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ON THE EXISTENCE OF GEODESIC VECTOR FIELDS ON CLOSED SURFACES

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Abstract. We construct an example of a Riemannian metric on the 2-torus such that its universal cover does not admit global Riemann normal coordinates.

1. Introduction

Definition 1.1. We call a vector field $v = v(x)$ on a Riemannian manifold (M^n, g) *geodesic*, if its length is identically 1 and if $\nabla^g_v v = 0$, where ∇^g is the Levi-Civita connection of g.

Clearly, a vector field is geodesic if and only if any orbit of its flow is an arc-length parameterised geodesic.

Example 1.1. For the metric

$$g = (dx^1)^2 + \sum_{i,j=2}^{n} g_{ij}(x)dx^i dx^j, \tag{1.1}$$

the vector field $\frac{\partial}{\partial x^1}$ is geodesic.

In dimension two the formula (1.1) reads

$$g = dx^2 + f(x,y)dy^2. \tag{1.2}$$

Coordinates such that the metric has the form (1.2) are called *Riemann normal* (*or geodesic normal or semi-geodesic*) coordinates. It is known [6] that, in dimension two, for any geodesic vector field there exists a local coordinate system (x, y) such that the metric has the form (1.2) and the vector field is $\frac{\partial}{\partial x}$. The goal of this paper is to construct an example of a Riemannian two-torus (T^2, g) such that its universal cover $(\mathbb{R}^2, \tilde g)$, where $\tilde g$ denotes the lift of g, has no geodesic vector field. Any sufficiently small C^2-perturbation of this metric has the same property. The example can be easily generalised to closed surfaces of negative Euler characteristic.

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We have two reasons for studying the problem. The first one is related to the very recent paper [7] studying conformal product structures on Kähler manifolds. [7, Corollary 4.6] guarantees the existence of a geodesic vector field on compact Kähler manifold of real dimension $n \geqslant 4$ carrying an orientable conformal product structure with non-identically zero Lee form. [7, Proposition 4.7] uses the results of the present paper to show the existence of direct product compact Kähler metrics with no orientable conformal product structure with non-identically zero Lee form.

Another reason comes from the theory of integrable geodesic flows on closed surfaces. [2, Theorem 1.6] implies that for any Riemannian 2-torus (T^2, g) such that the geodesic flow is integrable and the integral satisfies $\aleph$-condition (see [2, Definition 1.3]), there exists a geodesic vector field on the universal cover $(\mathbb{R}^2, \tilde{g})$, where $\tilde{g}$ denotes the lift of g. Our example is an "easy to construct" example of $\aleph$-nonintegrable geodesic flow. Recall that though generic geodesic flow is not integrable, proving that a geodesic flow is nonintegrable or constructing an example of an nonintegrable geodesic flow is not an easy task (see e.g. [4, §10] and [3, §3]).

Examples using a similar idea have already appeared in the literature; see, for instance, [1, pp. 46–47] and [5, p. 11]. In these papers, the authors primarily focused on minimal geodesic laminations. Their examples demonstrated the nonexistence of a smooth torus in T^*T^2 that is invariant under the geodesic flow and has the property that every trajectory of the geodesic flow lying on this torus projects to a minimal geodesic.

The examples from [1, 5] are also sufficient for constructing non-$\aleph$-integrable geodesic flows, see the second reason above. However, it is worth noting that in the first one, which relates to the results of [7], the minimality condition is not essential.

2. Example and proof of nonexistence of geodesic vector field

Take the standard sphere with the standard metric. Next, take a small $\varepsilon > 0$ and change the topology of the manifold in the ε-neighborhood of the south pole by gluing a handle in the neighborhood. The metric outside the neighborhood is not changed, the metric in the modified neighborhood can be chosen arbitrary such that the obtained metric on the two-torus is smooth (see Fig. 1).

We consider the universal cover $\mathbb{R}^2$ and denote by $\tilde{g}$ the lift of the metric. Let us show that $(\mathbb{R}^2, \tilde{g})$ does not admit a geodesic vector field. We assume it does, denote the geodesic vector field by v, and find a contradiction.

In order to do it, consider the circle of radius 2ε around the north pole of the initial sphere and consider one of its lifts $C_{2\varepsilon}(N_0) = \partial B_{2\varepsilon}(N_0)$. Let us show that our geodesic vector field v is necessary transversal to it. Arguing by the method of contradiction, assume there exists a point where the vector field v is tangent to the circle. Consider the geodesic γ starting from this point in the direction of v. This geodesic, and also geodesics close to γ, do not enter the "light gray" region where we changed the sphere. Therefore, any geodesic γ_1 starting from a nearby point in the direction of our vector field intersects γ, as any two geodesics on the sphere intersect each other. This gives a contradiction, since velocity vectors of both geodesics at the point of intersection should be v.

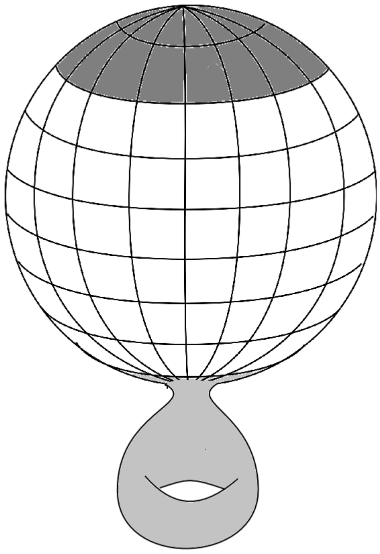

FIGURE 1. The torus made of the sphere: the dark-gray part is the 2ε-ball around the north pole. The surgery was made in the light-gray part.

Thus, our vector field is transversal to the circle at every point. Then, the index of the restriction of v to $B_{2\varepsilon}(N_0)$ is nonzero. But the index must be zero since v is never zero. The contradiction proves the nonexistence of a geodesic vector field.

Note also that in the proof we used the following properties of the standard metric of the sphere only:

1. Every arc-length parameterised geodesic starting at a point of the 2ε-circle and tangent to it does not reach the ε-neighborhood of the south pole within the time 2π.
2. Any two arc-length parameterised geodesics $\gamma_1, \gamma_2 \colon (0, 2\pi) \to S^2$ always intersect.

These properties are fulfilled for any sufficiently small perturbation, in the C^2-topology, of the standard metric of the sphere. This implies that one can construct such an example in the real-analytic category. Moreover, by attaching more than one handle in the "light gray" region (see Fig. 1), one can construct an example of a closed Riemannian surface of arbitrary negative Euler characteristic such that the universal cover does not admit a geodesic vector field.

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О ПОСТОЈАЊУ ГЕОДЕЗИЈСКИХ ВЕКТОРСКИХ ПОЉА НА ЗАТВОРЕНИМ ПОВРШИМА

Резиме. Конструишемо пример Риманове метрике на 2-торусу тако да њено универзално наткривање не допушта глобалне Риманове нормалне координате.

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