength (equal maximum stress in both leaves) and the weight of the spring. For $\gamma > [\lambda/(1-\lambda)]^{1/2}$, the stresses $\sigma_1^{\text{nip}} = \sigma_2^{\text{nip}}$ decrease below the maximum stress in the spring $\sigma_C = (1-\lambda)\sigma_*$, but the spring becomes stiffer and heavier (Fig. 3).

While nipping reduces the greatest stress in the spring (e.g., $\sigma_2^T$), it increases the other maximum stress ($\sigma_1^T$), depending on the values of $\lambda$ and $\gamma$, as can be seen from the plots shown earlier in Fig. 3. Figure 6a shows the variation of the initial gap $\delta_0/\delta_1$ with $\gamma$ for several values of $\lambda$, obtained from (3.7). The variation of $\delta_0/\delta_1$ with $\lambda$ for several values of $\gamma$ is shown in Fig. 6b.

Other nipping conditions could also be considered. For example, we have examined three additional nipping conditions: i) $\sigma_C = \sigma_2^T$ , ii) $|\sigma_1^T| + \sigma_C = 2\sigma_2^T$ , and iii) $F_C^T = F$ (leading to pure bending of the upper leaf from the clamped end to the contact point), but have found all of them to be less effective for the maximum stress reduction than the condition $\sigma_1^T = \sigma_2^T$.




(a) (b)

FIGURE 6. (a) The variation of the initial gap $\delta_0/\delta_1$ with $\gamma$ for several values of $\lambda$, required by the nipping condition $\sigma_1^T = \sigma_2^T$. (b) The corresponding variation of $\delta_0/\delta_1$ with $\lambda$ for several values of $\gamma$.

## 4. A two-leaf spring strengthened by an additional leaf

Additional full-length leaves are commonly placed on top of the master leaf of the graduated-length portion of the leaf spring to increase its strength and load capacity [10–12]. Figure 7a shows a two-leaf cantilever spring with an additional full-length leaf, having the thickness $h_a = \alpha h_1$ ($\alpha > 0$) and thus the cross-sectional moment of inertia $\alpha^3 I_1$, assuming the same width as for the two leaves below. All three leaves are made of the same material with the modulus of elasticity $E$. If the end-applied load at the top of the added leaf is $F$, the contact force $F_A$ develops between the added and the master leaf, as shown in Fig. 7b,c. This force is determined from the condition of equal deflections of two leaves at the contact point $A$,

$$v_A = \frac{(F - F_A)L_1^3}{3\alpha^3 EI_1} = \frac{F_A L_1^3}{3EI_1}\Big[1 - \frac{1}{2}\lambda(3-\lambda)f\Big]. \tag{4.1}$$

Expression(2.3) for the spring constant $k$ of a two-leaf spring was used to write the expression on the right-hand side of (4.1), with $f = f(\lambda, \gamma)$ defined by (2.2). Solving (4.1) for $F_A$ gives

$$F_A = \frac{F}{g}, \quad g = 1 + \alpha^3\Big[1 - \frac{\lambda}{2}(3-\lambda)f\Big]. \tag{4.2}$$

The parameter $g = g(\lambda, \gamma)$, expressed in terms of $f$, is conveniently introduced. The maximum deflection is consequently

$$v_{\max} = \frac{g-1}{\alpha^3 g}\, v_1, \quad v_1 = \frac{FL_1^3}{3EI_1},$$

and the spring constant of a strengthened spring becomes

$$\bar{k}_2 = \frac{F}{v_{\max}} = \frac{\alpha^3 g}{g-1}\, k_1,$$
$$k_1 = \frac{3EI_1}{L_1^3}.$$

This expression can also be derived from the expression for the equivalent spring constant of the parallel connection of the two-leaf spring and the added full-length leaf, i.e., from the sum of the spring constant $k_2$ given by (2.3) and $\alpha^3 k_1$ ($\bar{k}_2 = k_2 + \alpha^3 k_1$).

The maximum bending stresses (tensile at the upper-most points of the cross-sections) are

$$\sigma_a = \frac{g-1}{\alpha^2 g}\,\sigma_*, \qquad \sigma_1 = \frac{1-f}{g}\,\sigma_*,$$
$$\sigma_2 = \frac{f}{g\gamma^2}\,\sigma_*, \qquad \sigma_C = \frac{1-\lambda}{g}\,\sigma_*. \tag{4.3}$$

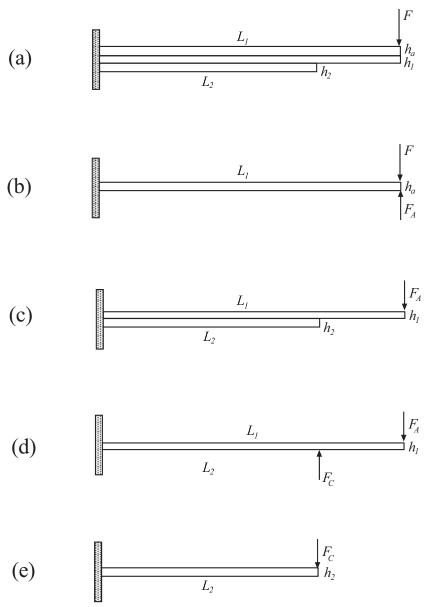
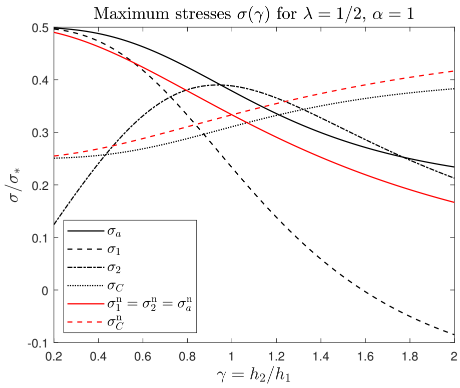
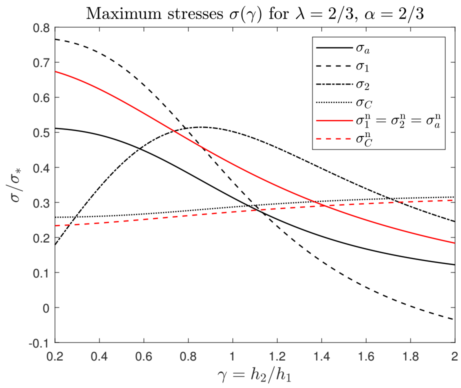
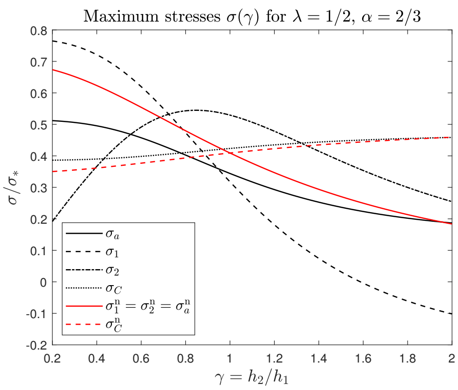


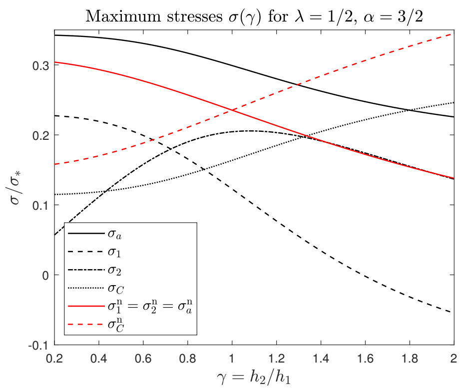

FIGURE 7. (a) A full-length leaf of thickness $h_a = \alpha h_1$ is placed atop the leaves of length $L_1$ and thickness $h_1$, and length $L_2 = \lambda L_1$ and thickness $h_2 = \gamma h_1$. The applied force at the end of the uppermost leaf is $F$. (b) and (c) The free-body diagrams of the added full-length leaf and two graduated-length leaves below showing their contact force $F_A$. (d) and (e) The free-body diagrams of the bottom two leaves showing their contact force $F_C$.




(a) (b)




(c) (d)

FIGURE 8. The variations of the maximum stresses (4.3) with $\gamma$ for: (a) $\lambda = 1/2$, $\alpha = 1$, and (b) $\lambda = 2/3$, $\alpha = 2/3$. Also shown are the variations of the maximum stresses $\sigma_1^n = \sigma_2^n = \sigma_a^n$ and $\sigma_C^n$ in the case of nipping (red curves), obtained from (5.10) and (5.11).

In deriving (4.3), expressions (2.4) and (2.6) were used for $\sigma_1$ and $\sigma_2$, with $F_A$ acting on the two-leaf spring in Fig. 7c, rather than $F$ as in Fig. 2. The contact force $F_C$ between the two bottom leaves, shown in Fig. 7d,e, is given by $F_C = (f/\lambda)F_A = fF/(\lambda g)$, which follows from (2.1) by replacing $F$ with $F_A$. When this is substituted into $\sigma_C = F_C(L_1 - L_2)h_1/(2I_1)$, the expression for $\sigma_C$ in (4.3) is obtained. The variations of the stresses $\sigma_a$, $\sigma_1$, $\sigma_2$, and $\sigma_C$ versus $\gamma$ are shown in Fig. 8 for $\lambda = 1/2$, $\alpha = 2$, and $\lambda = 2/3$, and $\alpha = 2/3$. This figure also shows the variation of the maximum stress in the case when the load is applied to the pre-stressed spring, created by the process of nipping considered in section 5 (dotted curves).

## 5. Nipping analysis of a strengthened two-leaf spring

The objective in this section is determine the initial gap between the bottom two leaves ($\delta_0$) and the initial gap ($\delta_a$) between the added full-length leaf and the two fastened leaves below (Fig. 10), in order that, upon nipping and external loading of the assembled spring, the maximum stresses at the clamped ends are the same in all three leaves. The gap $\delta_a$ depends on the initial gap $\delta_{ao}$ of the added full-length leaf relative to the horizontal direction (corresponding to a given initial curvature of the added leaf, which is to be determined), and the residual gap $\delta_{10}$ left upon fastening of the two bottom leaves, which is given in Fig. 5. Thus, referring to Fig. 10,

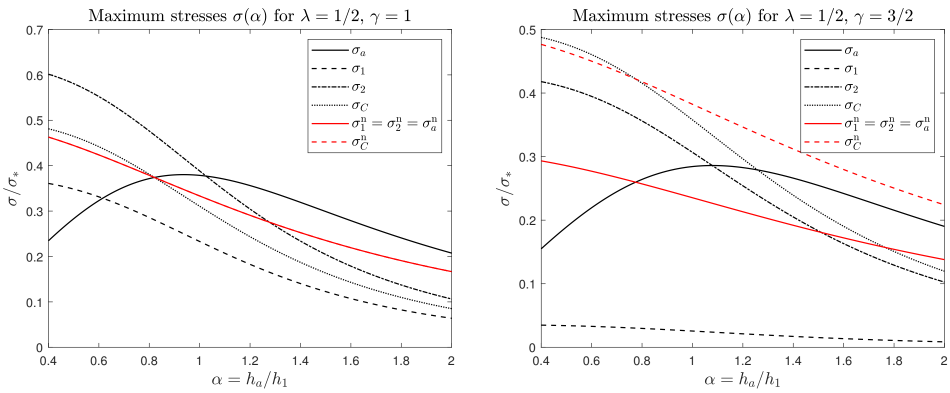
$$\delta_a = \delta_{10} - \delta_{ao},\qquad \delta_{10} = \frac{3-\lambda}{2\lambda}\,\frac{\gamma^3}{1+\gamma^3}\,\delta_0. \tag{5.1}$$

To close the gap $\delta_a$, the internal force $F'_A$ is applied and its value determined from the closing-of-gap condition

$$\frac{F'_A L_1^3}{3\alpha^3 E I_1} + \frac{F'_A L_1^3}{3EI_1}\left[1 - \frac{\lambda}{2}(3-\lambda)f\right] = \delta_a. \tag{5.2}$$

The expression for the spring constant $k_2$ of the two-leaf spring from (2.3) is used to write the second term on the left-hand side of (5.2). Thus,

$$F'_A = \frac{3\alpha^3}{g}\,\frac{EI_1\delta_a}{L_1^3}, \tag{5.3}$$




(a) (b)

FIGURE 9. The variations of the maximum bending stresses (4.3) with $\alpha$ for: (a) $\lambda = 1/2$, $\gamma = 1$, and (b) $\lambda = 1/2$, $\gamma = 3/2$. The variations of the maximum stresses $\sigma_1^n = \sigma_2^n = \sigma_a^n$ and $\sigma_C^n$ are also shown in the case of nipping (red curves), obtained from (5.10) and (5.11).

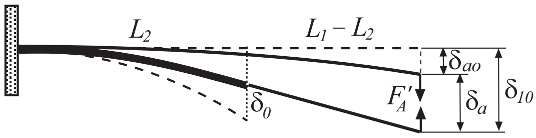



FIGURE 10. The residual end deflection of a pre-nipped two-leaf spring is $\delta_{10}$, corresponding to the initial gap $\delta_0$ between the two leaves. The added third leaf has a pre-curvature, so that its initial gap relative to the horizontal reference level is $\delta_{ao}$. The total initial gap between the added leaf and the nipped two leaves below is $\delta_a = \delta_{10} - \delta_{ao}$. To close this gap and assemble the spring, a pair of opposite internal forces $F_A'$ is applied, as shown.

with $g$ defined in (4.2). The maximum stresses in three leaves corresponding to $F_A'$, in analogy with (4.3), are

$$\sigma_a' = \frac{1}{\alpha^2}\,\sigma_\bullet, \qquad \sigma_1' = -(1-f)\sigma_\bullet, \qquad \sigma_2' = -\frac{f}{\gamma^2}\,\sigma_\bullet, \qquad \sigma_C' = -(1-\lambda)\sigma_\bullet, \tag{5.4}$$

expressed in terms of the stress parameter

$$\sigma_\bullet = \frac{F_A' L_1 h_1}{2I_1} = \frac{3\alpha^3}{2g}\,\frac{Eh_1\delta_a}{L_1^2}.$$

Because the bottom two leaves have already been fastened, they carry additional stress (prestress) corresponding to their initial gap $\delta_0$. By denoting them as $\sigma_1''$ and $\sigma_2''$, they are given from (3.3) by

$$\sigma_1'' = \frac{3}{2}\,\frac{Eh_1\delta_0}{L_1^2}\,\frac{1}{\lambda^2}\,\frac{\gamma^3}{1+\gamma^3}, \qquad \sigma_2'' = -\frac{3}{2}\,\frac{Eh_1\delta_0}{L_1^2}\,\frac{1}{\lambda^2}\,\frac{\gamma}{1+\gamma^3}. \tag{5.5}$$

The total internal stresses in the assembled spring are the sum of (5.4) and (5.5). If the so-assembled three-leaf spring is then loaded by an external force $F$ at the end of the added (upper-most) leaf, additional stresses arise, which are given, according to (4.3), by

$$\sigma_a^F = \frac{g-1}{\alpha^2 g}\,\sigma_*, \qquad \sigma_1^F = \frac{1-f}{g}\,\sigma_*, \qquad \sigma_2^F = \frac{f}{g\gamma^2}\,\sigma_*, \qquad \sigma_C^F = \frac{1-\lambda}{g}\,\sigma_*. \tag{5.6}$$

**5.1. Total stresses.** The total stresses in the assembled spring are the sum of the stresses from the external load $F$ and the prestresses due to double-nipping. Consequently, the maximum stresses at the clamped ends of three leaves and in the middle leaf above the contact point $C$ are

$$\sigma_1^T = \sigma_1^F + \sigma_1' + \sigma_1'', \qquad \sigma_2^T = \sigma_2^F + \sigma_2' + \sigma_2'',$$
$$\sigma_a^T = \sigma_a^F + \sigma_a', \qquad \sigma_C^T = \sigma_C^F + \sigma_C'.$$

Upon using (5.4), (5.5) and (5.6), these become

$$\begin{aligned}
\frac{\sigma_1^T}{\sigma_*} &= \frac{1}{\lambda^2}\,\frac{\gamma^3}{1+\gamma^3}\,\frac{\delta_0}{\delta_1} + \frac{1-f}{g}\Big(1-\alpha^3\,\frac{\delta_a}{\delta_1}\Big),\\
\frac{\sigma_2^T}{\sigma_*} &= -\frac{1}{\lambda^2}\,\frac{\gamma}{1+\gamma^3}\,\frac{\delta_0}{\delta_1} + \frac{f}{g\gamma^2}\Big(1-\alpha^3\,\frac{\delta_a}{\delta_1}\Big),\\
\frac{\sigma_a^T}{\sigma_*} &= \frac{1}{\alpha^2}\Big[1-\frac{1}{g}\Big(1-\alpha^3\,\frac{\delta_a}{\delta_1}\Big)\Big],\\
\frac{\sigma_C^T}{\sigma_*} &= \frac{1-\lambda}{g}\Big(1-\alpha^3\,\frac{\delta_a}{\delta_1}\Big),
\end{aligned} \tag{5.7}$$

where the identity was conveniently implemented,

$$\frac{3}{2}\,\frac{Eh_1\delta_1}{L_1^2} \equiv \sigma_*, \qquad \delta_1 = \frac{FL_1^3}{3EI_1}.$$

**5.2. Nipping condition.** The nipping-induced stress reduction depends on the imposed nipping condition used to specify the initial gaps $\delta_0$ and $\delta_a$. We examine in this section the nipping condition which requires that the maximum stresses at the clamped ends of all three leaves are equal to each other, i.e., $\sigma_1^T = \sigma_2^T$ and $\sigma_2^T = \sigma_a^T$. From (5.7), the following system of linear algebraic equations for $\delta_0/\delta_1$ and $\delta_a/\delta_1$ is then obtained

$$\begin{aligned}
\frac{g}{\lambda^2}\,\frac{\gamma(1+\gamma^2)}{1+\gamma^3}\,\frac{\delta_0}{\delta_1} + \alpha^3\Big(f-1+\frac{f}{\gamma^2}\Big)\frac{\delta_a}{\delta_1} &= f-1+\frac{f}{\gamma^2},\\
\frac{g}{\lambda^2}\,\frac{\gamma}{1+\gamma^3}\,\frac{\delta_0}{\delta_1} + \alpha\Big(1+\frac{f\alpha^2}{\gamma^2}\Big)\frac{\delta_a}{\delta_1} &= \frac{1}{\alpha^2}\,(1-g)+\frac{f}{\gamma^2}.
\end{aligned} \tag{5.8}$$

A closed-form solution to (5.8) is

$$\frac{1}{\lambda^2}\,\frac{\gamma}{1+\gamma^3}\,\frac{\delta_0}{\delta_1} = \frac{f-(1-f)\gamma^2}{\gamma^2(1+\alpha^2+\gamma^2)}, \qquad \alpha^3\,\frac{\delta_a}{\delta_1} = 1-\frac{(1+\gamma^2)g}{1+\alpha^2+\gamma^2}. \tag{5.9}$$

When (5.9) is substituted into (5.7), the maximum stresses at the clamped ends become

$$\sigma_1^{\mathrm{nip}} = \sigma_2^{\mathrm{nip}} = \sigma_a^{\mathrm{nip}} = \frac{\sigma_*}{1+\alpha^2+\gamma^2}. \tag{5.10}$$

This is a remarkably simple generalization of the expression for the maximum stress $\sigma_*/(1+\gamma^2)$ in the pre-nipped and loaded two-leaf spring, given by (3.8) of section 3.2, because in the case $\alpha = 1$, (5.10) reduces to $\sigma_*/(2+\gamma^2)$. The maximum stress in the middle leaf above the contact point $C$ is

$$\sigma_C^{\mathrm{nip}} = \frac{(1-\lambda)(1+\gamma^2)}{1+\alpha^2+\gamma^2}\,\sigma_*. \tag{5.11}$$

The plots of (5.10) and (5.11) versus $\gamma$ are shown in Fig. 8 (red curves). This figure also contains the plots of the maximum stresses $(\sigma_1, \sigma_2, \sigma_a)$ in the absence of nipping, quantifying the magnitude of the maximum stress reduction for various values of $(\lambda, \alpha, \gamma)$ achieved by double-nipping. From the practical point of view, the most appealing combination of the parameters $(\lambda, \alpha, \gamma)$ is that for which all four maximum stresses are equal to each other ($\sigma_1^T = \sigma_2^T = \sigma_a^T = \sigma_C^T$). This is because to the right of the $\gamma$-value corresponding to the intersection of the two nipping stress curves (red curves in Fig. 8), the stress $\sigma_C^T$ is increased, and to the left the stress $\sigma_1^T = \sigma_2^T = \sigma_a^T$ is increased, relative to their values at the intersection point $\gamma = [\lambda/(1-\lambda)]^{1/2}$ (see section 5.2.1 below).

Having determined $\delta_a$, the value of $\delta_{ao}$ due to initial curvature of the added leaf, follows from (5.1) as $\delta_{ao} = \delta_{10} - \delta_a$. Thus, upon using (5.9), the two required gaps for the described double-nipping are

$$\text{(5.12)} \quad \frac{\delta_{ao}}{\delta_1} = \frac{1-(1/\alpha)+[1-\lambda(3-\lambda)/2]\gamma^2}{1+\alpha^2+\gamma^2}, \quad \frac{\delta_0}{\delta_1} = \frac{\lambda^2(1+\gamma^3)[f-(1-f)\gamma^2]}{\gamma^3(1+\alpha^2+\gamma^2)}.$$

Figure 9 shows the variation of the nipping stresses with $\alpha$ at given values of $\lambda$ and $\gamma$. From Fig. 9a it can be seen that for $\lambda = 1/2$ and $\gamma = 1$, the nipping stress decreases with $\alpha$ and in this particular case all four stresses are equal to each other ($\sigma_1^T = \sigma_2^T = \sigma_a^T = \sigma_C^T$), for any $\alpha$. While the increase of $\alpha$ decreases the maximum nipping stress, thus increasing the strength of the spring, it also increases its stiffness, which may not be desirable regarding vibration considerations. On the other hand, from Fig. 9b we see that for $\lambda = 1/2$ and $\gamma = 3/2$, the nipping is not an effective means of stress reduction, because for $\alpha$ greater than about 0.8, $\sigma_C^T$ is greater than all stresses in the spring without nipping, while for smaller values of $\alpha$, the stress $\sigma_C^T$ is barely smaller than $\sigma_C$. Also, the stress $\sigma_C^T$ is greater than $\sigma_1^T = \sigma_2^T = \sigma_a^T$ for all values of $\alpha$, provided that $\gamma > [\lambda/(1-\lambda)]^{1/2}$. One can similarly examine the nipping effectiveness with respect to $\alpha$ for any other values of $\lambda$ and $\gamma$.

5.2.1. *Making maximum stresses equal to each other.* There is a unique relationship between $\lambda$ and $\gamma$ for which all four maximum stresses in the pre-nipped and loaded configuration of the spring become equal to each other. From (5.10) and (5.11), this is

$$\text{(5.13)} \qquad \sigma_1^{\text{nip}} = \sigma_2^{\text{nip}} = \sigma_a^{\text{nip}} = \sigma_C^{\text{nip}} = \frac{\sigma_*}{1+\alpha^2+\gamma^2} \quad \Leftrightarrow \quad \gamma^2 = \frac{\lambda}{1-\lambda}.$$

For $\lambda = 1/2$, (5.13) gives $\gamma = 1$ (Fig. 8a,c,d), while for $\lambda = 2/3$ it gives $\gamma = \sqrt{2} \approx 1.4142$ (Fig. 8b). The corresponding nipping stresses in the four cases shown in Fig. 8 are $\sigma^{\text{nip}} = \sigma_*/3$, $9\sigma_*/31 \approx 0.29\sigma_*$, $9\sigma_*/22 \approx 0.409\sigma_*$, and $4\sigma_*/17 \approx 0.235\sigma_*$, respectively. The required normalized gaps $(\delta_{ao}, \delta_0)/\delta_1$ for this stress reduction follow from (5.12) and are equal to $(0.125, 0.042)$, $(-0.016, 0.103)$, $(-0.051, 0.051)$, and $(0.167, 0.029)$. Other nipping conditions could also be examined, similarly to the analysis from section 3, but the nipping condition considered in this section appears to be the most effective for the stress reduction in the spring.

**5.3. Total contact forces.** The total contact forces $F_A^T$ (between the added leaf and the master leaf below), and $F_C^T$ (between the master leaf and the shorter leaf below), due to double nipping and applied external load, are

$$F_A^T = F_A^F - F'_A, \quad F_C^T = F_C^F - (F'_C + F''_C) = \frac{fF_A^F}{\lambda} - \left(\frac{fF'_A}{\lambda} + F''_C\right). \tag{5.14}$$

Using (4.2) for $F_A^F$, (5.3) for $F'_A$, (2.1), corresponding to $F_A^F$ and $F'_A$, for $F_C^F$ and $F'_C$, respectively, and using the right-hand side of (3.2) for $F''_C$, we obtain from (5.14),

$$F_A^T = \frac{F}{g}\left(1 - \alpha^3\,\frac{\delta_a}{\delta_1}\right), \quad F_C^T = \frac{F}{\lambda}\left[\frac{f}{g}\left(1 - \alpha^3\,\frac{\delta_a}{\delta_1}\right) - \frac{1}{\lambda^2}\,\frac{\gamma^3}{1+\gamma^3}\,\frac{\delta_0}{\delta_1}\right]. \tag{5.15}$$

The contact forces $F'_C$ and $F''_C$ are shown in Fig. 11. They can be used to reproduce the previously derived stress expressions by substituting (5.14) into

$$\begin{aligned}
\sigma_a &= \frac{M_a h_1}{2\alpha^2 I_1}, \quad M_a = (F - F_A^T)L_1,\\
\sigma_C &= \frac{M_C h_1}{2I_1}, \quad M_C = F_A^T(L_1 - L_2),\\
\sigma_1 &= \frac{M_1 h_1}{2I_1}, \quad M_1 = F_A^T L_1 - F_C^T L_2,\\
\sigma_2 &= \frac{M_2 h_2}{2I_2}, \quad M_2 = F_C^T L_2.
\end{aligned} \tag{5.16}$$

It is also noted that expressions (5.16) can be used to determine both $F_A^T$ and $F_C^T$ from them, if these contact forces were not previously determined, by imposing the nipping condition $\sigma_a = \sigma_1 = \sigma_2$, and then using (5.15) to calculate the corresponding gaps $\delta_a$ and $\delta_0$. Numerical values for the contact forces are given in Table 1. In an early study of nipping in multi-leaf springs [13], the gaps were assumed to exist between all leaves, but they were arbitrarily prescribed from the

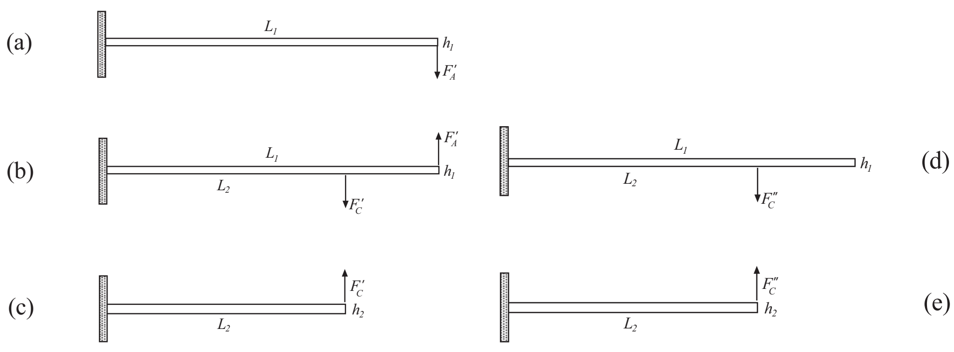
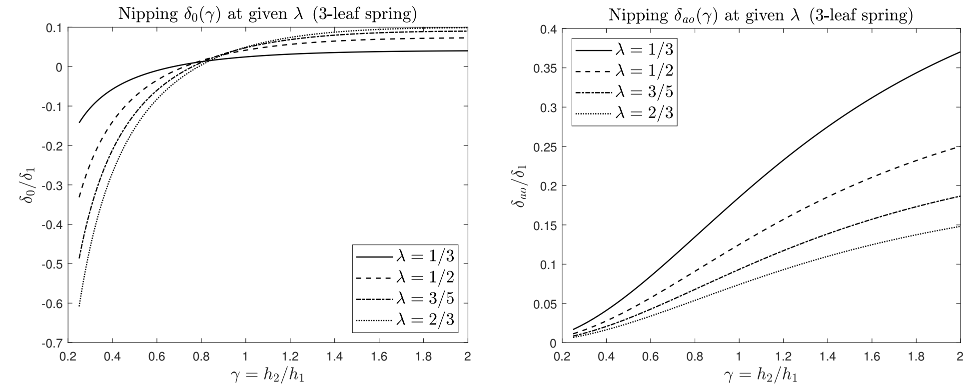



FIGURE 11. (a)–(c) The contact forces $F'_A$ and $F'_C$ due to gap $\delta_a$. (d)–(e) The contact force $F''_C$ between the bottom two leaves due to gap $\delta_0$.




(a) (b)

FIGURE 12. The variation of initial gaps: (a) $\delta_0/\delta_1$, and (b) $\delta_{ao}/\delta_1$ with $\gamma$ for several values of $\lambda$, as required by the nipping condition $\sigma_1^T = \sigma_2^T = \sigma_a^T$.

outset of the analysis, rather than being determined from the introduced physical conditions, such as equalizing the maximum stresses in leaves, as done in the present paper. A detailed nipping analysis of multi-leaf springs with two to five leaves of uniform length decrement and uniform thickness, strengthened by an additional full-length leaf of the same thickness, has been recently presented in [**14**].

TABLE 1. Contact forces for $\alpha = 1$ and different $(\lambda, \gamma)$

| $(\lambda,\gamma)$ | $F_A^F/F$ | $F_A'/F$ | $F_A^T/F$ | $F_C^F/F$ | $F_C'/F$ | $F_C''/F$ | $F_C^T/F$ |
|---|---|---|---|---|---|---|---|
| (1/3,2/3) | 0.5363 | -0.0546 | 0.5909 | 0.4904 | -0.0499 | -0.0052 | 0.5455 |
| (1/3,1) | 0.5870 | -0.0797 | 0.6667 | 1.1739 | -0.1594 | 0.3333 | 1.0000 |
| (1/3,3/2) | 0.6481 | -0.1166 | 0.7647 | 2.0000 | -0.3597 | 0.7714 | 1.5882 |
| (1/2,2/3) | 0.5490 | -0.0419 | 0.5909 | 0.3137 | -0.0239 | -0.0260 | 0.3636 |
| (1/2,1) | 0.6214 | -0.0453 | 0.6667 | 0.7767 | -0.0566 | 0.1667 | 0.6667 |
| (1/2,3/2) | 0.7157 | -0.0491 | 0.7647 | 1.3802 | -0.0946 | 0.4160 | 1.0588 |
| (2/3,2/3) | 0.5579 | -0.0331 | 0.5909 | 0.2231 | -0.0132 | -0.0364 | 0.2727 |
| (2/3,1) | 0.6467 | -0.0200 | 0.6667 | 0.5659 | -0.0175 | 0.0833 | 0.5000 |
| (2/3,3/2) | 0.7692 | 0.0045 | 0.7647 | 1.0385 | 0.0061 | 0.2382 | 0.7941 |

Figure 12a shows the variation of $\delta_0/\delta_1$ with $\gamma$, as determined by (5.12) in the case $\alpha = 1$, for several values of $\lambda$. Similarly, Fig. 12b shows the variation of the initial gap $\delta_{ao}/\delta_1$ with $\gamma$, also following from (5.12), for several values of $\lambda$ and $\alpha = 1$. The required gap $\delta_0$ can be either positive or negative, depending on the values of $(\lambda, \gamma)$ (second column of Table 2). Correspondingly, the nipping contact force $F_C''$ can be positive or negative, as can be seen from column 7 of Table 1.

Furthermore, in all but one case listed in Table 1, the value $\delta_a < 0$, which means that $\delta_{10} > \delta_{ao}$. Accordingly, the value of the nipping contact force $F'_A$ in these cases is negative, as shown in column 3 of Table 2.

TABLE 2. Nipping parameters for $\alpha = 1$ and different $(\lambda, \gamma)$

| $(\lambda, \gamma)$ | $\delta_0/\delta_1$ | $\delta_a/\delta_1$ | $\delta_{ao}/\delta_1$ | $\delta_{10}/\delta_1$ |
|---|---|---|---|---|
| (1/3,2/3) | -0.0008 | -0.1018 | 0.1010 | -0.0008 |
| (1/3,1) | 0.0247 | -0.1358 | 0.1852 | 0.0494 |
| (1/3,3/2) | 0.0370 | -0.1798 | 0.2941 | 0.1143 |
| (1/2,2/3) | -0.0142 | -0.0763 | 0.0682 | -0.0081 |
| (1/2,1) | 0.0417 | -0.0729 | 0.1250 | 0.0521 |
| (1/2,3/2) | 0.0674 | -0.0685 | 0.1985 | 0.1300 |
| (2/3,2/3) | -0.0471 | -0.0593 | 0.0404 | -0.0189 |
| (2/3,1) | 0.0494 | -0.0309 | 0.0741 | 0.0432 |
| (2/3,3/2) | 0.0915 | 0.0059 | 0.1176 | 0.1235 |

## 6. Conclusions

Nipping analysis of a two-leaf spring strengthened by an additional full-length leaf is presented, valid for any permissible values of the introduced length and thickness parameters $(\lambda, \alpha, \gamma)$. Two initial gaps between the pairs of consecutive leaves are needed to achieve a desired nipping-induced stress reduction. These gaps are determined from the nipping condition that the maximum stresses in all the leaves within the clamped cross-section of the loaded pre-nipped spring are equal to each other $(\sigma_1^T = \sigma_2^T = \sigma_a^T)$. The closed-form expressions for $\delta_0$ and $\delta_{ao}$ are given by (5.12), with the corresponding nipping stress given by (5.10). In the case $\alpha = 1$ (the same thicknesses of upper two leaves), the nipping stress is $\sigma_*(2 + \gamma^2)$, which represents a simple generalization of the expression $\sigma_*(1+\gamma^2)$ for the nipping stress in the absence of additional full-length leaf. The corresponding initial gap $\delta_0$ is specified by (3.7). The introduced normalizing stress factor $\sigma_*$ is defined by (2.6). The required initial gaps for the stress reduction may be either positive or negative, depending on the values of introduced geometric parameters $(\lambda, \alpha, \gamma)$. In some cases, nipping is not an effective means of stress reduction, because one leaf in the loaded and pre-nipped spring may have a higher stress than the maximum stress in the loaded spring produced without nipping. There is a particular relationship between $\lambda$ and $\gamma$ for which the maximum stresses in all critical cross-sections of the pre-nipped and loaded spring are equal to each other $(\sigma_1^T = \sigma_2^T = \sigma_a^T = \sigma_C^T)$. This is $\gamma^2 = \lambda/(1 - \lambda)$, as given by (5.13), independently of $\alpha$. Expressions for the total contact forces between the leaves are derived and discussed. The generalization of the presented analysis to leaf springs with more than three leaves is conceptually straightforward, albeit more tedious because of additional length and thickness parameters that need to be introduced. The presented analysis and results may be useful for multi-leaf spring design and corresponding optimization studies involving considerations of spring geometry, material selection, and weight.

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## АНАЛИЗА ПРЕДНАПРЕГНУТЕ ЛИСНАТЕ ОПРУГЕ ОЈАЧАНЕ ДОДАТНИМ ЛИСТОМ ПУНЕ ДУЖИНЕ

Резиме. У раду је изложена анализа преднапрегнуте дволисне опруге ојачане додатним листом пуне дужине. Сви листови су од истог материјала и истог правоугаононг попречног пресека. Изведени су изрази за почетне зазоре између листова чијим се затварењем изједначују максимални напони у уклештеним попречним пресецима свих листова оптерећене опруге. Ови изрази важе за све комбинације уведених геометријских параметара дужине и дебљине листова. У зависности од величине ових параметара, почетни зазори неопходни за снижење максималног напона могу бити позитивни или негативни. За одређене комбинације уведених параметара, коришћење листова различите почетне кривине не доводи до смањења максималног напона у опрузи, независно од величине почетних зазора. С друге стране, за опруге са листовима исте дебљине чија се дужина смањује равномерно, максимални напони у свим критичним пресецима опруге постају исти. Изложена анализа и добијени резултати могу бити од интереса за конструисање лиснатих опруга и њихову оптимизацију у односу на геометрију, избор материјала и тежину.

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