THEORETICAL AND APPLIED MECHANICS
Volume 52 (2025) Issue 1, 17–37 DOI: https://doi.org/10.2298/TAM241213010F

POINT SPECTRA AND NORMAL MODES OF THE RAYLEIGH LOADED STRING WITH DAMPING

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Abstract. We describe the point spectra of some dissipative version of the celebrated "Rayleigh loaded string", an elastic string of finite length carrying a number $n \geqslant 1$ of equally spaced, equal point masses, which is a basic model that exhibits a band structure and appears in many applied areas. We consider the case in which the dissipation is due to a viscous damping due to the interaction string-environment, a standard model for internal visco-elastic dissipation (the Kelvin–Voigt model), and their combined presence. We show that the point spectrum of each of these damped versions of the Rayleigh loaded string is a continuous deformation of the point spectrum of the unloaded elastic strings with that damping and that presents a band structure similar to that of the undamped case. We also provide explicit analytical expressions for the eigenfunctions, for any value of n.

1. Introduction

1.1. Aim and motivations. The *Rayleigh's loaded string*, or simply the *loaded string*, [4, 7, 20, 22, 25] is an elastic string with fixed ends and a certain non-homogeneous density distribution, which is assumed to perform small transverse oscillations under the influence of no forces except for the (constant) string tension. The non-homogeneity is due to the presence of a number $n \geqslant 1$ of localized point masses ("loads") along the string.

One of the reasons for interest in this system is that it is an instance of a "locally periodic system", a class of systems whose spectral properties are of significant interest in various areas of physics, and have received great attention in the physics literature (see [7] for an introduction). Cases with any number of loads, variously distributed along the string, have been considered in connection with the formation of conduction bands, Anderson localization and other classical and quantum phenomena [10, 20].

2020 *Mathematics Subject Classification:* Primary 35Q70; Secondary 74H45, 47J10.
Key words and phrases: Rayleigh loaded string, damped elastic string, viscous elastic string, Kelvin-Voigt viscoelastic string, band spectrum.

In the simplest case, which is the one we will consider here, the masses are equal and equally spaced and their introduction is well known to give the spectrum a band structure [7,12,22]. Specifically, the spectrum is a continuous deformation of that of the uniform ("unloaded") elastic string, with deformation parameter the ratio $\hat{\mu} := \frac{m}{M}$ between the mass m of the loads and the mass M of the string, and is organized in bands of $n+1$ eigenvalues. Each band is formed by one of those special eigenvalues of the unloaded string which have nodes at the locations of the point masses, and of other n eigenvalues which tend to it as $\hat{\mu}$ grows.

The aim of the present paper is to investigate the effect of various types of dissipation on the point spectrum of the loaded string. We are mainly interested in the case in which the elastic string is replaced by a viscoelastic string, with internal damping, but we will also consider the cases of viscous external damping and the combined effect of the two. Specifically, we will consider the standard "Kelvin–Voigt" model of viscoelastic string [8,23,24], which introduces in the wave equation for the transversal string displacement ψ a term $\hat{\gamma}\psi_{txx}$, with a constant $\hat{\gamma} > 0$, and the standard model (Heaviside's "telegrapher" equation [11]) of external damping which introduces in the wave equation a term $-\hat{\beta}\psi_t$, with a constant $\hat{\beta} > 0$. The string with both sources of dissipation has been studied, e.g., in [15]. The point spectra of the unloaded strings with these types of dampings, which depend on $\hat{\beta}$ and $\hat{\gamma}$, are known (see e.g. [17, 18, 23]). Their eigenvalues are given by two functions, parametrized by $\hat{\beta}$ and $\hat{\gamma}$, of those of the undamped elastic string. We will call these two functions the "spectral map".

More generally, the spectral properties of damped elastic strings and of similar systems, with various types of non-uniform dampings (particularly of the Kelvin–Voigt type) and non-homogeneous densities, have received ample attention in the mathematical literature in the last decades, and many general results are known, e.g. on the localization of the spectrum in the complex plane, the accumulations of the eigenvalues, the existence of a Riesz basis, etc. (see e.g. [3,9,13–15,17,18,23,26] and references therein). However, it seems to us that considering the limit case of a (uniformly) damped string with equal, concentrated inhomogeneities at equally spaced points, where everything can be made explicit, has some interest.

For instance, our interest for this problem originated from a study of the long-time dynamics of a chain of $n \geqslant 1$ pendula hanging from a viscoelastic string [6]. Upon linearization around the equilibrium configuration, such a system decouples into two linear subsystems, one of which is precisely the Kelvin–Voigt viscoelastic loaded string that we consider here. The other subsystem can be viewed as a Kelvin–Voigt viscoelastic string with n harmonic oscillators of a certain (non generic) type attached to it. A detailed knowledge of the spectra of these two systems is, thus, a prerequisite for the study of the string carrying the pendula. However, because of some relevant differences between them (and also the non-genericity of that specific Kelvin–Voigt case), we find it clearer to study them separately.

1.2. Results. We will focus on the dependence of the damped spectra on the mass of the loads, namely, on the parameter $\hat{\mu}$, and show that it has a well defined band structure.

Specifically, we will first show that, for each $\hat{\beta}$ and $\hat{\gamma}$, the eigenvalues of the damped loaded string with a certain $\hat{\mu}$ are given by the same "spectral map" of the unloaded damped string, but evaluated on the eigenvalues of the Rayleigh string with that $\hat{\mu}$.

As a consequence, for each $\hat{\beta}$ and $\hat{\gamma}$, the spectrum is a continuous deformation, with deformation parameter $\hat{\mu}$, of the spectrum of the unloaded string with those $\hat{\beta}$ and $\hat{\gamma}$. As $\hat{\mu}$ grows, the eigenvalues move, in the complex plane, on a "spectral locus" which is determined by n, $\hat{\beta}$ and $\hat{\gamma}$. (For given n, $\hat{\beta}$ and $\hat{\gamma}$, such a spectral locus can be seen as the union of the point spectra of the damped loaded string for all $\hat{\mu} \geqslant 0$). This process produces the formation of bands in the spectrum.

Each band contains one of those special eigenvalues of the, now damped, unloaded string with that $\hat{\beta}$ and $\hat{\gamma}$ whose eigenfunctions have nodes at the locations of the point masses, and other n eigenvalues which move towards it, making the band narrower, as $\hat{\mu}$ grows, and cluster to them as $\hat{\mu} \to +\infty$.

In addition, we will derive explicit expressions for the eigenfunctions, which are valid for all n (and, to our knowledge, are new even in the undamped case).

1.3. Organization of the paper. In Section 2 we describe the system under study. In Section 3 we review the spectra of the damped unloaded strings and determine their spectral loci which, as explained above, are the starting point for the description of the spectrum of the damped loaded strings. In Section 4 we study these spectra; preliminarily; however, we review in detail the structure of the spectrum of the Rayleigh loaded undamped string ($\hat{\beta} = \hat{\gamma} = 0$, $\hat{\mu} > 0$) whose eigenvalues determine those of the damped loaded strings with the same $\hat{\mu}$. Section 5 is devoted to the eigenfunctions.

Unless stated differently, by "spectrum" we mean "point spectrum".

2. The system

2.1. The undamped and damped loaded strings. Rayleigh's loaded string is a planar homogeneous elastic string of finite length and fixed ends which carries n point masses, which we assume to be equally spaced and with the same mass m. The only force acting on the system is the elasticity of the string, whose tension τ is assumed to be constant. We denote M the mass of the (unloaded) string, L its length, and $\rho = M/L$ its constant linear density. The string is supposed to perform transverse displacements and its configuration at each time t is, thus, described by an embedding (with standard regularity properties, see below) of the form

$$[0, L] \ni x \mapsto (x, \psi(x, t)) \in \mathbb{R}^2$$

where $\psi \colon [0, L] \times \mathbb{R} \to \mathbb{R}$ satisfies the boundary conditions $\psi(0, t) = \psi(L, t) = 0 \ \forall t$. The point masses are attached to the points of the string with material coordinate x given by $\frac{1}{n+1}L, \frac{2}{n+1}L, \ldots, \frac{n}{n+1}L$. The equation governing the dynamics of the system is given, e.g., in [4, equation (40)], and, after the inclusion of a viscous dissipative term $\hat\beta\psi_t$ and of a Kelvin–Voigt dissipative term $-\hat\gamma\psi_{txx}$, with positive constants $\hat\gamma$ and $\hat\beta$, takes the form

$$\rho\psi_{tt} - \tau\psi_{xx} + m\sum_{j=1}^{n}\psi_{tt}\delta_{\frac{jL}{n+1}} - \hat\gamma\psi_{txx} + \hat\beta\psi_t = 0, \quad \psi(0,t) = \psi(L,t) = 0. \tag{2.1}$$

Here, for any $a\in\mathbb{R}$, $\delta_a(x) := \delta(x-a)$ denotes the translated Dirac delta.

In order to slightly simplify the equations that determine the eigenvalues, we use dimensionless space and time coordinates $\tilde x := (n+1)\frac{x}{L}$, $\tilde t := (n+1)\frac{\pi}{L}\sqrt{\frac{\tau}{\rho}}t$ (which we will keep on denoting x,t) and parameters

$$\beta := \frac{L}{n+1}\frac{\hat\beta}{2\pi\sqrt{\rho\tau}}, \quad \gamma := \frac{n+1}{2}\frac{\pi\hat\gamma}{L\sqrt{\rho\tau}}, \quad \mu := \frac{n+1}{2}\frac{m}{M}.$$

After this (unusual) rescaling, the fundamental frequency $\frac{\pi}{L}\sqrt{\frac{\tau}{\rho}}$ of the elastic string of length L is $\frac{1}{n+1}$. The inclusion of the factor $n+1$ in the rescalings and parameters prevents its appearing in some conditions, particularly in the quantity $\xi_{\beta,\gamma}$ defined below. Also, after the rescaling, the string has length $n+1$, the point masses are at the points

$$x_j := j, \quad j = 0,\dots,n+1,$$

and equation (2.1) becomes

$$\pi^2\psi_{tt} - \psi_{xx} + 2\mu\pi^2\sum_{j=1}^{n}\psi_{tt}\delta_{x_j} + 2\beta\pi^2\psi_t - 2\gamma\psi_{txx} = 0, \quad x\in(0,n+1), \ \ t\in\mathbb{R}, \tag{2.2}$$

with the boundary conditions

$$\psi(0,t) = \psi(n+1,t) = 0 \quad \forall t\in\mathbb{R}. \tag{2.3}$$

We consider equations (2.2), (2.3) for $\beta\geqslant 0$, $\gamma\geqslant 0$ and $\mu\geqslant 0$. We call them the *damped* (*undamped, if* $\beta=\gamma=0$) *loaded* (*unloaded, if* $\mu=0$) *string* $S_{\beta,\gamma,\mu}$.

2.2. Weak formulation. Consider the space Σ of continuous real functions $f\colon[0,n+1]\to\mathbb{R}$ which are C^2 in $x\in(0,n+1)\smallsetminus\{x_1,\dots,x_n\}$, have bounded left and right first x-derivative at $x_1,\dots,x_n$, and vanish at 0 and $n+1$. Let $\tilde\Sigma$ be the space of real functions

$$\psi\colon[0,n+1]\times\mathbb{R}\to\mathbb{R}, \qquad (x,t)\mapsto\psi(x,t)$$

which are C^2 in t and are such that, for each t, $x\mapsto\psi(x,t)$ belongs to Σ.

Then, following a standard approach (see e.g. [25, Appendix III to Ch. II]), we define as (real) solution of equations (2.2)–(2.3), or of $S_{\beta,\gamma,\mu}$, any function $\psi\in\tilde\Sigma$ which satisfies, for all $t\in\mathbb{R}$, the equation

$$\pi^2\psi_{tt}(x,t) - \psi_{xx}(x,t) + 2\beta\pi^2\psi_t(x,t) - 2\gamma\psi_{txx}(x,t) = 0 \quad \forall x\in(0,n+1)\smallsetminus\{x_1,\dots,x_n\} \tag{2.4}$$

and the jump conditions

$$2\mu\pi^2\psi_{tt}(x_j,t) - (\psi_x+2\gamma\psi_{tx})(x_j^+,t) + (\psi_x+2\gamma\psi_{tx})(x_j^-,t) = 0 \quad \forall j=1,\dots,n. \tag{2.5}$$

Here $\psi_x(x_j^{\pm}, t)$, $\psi_{tx}(x_j^{\pm}, t)$ stand for $\lim_{x\to x_j^{\pm}} \psi_t(x,t)$, $\lim_{x\to x_j^{\pm}} \psi_{tx}(x,t)$. By a complex solution we refer to a complex function whose real and imaginary parts are real solutions. (The jump of the x derivatives at each point x_j is computed integrating (2.2) in $(x_j - \epsilon, x_j + \epsilon)$ and taking the limit $\epsilon \to 0$).

For $\mu = 0$, the jump conditions (2.5) imply the smoothness of ψ and (2.4) reduces to the wave equation (2.2) without Dirac deltas, which describes the damped (undamped, if $\beta = \gamma = 0$) unloaded string $S_{\beta,\gamma,0}$. We may, thus, include in this formulation the case of the unloaded strings of length $n+1$ with any $n \geqslant 1$.

2.3. The point spectrum. For this type of systems, eigenvalues can be found via separation of variables (or, equivalently, via matrix pencils [**13**]). Specifically, a complex number λ is an *eigenvalue* if there exist a nonzero complex solution ψ of the form

$$\psi(x,t) = f(x)e^{\lambda t};$$

we will call f the *eigenfunction* and the family of solutions $c\psi$, with $c \in \mathbb{C} \setminus \{0\}$, the (damped) normal mode with eigenvalue λ. The *point spectrum* $\mathrm{Sp}_{\beta,\gamma,\mu}$ of $S_{\beta,\gamma,\mu}$ is the set of all its eigenvalues. Obviously, the nonreal eigenvalues come in conjugate pairs.

Explicitly, from (2.3), (2.4) and (2.5), λ is an eigenvalue of $S_{\beta,\gamma,\mu}$ with eigenfunction f if and only if $f \in \Sigma$, $f \neq 0$ and

$$(1 + 2\gamma\lambda)f''(x) = \pi^2(\lambda^2 + 2\beta\lambda)f(x) \quad \forall x \in (0, n+1) \setminus \{x_1, \dots, x_n\}, \tag{2.6a}$$

$$(1 + 2\gamma\lambda)(f'(x_j^+) - f'(x_j^-)) = 2\mu\pi^2\lambda^2 f(x_j) \quad \forall j = 1, \dots, n, \tag{2.6b}$$

$$f(0) = f(n+1) = 0. \tag{2.6c}$$

Note that, for each $\beta \geqslant 0$ and $\gamma \geqslant 0$, $\mathrm{Sp}_{\beta,\gamma,0}$ is the spectrum of the unloaded (damped or undamped) string $S_{\beta,\gamma,0}$, and that for each $\mu > 0$, $\mathrm{Sp}_{0,0,\mu}$ is the spectrum of the undamped Rayleigh loaded string.

3. The spectra of the damped unloaded strings

3.1. The spectral map. We now review the spectra of the damped unloaded strings with the three types of dissipation that we consider. As already mentioned in the Introduction, all this is elementary and essentially known, and can also be viewed as particular cases of general situations discussed e.g. in [**3**,**9**,**13**,**14**,**17**,**23**,**26**]. However, our approach is finalized to the subsequent treatment of the damped loaded string.

For all $\beta \geqslant 0$, define the functions

$$\xi_{\beta,0} \colon \mathbb{C} \to \mathbb{C}, \quad \xi_{\beta,0}(\lambda) := \pi\sqrt{\lambda^2 + 2\beta\lambda},$$

and, for $\gamma > 0$,

$$\xi_{\beta,\gamma} \colon \mathbb{C} \setminus \{-\tfrac{1}{2\gamma}\} \to \mathbb{C}, \quad \xi_{\beta,\gamma}(\lambda) := \pi\frac{\sqrt{\lambda^2 + 2\beta\lambda}}{\sqrt{1 + 2\gamma\lambda}}.$$

Here and in the following $\sqrt{\cdot}$ denotes the real square root for real nonnegative arguments and (e.g. the choice of the branch is immaterial) the complex square root with nonnegative imaginary part for all other, real or complex, arguments. We make the tacit convention that, if $\gamma = 0$, then $\{-\frac{1}{2\gamma}\}$ is the empty set and use the symbol $\xi_{\beta,\gamma}$ for both $\gamma = 0$ and $\gamma > 0$. We will see below that 0, -2β and $-\frac{1}{2\gamma}$ are not eigenvalues, so in the rest of this section we tacitly exclude them from consideration.

It is well known that the spectrum of the damped unloaded string $S_{\beta,\gamma,0}$ is related to that of the elastic string $S_{0,0,0}$ by a simple relation: *for any* $\beta \geqslant 0$ *and* $\gamma \geqslant 0$, $\lambda \in \mathrm{Sp}_{\beta,\gamma,0}$ *if and only if* $\frac{1}{\pi}\xi_{\beta,\gamma}(\lambda) \in \mathrm{Sp}_{0,0,0}$ *and* $\lambda \neq -2\beta, -\frac{1}{2\gamma}$ This is due to the fact that, as an abstract equation, when $m = 0$ equation (2.1) has the structure $A\ddot{\psi} + B\dot{\psi} + C\psi = 0$ with linear operators A, B and C such that B is a linear combination of the other two (see e.g. [**16**, Section 7.5]). In fact, it is a simple check (integrate (2.2) with $\mu = 0$ in $[0, n+1]$ and impose the boundary conditions (2.3)) that a complex number λ belongs to $\mathrm{Sp}_{\beta,\gamma,0}$ if and only if it is different from -2β and from $-\frac{1}{2\gamma}$ and satisfies $\sinh\big((n+1)\xi_{\beta,\gamma}(\lambda)\big) = 0$ and belongs to $\mathrm{Sp}_{0,0,0}$ if and only if $\sinh((n+1)\pi\lambda) = 0$.

We define
$$\omega_k := \frac{k}{n+1}, \quad k \in \mathbb{N},$$
so that $\mathrm{Sp}_{0,0,0} = \{\pm i\omega_k : k \in \mathbb{Z}_+\}$ and $\lambda \in \mathbb{C} \setminus \{-2\beta, -\frac{1}{2\gamma}\}$ is in $\mathrm{Sp}_{\beta,\gamma,0}$ if and only if $\xi_{\beta,\gamma}(\lambda) = i\pi\omega_k$ for some $k \in \mathbb{Z}_+$. Solving these equations gives $\mathrm{Sp}_{\beta,\gamma,0} = \{\lambda_\pm^{\beta,\gamma}(\omega_k) \in \mathbb{C} \setminus \{-\frac{1}{2\gamma}, -2\beta\} : \, k \in \mathbb{Z}_+\}$ with the two functions

$$\lambda_\pm^{\beta,\gamma} \colon \mathbb{R}_+ \to \mathbb{C}, \quad \lambda_\pm^{\beta,\gamma}(\omega) = -(\beta + \gamma\omega^2) \pm \sqrt{(\beta + \gamma\omega^2)^2 - \omega^2}, \tag{3.1}$$

that we will call *spectral map*.

Note that the labeling of the eigenvalues is such that

$$\xi_{\beta,\gamma}(\lambda_\pm^{\beta,\gamma}(\omega_k)) = i\pi\omega_k \quad \forall \beta, \gamma, k \tag{3.2}$$

(the absence of the double sign at the left hand side is due to our convention on the square root).

3.2. The spectral locus. In view of the study of the damped loaded string we introduce, for each $\beta \geqslant 0$ and $\gamma \geqslant 0$, the *spectral locus* $L_{\beta,\gamma} \subset \mathbb{C}$ *of* $\mathrm{Sp}_{\beta,\gamma,0}$ as
$$L_{\beta,\gamma} := L_{\beta,\gamma,+} \cup L_{\beta,\gamma,-}$$
with
$$L_{\beta,\gamma,+} := \lambda_+^{\beta,\gamma}(\mathbb{R}_+), \quad L_{\beta,\gamma,-} := \lambda_-^{\beta,\gamma}(\mathbb{R}_+),$$
so that $\mathrm{Sp}_{\beta,\gamma,0} \subset L_{\beta,\gamma}$ for all $n \geqslant 1$. It is not difficult to check that:

1. If $\beta \geqslant 0$, then $L_{\beta,0} = (-2\beta, 0) \cup \{\lambda \in \mathbb{C} : \, \mathrm{Re}(\lambda) = -\beta\}$.
2. If $\gamma > 0$, then $L_{0,\gamma} = (-\infty, -\frac{1}{2\gamma}) \cup C_{0,\gamma}$.
3. If $\beta, \gamma > 0$, then there are three cases:
   - 3.1. $4\beta\gamma < 1$ (underdamped case): $L_{\beta,\gamma} = \big(-\infty, -\frac{1}{2\gamma}\big) \cup (-2\beta, 0) \cup C_{\beta,\gamma}$.
   - 3.2. $4\beta\gamma = 1$ (critical case): $L_{\beta,\gamma} = (-\infty, 0)$.
   - 3.3. $4\beta\gamma > 1$ (overdamped case): $L_{\beta,\gamma} = (-\infty, -2\beta) \cup \big(-\frac{1}{2\gamma}, 0\big)$.

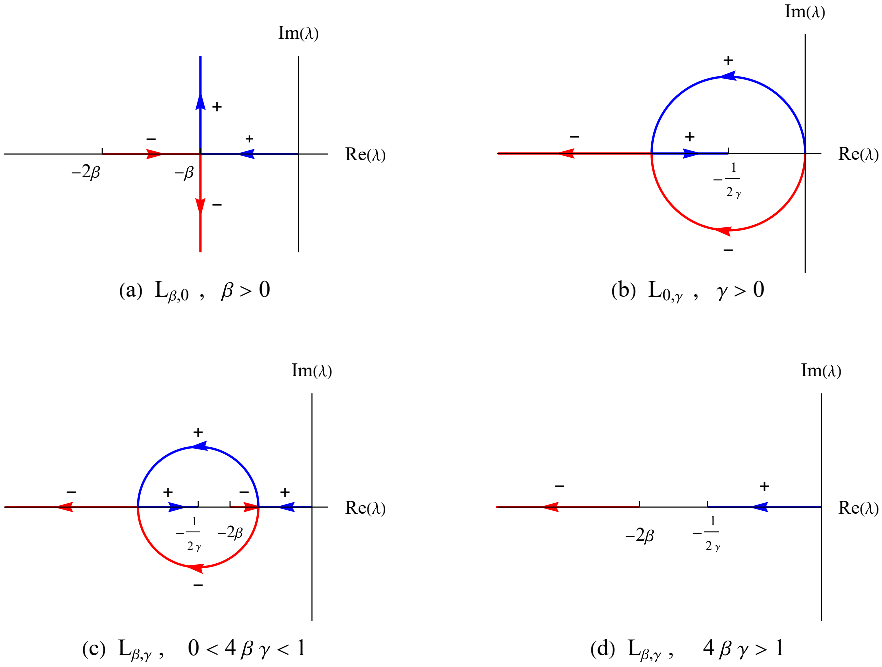


FIGURE 1. The spectral loci (in the non-critical case). The marks $\pm$ and the colors denote the subsets $L_{\beta,\gamma,+}$ (blue) and $L_{\beta,\gamma,-}$ (red) and the arrows denote the orientation on them, which is that of ω increasing in the parametrization of these sets by the functions $\lambda_\pm^{\beta,\gamma}$.

Here $C_{\beta,\gamma}$ is the circle of center $-\frac{1}{2\gamma}$ and radius $\frac{\sqrt{1-4\beta\gamma}}{2\gamma}$. The spectral loci are shown in Figure 1. In the case of a Kelvin–Voigt viscoelastic string without external damping ($\beta = 0$, $\gamma > 0$), the circle $C_{0,\gamma}$ is tangent to the imaginary axis at 0. In case 3.1., the point -2β lies inside the circle $C_{\beta,\gamma}$. We will not consider the critical case in the sequel.

It is also not difficult to show that, the critical case excluded, both functions $\lambda_\pm^{\beta,\gamma} \colon \mathbb{R}_+ \to L_{\beta,\gamma,\pm}$ are injective and, therefore, induce an orientation on the sets $L_{\beta,\gamma,\pm}$, according to increasing values of the parameter ω. These two subsets and their orientation are shown in Figure 1, and will be used later. We note also that $L_{\beta,\gamma,+}$ and $L_{\beta,\gamma,-}$ have empty intersection except, in cases 1., 2. and 3.3, at $\omega = \frac{1}{2\gamma}(1 \pm \sqrt{1-4\beta\gamma})$, which are the points of intersection between the negative real semi-axis and either the circle $C_{\beta,\gamma}$ or the line $\mathrm{Re}(\lambda) = -\beta$. These are the only values of ω for which $\lambda_\pm^{\beta,\gamma}$ are not smooth, but only continuous.

REMARK 3.1. In the critical case, $\lambda_\pm^{\beta,\gamma}$ are not injective because $\lambda_+^{\beta,\gamma}(\omega) = -2\beta$ for all $\omega \geqslant 2\beta$ and $\lambda_-^{\beta,\gamma}(\omega) = -2\beta$ for all $\omega \leqslant 2\beta$. As a consequence, in this case the spectral locus contains the point $-2\beta = -\frac{1}{2\gamma}$ (which is not in the point spectrum).

3.3. The unloaded spectra. We now quickly describe the structure of the spectra $\mathrm{Sp}_{\beta,\gamma,0}$, considering first the two limit cases $\gamma = 0$ and $\beta = 0$.

1. The spectrum $\mathrm{Sp}_{\beta,0,0}$, of the string with viscous damping, $\beta > 0$ and $\gamma = 0$, is completely elementary. The eigenvalues are the $\lambda_{\pm}^{\beta,0}(\omega_k) = -\beta \pm \sqrt{\beta^2 - \omega_k^2}$, $k \in \mathbb{Z}_+$. A finite number (zero, if $\beta < \frac{1}{n+1}$) of them belong to the real interval $(-2\beta, 0)$ and all other to the line $\mathrm{Re}(\lambda) = -\beta$. See Figure 5.a (which uses a notation that will be introduced later).

We add that, for each k, as β grows the two eigenvalues $\lambda_{\pm}^{\beta,0}(\omega_k)$ move along the circle of radius ω_k centered at zero, from $\pm i\omega_k$ to the point $-\beta$, where they meet and then move on the real axis one to the right and one to the left of $-\beta$.

2. The Kelvin–Voigt spectrum $\mathrm{Sp}_{0,\gamma,0}$, with $\gamma > 0$, is very well known, see e.g. [**17**, **23**]. The eigenvalues are $\lambda_{\pm}^{0,\gamma}(\omega_k) = -\gamma\omega_k^2 \pm \sqrt{\gamma^2\omega_k^4 - \omega_k^2}$, $k \in \mathbb{Z}_+$.

Those with $k < \frac{n+1}{\gamma}$, if present, form a finite number of pairs of non-real complex conjugate numbers which belong to the circle $C_{0,\gamma}$. The remaining ones belong to the interval $(-\infty, -\frac{1}{2\gamma})$ and accumulate to its boundaries. The $\lambda_{\pm}^{0,\gamma}(\omega_k)$ are all pairwise distinct except, if $\frac{n+1}{\gamma} \in \mathbb{Z}_+$, $\lambda^{0,\gamma,0}_{\frac{n+1}{\gamma},\pm} = -\frac{1}{\gamma}$.

The Kelvin–Voigt spectrum is illustrated in Figure 2.

Here too, as γ grows, the eigenvalues move on circles centered at zero until they reach the real axis at the point of intersection with the circle $C_{0,\gamma}$.

3. For $\beta > 0$ and $\gamma > 0$ there are two cases.

In the underdamped case $4\beta\gamma < 1$ the spectrum is similar to that of the Kelvin–Voigt system, but slightly more complex: there are infinitely many eigenvalues in the interval $(-\infty, -\frac{1}{2\gamma})$ which accumulate to its boundaries, a possibly zero finite number of eigenvalues in the interval $(-2\beta, 0)$, and a possibly zero finite number of conjugate pairs of eigenvalues in the circle $C_{\beta,\gamma}$. ([**18**, Lemma 1.7] treats this case, but with different boundary conditions, with $\beta = c\gamma$ for some $c > 0$ and with an additional term $c\psi$ in the equation, and this changes the radius of the circle $C_{\beta,\gamma}$).

In the overdamped case, with $4\beta\gamma > 1$, there are infinitely many eigenvalues in the interval $(-\infty, -2\beta)$ which accumulate to $-\infty$ and infinitely many eigenvalues in the interval $(-\frac{1}{2\gamma}, 0)$ which accumulate to $-\frac{1}{2\gamma}$.

See Figures 6a and 6c.

Remark 3.2. Since its eigenvalues depend continuously on $\gamma \geqslant 0$, the spectrum $\mathrm{Sp}_{0,\gamma,0}$ of the Kelvin–Voigt string is a continuous deformation of the spectrum $\mathrm{Sp}_{0,0,0}$ of the elastic string. However, the continuity at $\gamma = 0$ is not geometrically evident if these spectra are plotted in the complex plane, because, at infinity, the imaginary axis is "very far" from the real one. This is one of the reasons why we will later prefer regarding the spectrum $\mathrm{Sp}_{\beta,\gamma,\mu}$ of the damped loaded string as a deformation of the spectrum $\mathrm{Sp}_{\beta,\gamma,0}$ of the damped unloaded strings, rather than of the spectrum $\mathrm{Sp}_{0,0,\mu}$ of the undamped loaded string. Another reason is that the spectral locus $L_{\beta,\gamma}$ does not change with μ. We note, however, that the continuity of the Kelvin–Voigt spectrum at $\gamma = 0$ becomes geometrically evident if the spectra are plotted on the Riemann sphere, see Figure 2. Similarly, when $\beta > 0$, the relation

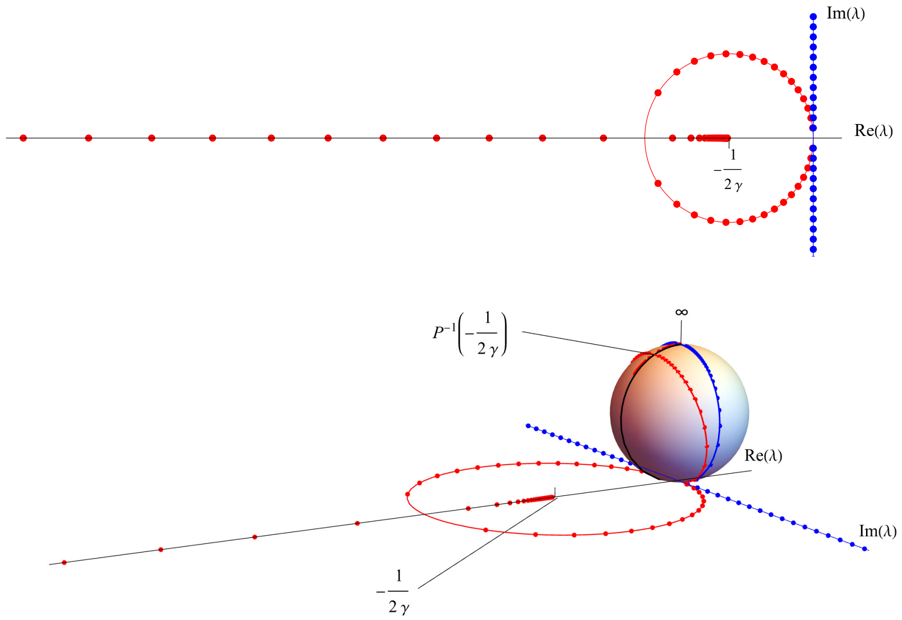


FIGURE 2. The point spectra of the Kelvin–Voigt string (red points) and of the elastic string (blue points) in the complex plane (top) and in the complex plane and on the Riemann sphere (bottom). The red, black and blue lines are, respectively, the circle $C_{0,\gamma}$, the real axis and the imaginary axis in the complex plane and their preimages on the Riemann sphere (In both figures, $n = 3$, $\gamma = 0.24$).

between $\mathrm{Sp}_{\beta,\gamma,0}$ for small γ and $\mathrm{Sp}_{\beta,0,0}$ could be geometrically understood plotting the spectra on the Riemann sphere.

3.4. Notation for the bands. For $\beta, \gamma \geqslant 0$, the eigenfunction of the string $S_{\beta,\gamma,0}$ relative to an eigenvalue λ is

$$f_\lambda^{\beta,\gamma,0}(x) = \sinh(\xi_{\beta,\gamma}(\lambda)x), \quad x \in [0, n+1] \tag{3.3}$$

(integrate (2.6a) with $\mu = 0$ and use the boundary condition (2.6c)). Thus, the two eigenvalues of each pair $\lambda_\pm^{\beta,\gamma}(\omega_k)$ have the same eigenfunction.

Moreover, identities (3.2) imply that the eigenfunctions are independent of β and γ, in the sense that the eigenfunction of $\mathrm{Sp}_{\beta,\gamma,0}$ relative to the pair of eigenvalues $\lambda_\pm^{\beta,\gamma}(\omega_k)$ equals the eigenfunction of $\mathrm{Sp}_{0,0}$ relative to the pair of eigenvalues $\lambda_\pm^{0,0}(\omega_k) = \pm i\omega_k$, namely $\sin(\pi\omega_k x)$.

Consequently, the eigenfunctions relative to the eigenvalues $\lambda_\pm^{\beta,\gamma}(\omega_{(n+1)\ell})$ have nodes at the points $x_1, \ldots, x_n$. We will see below—and it is well known at least for $\beta = \gamma = 0$ [22]—that these eigenvalues also belong to $\mathrm{Sp}_{\beta,\gamma,\mu}$ for all $\mu > 0$, and that they organize its band structure. For this reason, we will call them the *string eigenvalues* of $\mathrm{Sp}_{\beta,\gamma,0}$ and of $\mathrm{Sp}_{\beta,\gamma,\mu}$, and we will denote them

$$\lambda^{\beta,\gamma}_{\ell,0,\pm} := \lambda^{\beta,\gamma}_{\pm}(\omega_{\ell,0}), \quad \ell \in \mathbb{Z}_+.$$

In particular, $\lambda^{0,0}_{\ell,0,\pm} = \pm i\omega_{\ell,0}$.

To facilitate, later, the description of the band structure of $\mathrm{Sp}_{\beta,\gamma,\mu}$ we relabel the frequencies of the elastic string as

$$\omega_{\ell,q} := \omega_{(n+1)\ell+q}, \quad \ell \in \mathbb{N}, \ \ q = 0,\dots,n.$$

Here, properly, $\omega_{0,0}$ is not defined; however, in order to simplify the notation, we make the convention $\omega_{0,0} = 0$.

4. The point spectra of the damped loaded strings

4.1. The spectral map. The key fact for our analysis is that between the spectra of the damped and undamped loaded strings there is the same relation as in the unloaded case, namely, the spectral map (3.1). This, together with some consequences and other properties, is the content of the following Proposition:

PROPOSITION 4.1. *For all $n \geqslant 1$, $\beta \geqslant 0$, $\gamma \geqslant 0$ and $\mu \geqslant 0$:*

i. 0, -2β *and, if* $\gamma > 0$, $-\frac{1}{2\gamma}$ *do not belong to* $\mathrm{Sp}_{\beta,\gamma,\mu}$.

ii. *If* $\lambda \in \mathrm{Sp}_{\beta,\gamma,\mu}$ *then* $\mathrm{Re}(\lambda) = 0$ *if* $\beta = \gamma = 0$ *and* $\mathrm{Re}(\lambda) < 0$ *otherwise.*

iii. $\lambda \in \mathbb{C} \setminus \{0, -2\beta, -\frac{1}{\gamma}\}$ *belongs to* $\mathrm{Sp}_{\beta,\gamma,\mu}$ *if and only if* $\frac{1}{\pi}\xi_{\beta,\gamma}(\lambda)$ *belongs to* $\mathrm{Sp}_{0,0,\mu}$, *that is, if and only if* $\lambda = \lambda^{\beta,\gamma}_{\pm}(\omega)$ *for some* $\omega > 0$ *such that* $\pm i\omega \in \mathrm{Sp}_{0,0,\mu}$.

iv. $\mathrm{Sp}_{\beta,\gamma,\mu} \subset L_{\beta,\gamma}$.

PROOF. (i.) It follows from (2.6) that if λ is such that $\lambda = 0$, $\lambda^2 + 2\beta\lambda = 0$ or $1 + 2\beta\lambda = 0$ then $f = 0$.

(ii.) Define the total energy at time t of a complex solution ψ as

$$E_t(\psi) := \frac{1}{2}\sum_{j=0}^{n}\int_{x_j}^{x_{j+1}} \Big(|\psi_t(x,t)|^2 + \frac{1}{\pi^2}|\psi_x(x,t)|^2\Big)dx + \mu\sum_{j=1}^{n}|\psi_t(x_j,t)|^2.$$

For any solution ψ, using (2.4), the vanishing of ψ_t at $x = 0, n{+}1$ and an integration by parts of all terms containing $\psi_x\bar{\psi}_{tx}$, $\psi_x\bar{\psi}_{txx}$ and of their complex conjugates which arise in the computation, gives $\frac{d}{dt}E_t(\psi) = -\Delta(\psi)$ with

$$\Delta(\psi) := \sum_{j=0}^{n}\int_{x_j}^{x_{j+1}} \Big(\frac{2\gamma}{\pi^2}|\psi_{tx}(x,t)|^2 + 2\beta|\psi_t(x,t)|^2\Big)dx.$$

For a damped normal mode $\psi_\lambda(x,t) = e^{\lambda t}f(x)$,

$$\Delta(\psi_\lambda) = |\lambda e^{\lambda t}|^2\sum_{j=0}^{n}\int_{x_j}^{x_{j+1}} \Big(\frac{2\gamma}{\pi^2}|f'(x)|^2 + 2\beta|f(x,t)|^2\Big)dx > 0$$

if at least one among β and γ is positive. But, clearly, $E_t(\psi_\lambda) = e^{2t\mathrm{Re}(\lambda)}E_0(\psi_\lambda)$ and thus $\frac{d}{dt}E_t(\psi_\lambda) = 2\mathrm{Re}(\lambda)E_t(\psi_\lambda)$. This implies $2\mathrm{Re}(\lambda)E_t(\psi_\lambda) = -\Delta(\psi_\lambda) < 0$ and proves the statement because $E_t(\psi) > 0$ if the solution ψ is nonzero.

(iii.) f satisfies (2.6) with some $\beta, \gamma > 0$ and $\lambda \neq 0, -2\beta, -\frac{1}{2\gamma}$ if and only if it satisfies them with λ replaced by $\frac{1}{\pi}\xi_{\beta,\gamma}(\lambda)$ and β and γ replaced by 0. By ii., the eigenvalues of $\mathrm{Sp}_{0,0,\mu}$ are purely imaginary. Solving the equation $\xi_{\beta,\gamma}(\lambda) = i\pi\omega$ with $i\omega \in \mathrm{Sp}_{0,0,\mu}$ gives $\lambda = \lambda_\pm^{\beta,\gamma}(\omega)$.

(iv.) For each $\omega \in \mathbb{R}_+$, $\lambda_\pm^{\beta,\gamma}(\omega) \in L_{\beta,\gamma}$. □

REMARK 4.1. When $m = 0$, namely, for the unloaded string, item ii. could also be deduced with an algebraic argument from (2.6a) (see [16, Theorem 7.1]).

4.2. Review of $\mathbf{Sp_{0,0,\mu}}$. Item iii. of Proposition 4.1 allows to deduce the spectrum $\mathrm{Sp}_{\beta,\gamma,\mu}$ of the damped loaded string from that of the undamped loaded string $\mathrm{Sp}_{0,0,\mu}$, which is known: a complex number $\lambda \neq 0$ belongs to $\mathrm{Sp}_{0,0,\mu}$ if and only if it satisfies

$$\sinh(\pi\lambda)U_n(\mu\pi\lambda\sinh(\pi\lambda) + \cosh(\pi\lambda)) = 0 \tag{4.1}$$

(see [7, 22] for $n = 1$ and [20] for an equivalent formulation valid for any n, where, however, there are no detailed proofs).

Here the U_k, $k \in \mathbb{N}$, are the Chebyshev polynomials of the second kind, which are the complex polynomials defined by the recurrence

$$U_0(x) := 1, \quad U_1(x) := 2x, \quad U_k(x) := 2xU_{k-1}(x) - U_{k-2}(x) \qquad (k \geqslant 2,\ x \in \mathbb{C}) \tag{4.2}$$

(see e.g. [19]). Each $U_k\colon \mathbb{C} \to \mathbb{C}$ is a polynomial of degree k and has the k simple zeroes

$$c_{k,p} := \cos\left(\frac{p\pi}{k+1}\right), \quad p = 1, \dots, k. \tag{4.3}$$

The appearance of the Chebyshev polynomials in the eigenvalue equations is typical of locally periodic systems [7].

The zeroes of the first factor in (4.1) are the "string" eigenvalues $\pm i\omega_{\ell,0} = \pm i\ell$, $\ell \in \mathbb{Z}_+$, of the elastic string, whose eigenfunctions have nodes at the positions of the point masses.

Since the zeroes of U_n are the n numbers $c_{n,p}$, $p = 1, \dots, n$, the zeroes of the second factor in (4.1) are the solutions of the n equations

$$\mu\pi\lambda\sinh(\pi\lambda) + \cosh(\pi\lambda) = c_{n,p}, \quad p = 1, \dots, n, \tag{4.4}$$

and thus come in groups of n. Their properties are well known and are recalled in the following Proposition (of which we provide for completeness a proof, because we could not find a detailed one in the literature).

PROPOSITION 4.2. *For any $n \geqslant 1$ and $\mu > 0$, $\mathrm{Sp}_{0,0,\mu}$ consists of the string eigenvalues $\pm i\omega_{\ell,0}$, $\ell \in \mathbb{Z}_+$, and, for each $\ell \in \mathbb{N}$, of n pairs of other eigenvalues $\pm i\omega^\mu_{\ell,q}$, $q = 1, \dots, n$, with*

$$\omega_{\ell,0} < \omega^\mu_{\ell,1} < \cdots < \omega^\mu_{\ell,n} < \omega_{\ell+1,0}. \tag{4.5}$$

Each $\omega^\mu_{\ell,q}$ is a smooth decreasing function of μ which tends to $\omega_{\ell,q}$ for $\mu \to 0$ and to $\omega_{\ell,0}$ for $\mu \to +\infty$.

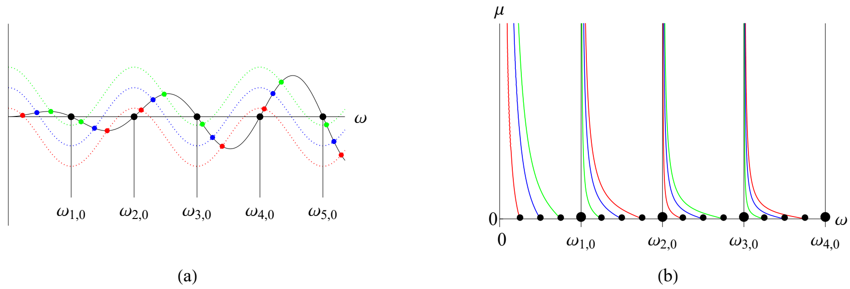


FIGURE 3. The spectrum $\mathrm{Sp}_{0,0,\mu}$ of the undamped Rayleigh string, for $n = 3$. (a) The frequencies $\omega^\mu_{\ell,p}$ are the abscissas of the intersection points between the graphs of the function at the l.h.s. of equation (4.6) (solid curve) and of the $n = 3$ functions at its r.h.s. (dashed curves; $p = 1$ red, $p = 2$ blue, $p = 3$ green), reordered as in (4.5); the black points denote the string frequencies. (b) The frequencies as functions of μ; the vertical black lines denote the string frequencies; the other lines denote the frequencies $\omega^\mu_{\ell,q}$, with the coloring as in (a). (In (a) $\mu = 0.1$, in (b) $\mu \in (0, 3)$).

PROOF. Fix $\mu > 0$. The non-string eigenvalues of $\mathrm{Sp}_{0,0,\mu}$ are the solutions $\pm i\omega$ with non-integer positive ω of the n equations (4.4). Written for $\omega \in \mathbb{R}_+ \setminus \mathbb{Z}_+$, these equations are

$$\mu\pi\omega \sin(\pi\omega) = \cos(\pi\omega) - c_{n,p}, \quad p = 1, \dots, n, \tag{4.6}$$

or else

$$F(\omega) = f_p(\omega), \quad p = 1, \dots, n, \tag{4.7}$$

with

$$F(\omega) := \pi\mu\omega, \quad f_p(\omega) = \frac{\cos(\pi\omega) - c_{n,p}}{\sin(\pi\omega)}.$$

Since $|c_{n,p}| < 1$, $f'_p(\omega) = \pi\frac{c_{n,p}\cos(\pi\omega)-1}{\sin(\pi\omega)^2} < 0$, f_p is strictly decreasing in each interval $(\ell, \ell+1)$, $\ell \in \mathbb{N}$, which it maps diffeomorphically onto $\mathbb{R}$. Together with the fact that, for $\mu > 0$, $\mathbb{R}_+ \ni \omega \mapsto F(\mu, \omega)$ is strictly increasing and onto $\mathbb{R}_+$, this ensures that each equation (4.6) or (4.7) has exactly one positive solution $\tilde\omega_{\ell,p}(\mu)$ in each such interval. Solutions with different p are (at the same μ) obviously different. A look at Figure 3.a, which plots the two functions at the two sides of (4.6), shows that at fixed μ, the $\tilde\omega_{\ell,p}(\mu)$ increase (decrease) with p if ℓ is even (odd). We thus relabel them as

$$\omega^\mu_{\ell,q} := \tilde\omega_{\ell,q} \quad \text{if } \ell \text{ is even}, \quad \omega^\mu_{\ell,q} := \tilde\omega_{\ell,n-q+1} \quad \text{if } \ell \text{ is odd}$$

so as to get the ordering (4.5).

Consider now the μ-dependence of the $\omega^{\mu}_{\ell,q}$. An inspection of Figure 3.a shows that they are decreasing functions of μ and that each of them tends to $\omega_{\ell,0}$ as $\mu \to +\infty$ and (after the relabeling) to $\omega_{\ell,q}$ for $\mu \to 0$ (the solid curve becomes steeper if μ increases and flatter if μ decreases). Their smoothness follows from the implicit function theorem, given that $\frac{\partial}{\partial\omega}(F(\mu,\omega) - f_p(\omega)) = \pi\mu - f_p'(\omega) > 0$ for all $\mu \geqslant 0$ and $\omega \notin \mathbb{Z}_+$. $\square$

Thus, the undamped Rayleigh spectrum $\mathrm{Sp}_{0,0,\mu}$ is a global, smooth μ-deformation of the spectrum $\mathrm{Sp}_{0,0,0}$ of the elastic string. As μ increases, the string frequencies $\omega_{\ell,0}$ remain fixed while the other frequencies $\omega^{\mu}_{\ell,q}$ move to their left towards the immediately lower $\omega_{\ell,0}$ (towards zero, if $\ell = 0$), forming bands which, for each μ, become narrower with ℓ. See Figure 3b.

It is also possible to give a quantitative, asymptotic estimate on the size $\omega^{\mu}_{\ell,n} - \omega_{\ell,0}$ of the bands. From Figure 3a it is clear that, if $g(\omega) := \pi\mu\omega\sin(\pi\omega)$, then for sufficiently large μ and ℓ $1 \approx |g(\omega^{\mu}_{\ell,n}) - g(\omega_{\ell,0})| \approx |g'(\omega_{\ell,0})||\omega^{\mu}_{\ell,n} - \omega_{\ell,0}| = \pi^2\mu\ell|\omega^{\mu}_{\ell,n} - \omega_{\ell,0}|$. Thus, for large μ and ℓ, the size of the bands decrease as $\frac{1}{\mu\ell}$.

Remark 4.2. (i) The fact that, in absence of damping, the frequencies decrease when μ grows is in agreement with a general theorem by Rayleigh on the dependence of the frequencies of mechanical systems on the rigidity [**1**, **22**]. On this regard, we mention that in [**12**] the band structure is qualitatively deduced from such a theorem and a limit argument; clearly, this type of argument does not apply in the presence of viscoelastic damping.

(ii) It follows from (4.1) and Proposition 4.2 that, for any $n \geqslant 1$, $\beta \geqslant 0$, $\gamma \geqslant 0$ and $\mu > 0$, a complex number λ belongs to $\mathrm{Sp}_{\beta,\gamma,\mu}$ if and only if $\lambda \neq 0$, $\lambda \neq -2\beta$, $\lambda \neq -\frac{1}{2\gamma}$ (if $\gamma > 0$) and

$$\sinh(\xi_{\beta,\gamma}(\lambda))U_n(\mu\xi_{\beta,\gamma}(\lambda)\sinh(\xi_{\beta,\gamma}(\lambda)) + \cosh(\xi_{\beta,\gamma}(\lambda))) = 0. \tag{4.8}$$

We give a direct proof of this equation, and of its particular case (4.1), within the proof of Proposition 5.1 on the eigenfunctions, because some of its details are needed to compute the eigenfunctions.

4.3. The point spectra of the damped Rayleigh strings.

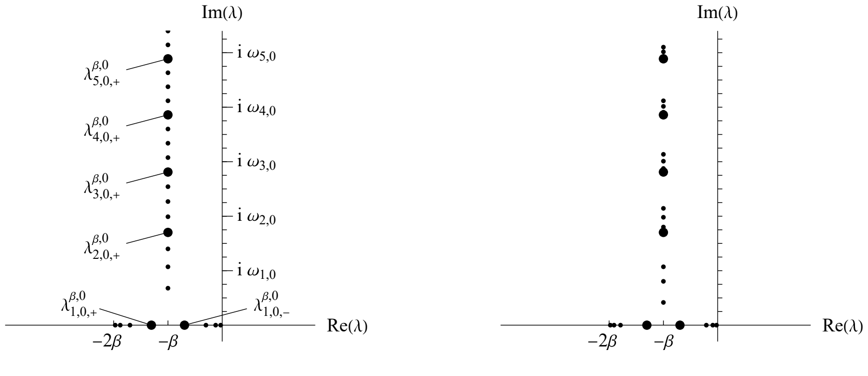
At this point, the structure of the spectra of the damped Rayleigh strings should be clear. Recall that the intervals and arcs of the spectral locus are oriented according to ω increasing.

Proposition 4.3. *Consider any $\beta \geqslant 0$ and $\gamma \geqslant 0$ such that $4\beta\gamma \neq 1$.*

i. *For any $\mu > 0$, $\mathrm{Sp}_{\beta,\gamma,\mu}$ consists of the string eigenvalues $\lambda^{\beta,\gamma}_{\ell,0,\pm}$, $\ell \in \mathbb{Z}_+$, and of the eigenvalues $\lambda^{\beta,\gamma}_{\pm}(\omega^{\mu}_{\ell,1}), \dots, \lambda^{\beta,\gamma}_{\pm}(\omega^{\mu}_{\ell,n})$, $\ell \in \mathbb{N}$.*

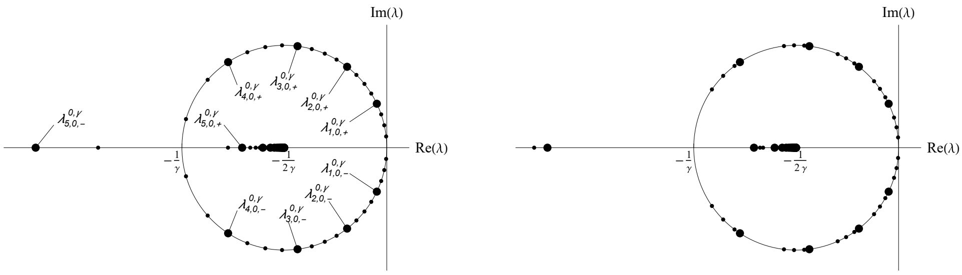
ii. *As μ grows, each $\lambda^{\beta,\gamma}_{\pm}(\omega^{\mu}_{\ell,p})$, $p \neq 0$, moves continuously and monotonically along $L_{\beta,\gamma,\pm}$, in the direction opposed to that induced by ω increasing, and satisfies*

$$\lim_{\mu\to 0} \lambda^{\beta,\gamma}_{\pm}(\omega^{\mu}_{\ell,p}) = \lambda^{\beta,\gamma}_{\pm}(\omega_{\ell,p}), \qquad \lim_{\mu\to+\infty} \lambda^{\beta,\gamma}_{\pm}(\omega^{\mu}_{\ell,p}) = \lambda^{\beta,\gamma}_{\ell,0,\pm}.$$



(a) $\mathrm{Sp}_{\beta,0,0}$, $\beta > 0$ (b) $\mathrm{Sp}_{\beta,0,\mu}$, $\beta > 0$, $\mu > 0$

FIGURE 4. (a) The point spectrum $\mathrm{Sp}_{\beta,0,0}$, $\beta > 0$, of the elastic string with viscous damping and (b) the point spectrum $\mathrm{Sp}_{\beta,0,\mu}$ of the Rayleigh string with viscous damping, $\mu > 0$. For comparison, the eigenvalues of the elastic string without damping are shown in (a) by the ticks on the imaginary axis. In this and the next figures the thicker dots are the string eigenvalues and the thinner ones are the other eigenvalues. The string eigenvalues are labeled in (a). Only the upper complex halfplane is shown. (Values used: $n = 3$, $\beta = 1.05$ and, in (b), $\mu = 0.15$).



(a) $\mathrm{Sp}_{0,\gamma,0}$, $\gamma > 0$ (b) $\mathrm{Sp}_{0,\gamma,\mu}$, $\gamma > 0$, $\mu > 0$

FIGURE 5. The spectra $\mathrm{Sp}_{0,\gamma,0}$ of the unloaded (a) and $\mathrm{Sp}_{0,\gamma,\mu}$ (b) of the loaded viscoelastic string. The string eigenvalues (thicker dots) are labeled in (a). Note the narrowing of the bands and the passage of some eigenvalues from $(-\infty, -\frac{1}{2\gamma})$ to $C_{0,\gamma}$ as μ grows. (Numerical values: $n = 3$, $\gamma = 0.22$ and, in (b), $\mu = 0.1$).

PROOF. (i.) This follows from item iii. of Proposition 4.1 and from the description of $\mathrm{Sp}_{0,0,\mu}$ in Proposition 4.2.

(ii.) Since the $\omega^{\mu}_{\ell,q}$ are continuous functions of $\mu \geqslant 0$, the $\lambda^{\beta,\gamma}_{\pm}(\omega^{\mu}_{\ell,q})$ depend continuously on $\beta \geqslant 0$, $\gamma \geqslant 0$ and $\mu \geqslant 0$. As μ grows, for fixed $\beta \geqslant 0$ and $\gamma \geqslant 0$,

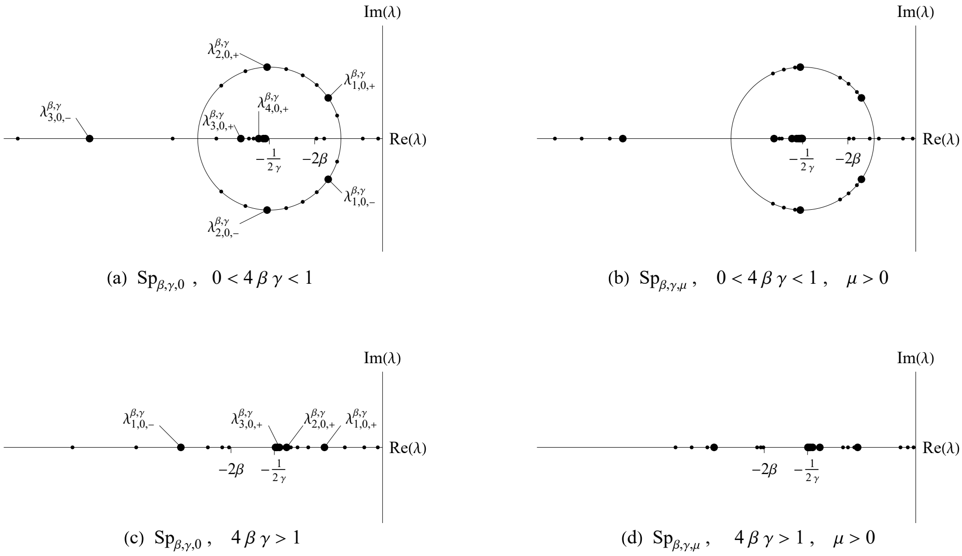


FIGURE 6. The spectrum of the viscoelastic string with viscous damping in the underdamped (a) and overdamped (c) cases compared to the spectrum of the Rayleigh string with the same damping (b,c). (Used values: $n = 3$; (a) $\beta = 0.5$, $\gamma = 0.3$, $\mu = 0$; (b) $\beta = 0.5$, $\gamma = 0.3$, $\mu = 0.21$; (c) $\beta = 0.7$, $\gamma = 0.5$, $\mu = 0$; (d) $\beta = 0.5$, $\gamma = 0.3$, $\mu = 0.6$).

the string eigenvalues remain fixed and the other eigenvalues move continuously on the spectral locus $L_{\beta,\gamma}$. Since each $\omega^{\mu}_{\ell,q}$ is a decreasing function of μ and belongs to the interval $(\omega_{\ell,0}, \omega_{\ell,q})$, see (4.5), each $\lambda^{\beta,\gamma}_{\pm}(\omega^{\mu}_{\ell,q})$ moves monotonically between $\lambda^{\beta,\gamma}_{\ell,0,\pm}$ and $\lambda^{\beta,\gamma}_{\pm}(\omega_{\ell,q})$. The limit behaviours follow from those of the $\omega^{\mu}_{\ell,q}$. □

Globally, $\mathrm{Sp}_{\beta,\gamma,\mu}$ is a continuous deformation of $\mathrm{Sp}_{\beta,\gamma,0}$ in which, for each $\ell \in \mathbb{N}$, each eigenvalue $\lambda^{\beta,\gamma}_{\pm}(\omega^{\mu}_{\ell,q})$ is closer to the string eigenvalue $\lambda^{\beta,\gamma}_{\ell,0,\pm}$ than $\lambda^{\beta,\gamma}_{\pm}(\omega_{\ell,q})$. Therefore, bands of $n+1$ eigenvalues are present, which, as in the undamped case, are separated by the string eigenvalues, but are narrower than those of the undamped case. They become narrower as ℓ and μ grow (because of those of $\mathrm{Sp}_{0,0,\mu}$ do so).

The resulting structure is illustrated in Figures 4, 5 and 6 for the three cases of the Kelvin–Voigt dissipation, viscous dissipation, and the combination of the two.

5. The eigenfunctions

We conclude this study by providing expressions for the eigenfunctions. To our knowledge, these expressions are new even in the case $\gamma = 0$: the best we could find in the literature are numerical investigation of the eigenfunctions [12, 21].

Preliminarily note that, in view of (4.3), the zeroes of the factor U_n in the eigenvalue equation (4.8), namely the non-string eigenvalues, are the zeroes of the n equations

$$\mu\pi\lambda \sinh(\xi_{\beta,\gamma}(\lambda)) + \cosh\big(\xi_{\beta,\gamma}(\lambda)\big) = c_{n,p}, \quad p = 1,\dots,n. \tag{5.1}$$

By the *multiplicity* of an eigenvalue we mean the number of damped normal modes with that eigenvalue and linearly independent eigenfunctions.

PROPOSITION 5.1. *Consider any $n \geqslant 1$, $\beta \geqslant 0$, $\gamma \geqslant 0$ and $\mu > 0$ and assume $4\beta\gamma \neq 1$. Then:*

i. *All eigenvalues in $\mathrm{Sp}_{\beta,\gamma,\mu}$ have multiplicity 1.*
ii. *The eigenfunction relative to a string eigenvalue λ is as in (3.3).*
iii. *The eigenfunction f of an eigenvalue λ which is solution of the p-th equation (4.4) satisfies, for all $x \in [x_{j-1}, x_j]$, $j = 1,\dots,n+1$,*

$$f(x) = U_{j-1}(c_{n,p}) \sinh(\xi_{\beta,\gamma}(\lambda)(x - x_{j-1})) - U_{j-2}(c_{n,p}) \sinh(\xi_{\beta,\gamma}(\lambda)(x - x_j)). \tag{5.2}$$

PROOF. Since we need some informations on the eigenfunctions which can be obtained within a proof of the characteristic equation (4.8), we begin giving such a proof (using the standard "transfer matrix" method [7]). Consider a normal mode $\psi^\lambda(x,t) = f(x)e^{\lambda t}$. The eigenfunction $f \in \Sigma$ satisfies (2.6a), (2.6b) and (2.6c). Define $f_j := f|_{[x_{j-1},x_j]} \colon [x_{j-1}, x_j] \to \mathbb{C}$, $j = 1,\dots,n+1$ and

$$\xi := \xi_{\beta,\gamma}(\lambda), \quad c := \cosh(\xi), \quad s := \sinh(\xi).$$

By (2.6a), each f_j satisfies

$$f_j''(x) = \xi^2 f_j(x), \quad x \in (x_{j-1}, x_j),$$

and thus

$$f_j(x) = a_j \cosh((x - x_{j-1})\xi) + b_j \sinh((x - x_{j-1})\xi) \quad \forall x \in [x_{j-1}, x_j] \tag{5.3}$$

with $a_j := f_j(x_{j-1})$ and $b_j := \frac{1}{\xi} f_j'(x_{j-1}^+)$. Thus, $f_j(x_j) = ca_j + sb_j$ and $f_j'(x_j^-) = \xi(sa_j + cb_j)$. By the continuity of f, $f_{j+1}(x_j) = f_j(x_j)$, namely

$$a_{j+1} = ca_j + sb_j, \quad j = 1,\dots,n, \tag{5.4}$$

and (2.6b) gives

$$b_{j+1} = 2\mu\xi a_{j+1} - (sa_j + cb_j), \quad j = 1,\dots,n. \tag{5.5}$$

The last two sets of equations can be written as

$$\begin{pmatrix} a_{j+1} \\ b_{j+1} \end{pmatrix} = M \begin{pmatrix} a_j \\ b_j \end{pmatrix}, \quad j = 1,\dots,n,$$

with the 2×2 ("transfer") matrix

$$M := \begin{pmatrix} c & s \\ s + 2\mu\xi c & c + 2\mu\xi s \end{pmatrix}$$

and thus

$$\begin{pmatrix} a_{j+1} \\ b_{j+1} \end{pmatrix} = M^j \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}, \quad j = 1,\dots,n. \tag{5.6}$$

The boundary conditions (2.6c) give

$$a_1 = 0, \quad ca_{n+1} + sb_{n+1}.$$

Since $a_1 = 0$, necessarily $b_1 = \frac{1}{\xi} f'(0^+) \neq 0$ (otherwise all $a_j, b_j = 0$ and $\psi^\lambda = 0$). Given that the eigenfunction is defined up to a factor we may assume $b_1 = 1$. The vanishing of $ca_{n+1} + sb_{n+1}$ implies

$$c(M^n)_{12} + s(M^n)_{22} = 0,$$

which is,thus, a necessary condition for λ to belong to the spectrum. The following standard argument [2, 7] shows that this condition is exactly the eigenvalue equation (4.8).

First, note that M has determinant one. Thus, by the Cayley–Hamilton theorem and the recursion (4.2), its powers are given by

$$M^j = U_{j-1}(y)M - U_{j-2}(y)\mathbb{I}, \quad j \in \mathbb{N}, \tag{5.7}$$

where $y := \frac{1}{2}\operatorname{Tr}(M) = c + \mu\xi s$ and $\mathbb{I}$ is the 2×2 unit matrix (see [7]). Therefore, $c(M^n)_{12} + s(M^n)_{22} = (cM_{12} + sM_{22})U_{n-1}(y) - sU_{n-2}(y) = 2(cs + \mu\xi s^2)U_{n-1}(y) - sU_{n-2}(y) = s\big(2yU_{n-1}(y) - U_{n-2}(y)\big) = sU_n(y)$, where the last equality follows again from (4.2). This proves that every eigenvalue is a zero of $sU_n(y)$, that is, satisfies (4.8).

Conversely, it is easy to prove that any $\lambda \in \mathbb{C}$ which is $\neq 0, -2b, -\frac{1}{2\gamma}$ and which satisfies $sU_n(y) = 0$ belongs to $\operatorname{Sp}_{\beta,\gamma,\mu}$.

Remark 5.1. The unimodularity of the transfer matrix is usually related to time reversal and energy conservation [7]. In our case, energy is not conserved, but the dissipation does not act on the system at fixed t, hence on the eigenfunctions f.

We may now prove Proposition 5.1. We keep using the notation introduced so far in this proof.

(i.) This follows from ii. and iii.

(ii.) For a string eigenvalue, $s = 0$. Hence $c^2 = 1$ and a trivial induction gives $\sinh(\xi x_j) = 0$, $\cosh(\xi x_j) = c^j$ for all $j = 1, \dots, n+1$. Conditions (5.4) and (5.5) give all $a_j = 0$ and $b_j = c^{j-1}b_1$. The choice $b_1 = 1$ leads to $f_1(x) = \sinh(\xi(x - x_0)) = \sinh(\xi x)$ $\forall x \in [0, n+1]$ and, for each $j = 1, \dots, n+1$, $f_j(x) = b_j(\sinh(\xi x)\cosh(\xi x_{j-1}) - \cosh(\xi x)\sinh(\xi x_{j-1})) = c^{2(j-1)}\sinh(\xi x) = \sinh(\xi x)$ for all $x \in [x_{j-1}, x_j]$. Thus, $f(x) = \sinh(\xi x)$ for all $x \in [0, n+1]$.

(iii.) A non-string eigenvalue $\lambda \in \operatorname{Sp}_{\beta,\gamma,\mu}$ is a solution of the p-th equation (5.1) for some $p = 1, \dots, n$. In such a case, $\frac{1}{2}\operatorname{Tr}(M) = c + \mu\xi s = c_{n,p}$ and (5.6), (5.7) and $a_1 = 0$ give

$$\begin{pmatrix} a_{j+1} \\ b_{j+1} \end{pmatrix} = \begin{pmatrix} cU_{j-1}(c_{n,p}) - U_{j-2}(c_{n,p}) & sU_{j-1}(c_{n,p}) \\ (s + 2\mu\xi c)U_{j-1}(c_{n,p}) & (c + 2\mu\xi s)U_{j-1}(c_{n,p}) - U_{j-2}(c_{n,p}) \end{pmatrix} \begin{pmatrix} 0 \\ b_1 \end{pmatrix}.$$

This implies that there is a one-parameter choice of the a_j's and b_j's, parametrized by b_1, and thus a unique eigenfunction relative to λ. Choosing $b_1 = 1$ and observing that, since $c+2\mu\xi s = 2c_{n,p} - c$, $(c+2\mu\xi s)U_{j-1}(c_{n,p}) - U_{j-2}(c_{n,p}) = 2c_{n,p}U_{j-1}(c_{n,p}) -$

$U_{j-2}(c_{n,p}) - cU_{j-1}(c_{n,p}) = U_j(c_{n,p}) - cU_{j-1}(c_{n,p})$ (see (4.2)), we conclude that

$$a_{j+1} = sU_{j-1}(c_{n,p}), \quad b_{j+1} = U_j(c_{n,p}) - cU_{j-1}(c_{n,p}), \quad j = 0, \dots, n.$$

Therefore, from (5.3), $f_1(x) = \sinh(\xi x)$ and, for $j = 1, \dots, n$,

$$\begin{aligned} f_{j+1}(x) &= sU_{j-1}(c_{n,p})\cosh(\xi x - \xi x_j) + (U_j(c_{n,p}) - cU_{j-1}(c_{n,p}))\sinh(\xi x - \xi x_j) \\ &= U_{j-1}(c_{n,p})(s\cosh(\xi x - \xi x_j) - c\sinh(\xi x - \xi x_j)) + U_j(c_{n,p})\sinh(\xi x - \xi x_j) \\ &= -U_{j-1}(c_{n,p})\sinh(\xi x - \xi x_{j+1}) + U_j(c_{n,p})\sinh\left(\xi x - \xi x_j\right) \end{aligned}$$

where the last equality follows from the fact that

$$s = \sinh(\xi x_1) \quad \text{and} \quad c = \cosh(\xi x_1). \qquad \square$$

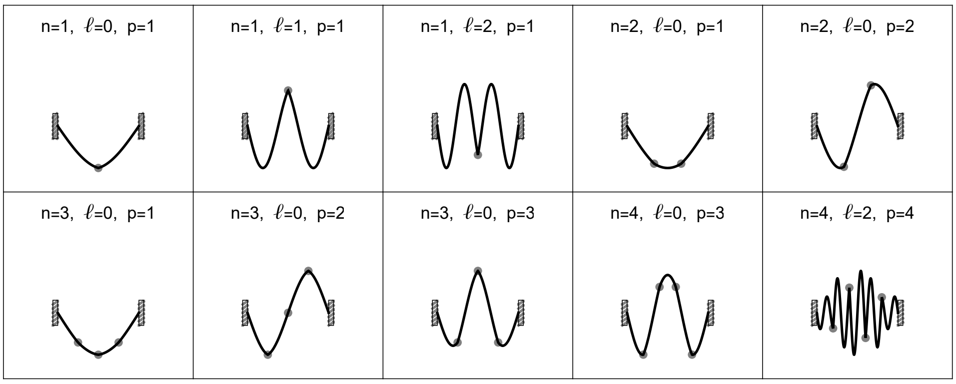


FIGURE 7. The eigenfunctions of the (damped or undamped) Rayleigh's system relative to a few eigenvalues $\lambda_{\pm}^{\beta,\gamma}(\omega_{\ell,p}^{\mu})$. The string profile is the graph of the function $-if$, with f as in (5.2) and normalized to 1. Note the discontinuities in the first derivative of the string profile. In all pictures, $\mu = 0.15$.

We note very quickly some properties of the eigenfunctions, omitting for shortness the simple checks. First, as it happens for the Kelvin–Voigt and elastic strings (see Section 3.4), for $\beta \geqslant 0$, $\gamma \geqslant 0$, $\ell \in \mathbb{Z}_+$ and $p = 1, \dots, n$, the eigenfunction relative to the pair of eigenvalues $\lambda_{\ell,p,\pm}^{\beta,\gamma,\mu}$ equals that relative to $\lambda_{\ell,p,\pm}^{0,0,\mu}$.

Second, if λ is a non-string eigenvalue in $\mathrm{Sp}_{\beta,\gamma,\mu}$, which is the solution of the p-th equation (5.1), $p = 1, \dots, n$, then its eigenfunction f satisfies the reflectional symmetry

$$f(x_{n+1-j} - y) = (-1)^{p+1} f(x_j + y) \quad \forall y \in [0, 1), \quad j = 0, \dots, \lfloor \tfrac{n}{2} \rfloor$$

and is such that $f_1(x_1) \neq 0$ and, for all $j = 1, \dots, n$:

$$\frac{f(x_j)}{f(x_1)} = U_{j-1}(c_{n,p}). \tag{5.8}$$

Identities (5.8) indicate that, during the oscillations, the relative displacements from the equilibrium configuration of the point masses (determined by the real parts of $f(x_1), \dots, f(x_n)$) are independent of the band (namely, of the index ℓ).

Lastly, using the fact that the Rayleigh's eigenvalues tend to those of the Kelvin–Voigt string for $\mu \to 0$, it is not difficult to verify that so do, pointwisely, the eigenfunctions. Thus, for small μ, the eigenfunctions are small deformations of those of the Kelvin–Voigt string, but with discontinuities in the first derivative of the string's profile function at the point masses. Figure 7 depicts a sample of these eigenfunctions, computed with the formulas of Proposition 5.1.

Remark 5.2. Since $\xi_{\beta,\gamma}(\lambda_+^{\beta,\gamma}(\omega_{\ell,p}^\mu)) = \xi_{\beta,\gamma}(\lambda_-^{\beta,\gamma}(\omega_{\ell,p}^\mu))$ for all ℓ and p, the two eigenvalues $\xi_{\beta,\gamma}(\lambda_\pm^{\beta,\gamma}(\omega_{\ell,p}^\mu))$ have the same eigenfunction. This explains why the eigenvalue $-\frac{1}{2\gamma}$, when present, has multiplicity one even though it can be seen as the coalesce of a pair of such eigenvalues.

Acknowledgments. F.F. has been partially supported by the MIUR-PRIN project 2022FPZEES *Stability in Hamiltonian dynamics and beyond*. The authors are members of *GNFM-INDAM*.

References

1. V. I. Arnold, *Mathematical Methods of Classical Mechanics*, Springer-Verlag, New York, 1995.
2. J. M. Bendickson, J. P. Dowling, M. Scalora, *Analytic expressions for the electromagnetic mode density in finite, one-dimensional, photonic band-gap structures*, Phys. Rev. E **53** (1996), 4107–4121.
3. S. Chen. K. Liu, Z. Liu, *Spectrum and stability for elastic systems with global or local Kelvin–Voigt damping*, SIAM J. Appl. Math. **59** (1998), 651-668.
4. Y. Chen, *Vibration of a string with attached concentrated masses*, J. Franklin Inst. **276** (1963), 191–196.
5. E. B. Davies, *Linear Operators and Their Spectra*, Cambridge Univ. Press, Cambridge, 2007.
6. S. Galasso, *Asymptotic dynamics and synchronization of dissipative systems of coupled pendula*, PhD thesis, University of Padova, 2022.
7. D. J. Griffiths, C. Steinke, *Waves in locally periodic media*, Am. J. Phys. **69** (2001), 137–154.
8. R. B Guenther, J. W. Lee, *Partial Differential Equations of Mathematical Physics and Integral Equations*, Dover Publications, Mineola, 2012.
9. B. Z. Guo, J. M. Wang, G. D. Zhang, *Spectral analysis of a wave equation with Kelvin–Voigt damping*, ZAMM Z. Angew. Math. Mech. **90** (2010), 323–342.
10. S. He, D. Maynard, *Detailed measurements of inelastic scattering in Anderson localization*, Phys. Rev. Lett. **57** (1986), 3171–3174.
11. O. Heaviside, *On the extra current*, Phil. Mag. (5) **2** (1876), 135–145.
12. C. H. Hodges, J. Woodhouse, *Vibration isolation from irregularity in a nearly periodic structure: Theory and measurements*, J. Acoust. Soc. Am. **74** (1983), 894–905.
13. B. Jacob, C. Trunk, *Location of the spectrum of operator matrices which are associated to second order equations*, Oper. Matrices **1** (2007), 45-60.
14. B. Jacob, C. Trunk, M. Winklemeier, *Analyticity and Riesz basis property of semigroups associated to damped vibrations*, J. Evol. Equ. **8** (2008), 263–281.
15. I. Karafyllis, M. Kontorinaki, M. Krstic, *Wave equations with kelvin-voigt damping and boundary disturbance: A study of the asymptotic gain properties*, 2019 18$^{\text{th}}$ European Control Conference (ECC), Naples, Italy, 2019, 1926–1931, doi: 10.23919/ECC.2019.8795857.
16. P. Lancaster, *Lambda-matrices and Vibrating Systems*, Pergamon Press, Oxford, 1966.

17. P. Lancaster, A. Shkalikov, *Damped vibrations of beam and related spectral problems*, Can. Appl. Math. Q. **2** (1994), 45–90.
18. L. Lu, L. Zhao, J. Hu, *Effect of Kelvin–Voigt damping on spectrum analysis of a wave equation*, Electron. J. Differ. Equ. **2018** (2018), 186.
19. J. C. Mason, D. C. Handscomb, *Chebyshev Polynomials*, Chapman and Hall/CRC, New York, 2002.
20. G. Óttarsson, C. Pierre, *Vibration and wave localization in a nearly periodic beaded string*, J. Acoust. Soc. Am. **101** (1997), 3430–3442.
21. S. Parmley, T. Zobrist, T. Clough, A. Perez-Miller, M. Makela, R. Yu, *Vibrational properties of a loaded string*, Am. J. Phys. **63** (1995) 547–553.
22. J. W. S. Rayleigh, *The Theory of Sound, Volume One*, Dover Publications, Mineola, 1945.
23. D. L. Russell, *On mathematical models for the elastic beam with frequency-proportional damping*, In: H. T. Banks (ed.), *Control and Estimation in Distributed Parameter Systems*, SIAM, 1992, 125–169.
24. S. Salsa, *Partial Differential Equations in Action: From Modelling to Theory*, Springer, New York, 2016.
25. A. N. Tikhonov, A. A. Samarskii, *Equations of Mathematical Physics*, Dover Publications, Mineola, 1990.
26. G. Q. Xu, N. E. Mastorakis, *Spectrum of an operator arising elastic system with local K-V damping*, ZAMM Z. Angew. Math. Mech. **88** (2008), 483–496.

ТАЧКАСТИ СПЕКТРИ И НОРМАЛНИ МОДОВИ РЕЈЛИЈЕВЕ ОПТЕРЕЋЕНЕ СТРУНЕ СА ПРИГУШЕЊЕМ

Резиме. Описујемо тачкасте спектре дисипативне верзије чувене "Рејлијеве оптерећене струне", еластичне струне коначне дужине која носи $n \geqslant 1$ једнако размакнутих, једнаких тачкастих маса, што је основни модел који показује тракасту структуру и појављује се у многим примењеним областима. Разматрамо случај у коме је дисипација последица вискозног пригушења услед међудејства струна-окружење, стандардни модел за унутрашњу вискоеластичну дисипацију (Келвин-Фојтов модел), као и њихову комбинацију. Показујемо да је тачкасти спектар сваке од ових пригушених верзија Рејлијеве оптерећене струне непрекидна деформација тачкастог спектра неоптерећених еластичних струна са тим пригушењем и која представља тракасту структуру сличну оној у непригушеном случају. Такође дајемо експлицитне аналитичке изразе за сопствене функције, за било коју вредност n.

(Received 13.12.2024)
(Available online 05.06.2025)

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