THEORETICAL AND APPLIED MECHANICS
Volume 52 (2025) Issue 1, 1–7 DOI: https://doi.org/10.2298/TAM250110011K

GENERALIZED COMPLETELY INTEGRABLE SYSTEMS

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Abstract. Dynamical systems more general than Hamiltonian systems are considered. The role of the Hamiltonian function is played by a 1-form (not necessarily closed) on a symplectic phase space. A bracket of such forms is introduced and a generalized Liouville theorem on the complete integrability is formulated. This generalization allows us to better understand the meaning of the conditions of the classical theorem on the complete integrability of the Hamilton equations and to reveal the role of tensor invariants.

1.

The modern point of view on Hamiltonian dynamical systems is formulated by E. Cartan [1] (a detailed exposition can be found in the book [2]; notations from this book will be used below).

Let (M^{2n}, Ω) be a symplectic manifold and Ω be a closed nondegenerate 2-form on M^{2n} (symplectic structure). The smooth function $H\colon M^{2n} \to \mathbb{R}$ generates a Hamiltonian vector field as follows

$$i_v\Omega = -dH. \tag{1.1}$$

The field v itself defines a Hamiltonian system of differential equations on M:

$$\dot{x} = v(x), \quad x \in M. \tag{1.2}$$

Let $F, G\colon M^{2n} \to \mathbb{R}$ be two smooth functions; According to formula (1.1), they correspond to the Hamiltonian vector fields u, w. The Lie derivative L_uG is called the Poisson bracket of these functions. It is easy to check that $L_uG = -L_vF$. In other words, $\{F, G\} = -\{G, F\}$.

The famous Liouville theorem states that if n independent first integrals of a Hamiltonian system (1.2), with pairwise zero Poisson brackets, are known, then this system of differential equations is integrable by quadratures. A discussion of various aspects of this theorem (both analytic and geometric) can be found, for example, in [2–4].

The integrability by quadratures is only related to local consideration. Therefore, Liouville's theorem is transferred to the more general case where the equation (1.1) is replaced by the following relation:

$$i_v\Omega = \varphi, \tag{1.3}$$

where φ is a closed (but not necessarily exact) 1-form. In this case, one can define the 'Poisson bracket' of closed 1-forms and formulate Liouville's theorem on the integrability by quadratures in a more general form [2]. Geometric aspects of

2020 *Mathematics Subject Classification:* 37J35, 70G65.

Key words and phrases: symplectic manifold, differential forms, distributions, Hamilton equations, Lie bracket, Poisson bracket, tensor invariants, complete integrability.

integrable dynamical systems of the form (1.3) with condition $d\varphi = 0$ are discussed in [5]. For other generalizations of integrable systems, see, e.g., [6–10].

2.

Let us consider the most general case when the vector field v is defined by a given 1-form φ (not necessarily closed). Since the 2-form Ω is assumed to be nondegenerate, there is a natural isomorphism $v \mapsto i_v\Omega$ of the space $\mathcal{T}(M)$ (tangent vector fields on M) and the space $\Lambda^1(M)$ (1-form on M). Therefore, without additional assumptions, the vector field v of (1.3) defines an arbitrary dynamical system on a symplectic manifold.

Having in mind a generalization of integrable Hamiltonian systems, we assume that there are n 1-forms $\varphi_1, \dots, \varphi_n$ linearly independent at each point of the manifold M^{2n}. They define n independent vector fields $v_1, \dots, v_n$ such that

$$i_{v_k}\Omega = -\varphi_k, \quad q \leqslant k \leqslant n. \tag{2.1}$$

There's a simple

PROPOSITION 2.1. *$i_{v_k}\varphi_j = -i_{v_j}\varphi_k$ for all $1 \leqslant j, k \leqslant n$.*

Indeed, according to (2.1),

$$i_{v_k}\varphi_j = i_{v_k} i_{v_j}\Omega = -i_{v_j} i_{v_k}\Omega = -i_{v_j}\varphi_k.$$

What is the meaning of the expression $i_{v_k}\varphi_j$? Let φ_j be closed 1-forms, that is, locally $\varphi_j = df_j$. Then

$$i_{v_k}\varphi_j = i_{v_k} df_j = L_{v_k} f_j = \{f_k, f_j\}, \tag{2.2}$$

where $df_k = \varphi_k$.

With this remark in mind, it is useful to introduce the n-dimensional distribution of tangent planes

$$\Pi^n = \{\varphi_1 = \dots = \varphi_n = 0\} \tag{2.3}$$

and assume that for all $1 \leqslant j, k \leqslant n$.

$$i_{v_k}\varphi_j = 0. \tag{2.4}$$

This means that at each point $x \in M^{2n}$ the linearly independent vectors $v_1(x), \dots, v_n(x)$ lie in the n-dimensional tangent plane Π^n_x.

According to (2.2), for usual Hamiltonian systems the condition (2.4) means that all pairwise Poisson brackets of the functions $f_1, \dots, f_n$ are zero, which will be the first integrals of the original Hamiltonian system (1.1)–(1.2). In the considered general case, condition (2.4) does not have such a meaningful sense yet.

3.

Let's introduce a bracket of two 1-forms:

$$[\varphi_k, \varphi_j] = L_{v_k}\varphi_j - L_{v_j}\varphi_k; \quad 1 \leqslant k, j \leqslant n.$$

This bracket is also defined for a degenerate 2-form Ω. More precisely, the bracket is defined in the subspace $\Lambda^1(M)$, which is the image of the space $\mathcal{T}(M)$ under the mapping $v \mapsto i_v\Omega$.

PROPOSITION 3.1.

$$[\varphi_k, \varphi_j] = i_{[v_k, v_j]}\Omega + d(i_{v_k} i_{v_j})\Omega. \tag{3.1}$$

COROLLARY 3.1. $[\varphi_k, \varphi_j] = -[\varphi_j, \varphi_k]$.

Proposition 3.1 is proved using the well-known formula $[L_u, i_w] = i_{[u,w]}$ (see, e.g., [2], Chapter IV, Section 3.4). Indeed, $L_{v_k}\varphi_j = L_{v_k} i_{v_j}\Omega = i_{v_j} L_{v_k}\Omega + i_{[v_k,v_j]}\Omega = i_{v_j} d\varphi_k + i_{[v_k,v_j]}\Omega = L_{v_j}\varphi_k - d i_{v_j}\varphi_k + i_{[v_k,v_j]}\Omega = L_{v_j}\varphi_k + i_{[v_k,v_j]}\Omega + d(i_{v_k} i_{v_j}\Omega)$.

Remark. In [2], the following Poisson bracket of two 1-forms is introduced

$$(\varphi_k, \varphi_j) = i_{[v_k,v_j]}\Omega \tag{3.2}$$

(according to the natural isomorphism of the spaces $\mathcal{T}(M)$ and $\Lambda^1(M)$). Unfortunately, the formula (3.2) does not reveal the meaning of this operation. Our way of introducing the bracket is closely related to tensor invariants of dynamical systems.

As an example, we turn again to Hamiltonian systems and put $\varphi_k = df_k$. Then, as it is easy to calculate,

$$[df_k, df_j] = -2d\{f_k, f_j\}. \tag{3.3}$$

Therefore, if the functions $f_1, \dots, f_n$ are in pairwise involution, then their differentials commute. In particular, the well-known fact follows from (3.1) and (3.3): if the functions are in involution, then their corresponding Hamiltonian vector fields commute. Here, the expression $i_{v_k} i_{v_j}\Omega$ in (3.1) equals zero since the functions f_k and f_j are in involution.

4.

At this point, we can formulate a general theorem on integrable dynamical systems on a symplectic phase space (M^{2n}, Ω).

Theorem 4.1. *Suppose that there are n independent 1-forms $\varphi_1, \dots, \varphi_n$ such that for all $1 \leqslant i, j \leqslant n$*

(1) $\varphi_i(v_j) = 0$,
(2) $[\varphi_i, \varphi_j] = 0$.

Then the distribution (2.3) is integrable and if its n-dimensional integral manifolds are found, then each of the systems of differential equations

$$\dot{x} = v_i(x), \quad x \in M^{2n} \tag{4.1}$$

is integrable by quadratures.

Further, if the vector fields $v_1, \dots, v_n$ are complete on M^{2n}, then

3. *integral manifolds of distribution Π are diffeomorphic to $\mathbb{T}^s \times \mathbb{R}^{n-s}$ ($\mathbb{T}^s$ is an s-dimensional torus),*
4. *on these invariant manifolds, we can choose s angular and $n - s$ linear coordinates such that these variables change uniformly along the solutions of each of the systems of differential equations (4.1).*

Indeed, from the first assumption, at each point $x \in M^{2n}$, the vectors $v_1(x), \dots, v_n(x)$ belong to the n-dimensional tangent planes Π_x^n at each point $x \in M^{2n}$. In particular, $i_{v_i} i_{v_j}\Omega = 0$ and therefore the last summand in (3.1) equals zero. From the second assumption (taking into account the nondegeneracy of Ω) follows the commuting of the vector fields $v_1, \dots, v_n$. But then the distribution Π is integrable: through each point $x \in M^{2n}$ passes the only n-dimensional integral manifold of this distribution. If these manifolds are found, then the integrability of each of the systems (4.1) follows from the Lie theorem applied to commuting vector fields $v_1, \dots, v_n$ which are tangent to the integrable manifolds. Conclusions 3 and 4 on the structure of integral manifolds and on the phase flows on them follow from well-known results on the free action of the group $\mathbb{R}^n$ on n-dimensional smooth manifolds (see, e.g., [4]).

Theorem 4.1, of course, contains as a special case the classical Liouville theorem on the complete integrability of Hamiltonian systems (together with its geometric aspect). Instead of the integrals $f_1, \dots, f_n$ we should consider their differentials $df_1, \dots, df_n$. The integral manifolds of the integrable distribution Π are already known: they are manifolds of joint levels of functions $f_1, \dots, f_n$. The only condition of Liouville's theorem—the equality to zero of the Poisson brackets of the first integrals—is here split into two conditions: the first condition fixes the integral manifolds of distribution Π, and the second condition guarantees the commutability of the corresponding Hamiltonian vector fields.

We typically consider the case of compact integral manifolds; then $s = n$. In the general case integral manifolds of distribution Π are more complicated. In particular, they can densely fill M^{2n}.

5.

Let the conditions of Theorem 1 be satisfied. Is it then possible to find explicitly (e.g., using quadratures) the integral manifolds of the distribution Π? It turns out that even in the simplest case, when $n = 1$, the answer to this question is negative.

Indeed, let $v = (v_1(x), v_2(x))$ be an arbitrary vector field in the plane $\mathbb{R}^2$ with the standard symplectic structure $\Omega = dx_1 \wedge dx_2$ (area 2-form). This field can, of course, be represented as (1.3). The first condition of Theorem 1 is automatically satisfied since

$$\varphi(v) = i_v(i_v\Omega) = 0.$$

Since $n = 1$, the second condition is also satisfied. The integral manifolds of the distribution (2.3) are the phase curves of the vector field v on the plane $\mathbb{R}^2$. Finding them is equivalent to the explicit integration of the system of differential equations in the plane, which is, of course, impossible in the general case.

Let us now assume that the 1-form φ is an invariant of the system

$$\dot{x}_1 = v_1(x_1, x_2), \quad \dot{x}_2 = v_2(x_1, x_2) \tag{5.1}$$

in the plane $\mathbb{R}^2$:

$$0 = L_v\varphi = i_v d\varphi + d(i_v\varphi) = i_v d\varphi. \tag{5.2}$$

Let us suppose that at some point in the phase plane, $d\varphi \neq 0$. Then this inequality holds in a neighborhood of that point. Consequently, according to (5.2) we have $v = 0$ in this neighborhood. If $d\varphi = 0$ in some region of the phase plane, then $\varphi = df$, where f is a smooth function. Since in this case

$$i_v\Omega = df,$$

then the system (5.1) is Hamiltonian with the Hamiltonian function $-f$. In particular, f is a first integral of the system. If $v \neq 0$, then obviously $df \neq 0$ and hence f in a non-constant function. We obtain the integrability by quadratures of the system (5.1).

Therefore, if $L_v\varphi = 0$ (that is, the 1-form φ is a conservation law), then the equations (5.1) are integrable by quadrature. This simple fact is in agreement with general observations about integrability of systems of differential equations with a sufficient number of independent tensor invariants [11].

6.

The observations of the previous section can be generalized. Let the 1-forms $\varphi_1, \dots, \varphi_n$ once again satisfy the conditions

$$i_{v_k}\varphi_j = \varphi_j(v_k) = 0 \tag{6.1}$$

and, in addition,

$$i_{v_k} d\varphi_j = 0, \quad 1 \leqslant k, j \leqslant n. \tag{6.2}$$

Then these forms will be invariants of each of the systems of differential equations (4.1) since

$$L_{v_k}\varphi_j = i_{v_k} d\varphi_j + d(i_{v_k}\varphi_j) = 0. \tag{6.3}$$

For ordinary fully integrable Hamiltonian systems, condition (6.2) is obviously satisfied (since the 1-form φ_j is closed). Moreover, the 1-forms φ_j satisfying the two conditions (6.1) and (6.2) are called integral forms by Cartan; they give rise to absolute integral invariants of dynamical systems (4.1).

From (6.3) it follows that the pairwise brackets of the 1-forms $\varphi_1, \dots, \varphi_n$ are zero. In particular, the vector fields $v_1, \dots, v_n$ are pairwise commutative (Proposition 2). Hence, the assumptions of the theorem in Proposition 4 are satisfied. Therefore, in this case the dynamical systems (4.1) can also be considered as completely integrable in the generalized sense.

It is worth noting that here each of the systems of differential equations (4.1) in the $2n$-dimensional phase space M^{2n} admits $2n$ independent tensor invariants: vector fields $v_1, \dots, v_n$ (among them the field v_i is a trivial invariant) and the forms $\varphi_1, \dots, \varphi_n$. According to [**11**], the systems (4.1) can be integrable by quadratures. However, is this really the case? As shown in Section 5, for $n = 1$ this is obviously not the case.

Let us discuss the question of integrability by quadratures of the dynamical system (4.1) satisfying the relations (2.1) if the conditions (6.1) and (6.2) are satisfied. Let the rank of the closed 2-form $d\varphi_j$ in some region M^{2n} be constant and equal to ρ (this means that $(d\varphi_j)^\rho \neq 0$, and $(d\varphi_j)^{\rho+1} = 0$). Then by Darboux's theorem, in some local coordinates $p_1, \dots, p_n, q_1, \dots, q_n$ on M^{2n}

$$d\varphi_j = \sum_{i=1}^{\rho} dp_i \wedge dq_i. \tag{6.4}$$

From the condition (6.2) it follows that the first ρ components of each of the vector fields $v_1, \dots, v_n$ in p and q coordinates are zero. From (6.4) we obtain the following local equation

$$\varphi_j = \sum_{i=1}^{\rho} p_i \wedge dq_i + dS(p, q), \tag{6.5}$$

where S is a smooth function. The first 1-form on the right-hand side of (6.5) equals zero on vectors $v_1, \dots, v_n$. But then, according to (6.1), the function S will be the first integral of each of the n systems of differential equations (4.1).

Thus, the problem of integration by quadratures for the dynamical system (4.1) reduces to the question of an efficient (using quadratures) reduction of the 2-forms $d\varphi_1, \dots, d\varphi_n$ to the 'canonical' form (6.4). However, known proofs of Darboux's theorem demonstrate that this is hardly possible. Although, if the closed 2-form $d\varphi_j$ is already reduced to the form (6.4), then the function S in (6.5) can be found by simple quadratures.

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УОПШТЕНИ ПОТПУНО ИНТЕГРАБИЛНИ СИСТЕМИ

Резиме. Разматрају се динамички системи општији од Хамилтонових система на симплектичком фазном простору. Улогу Хамилтонове функције игра 1-форма (не нужно затворена). Уводи се комутатор таквих форми и формулише се уопштена Лиувилова теорема о потпуној интеграбилности. Ово уопштење нам омогућава да боље разумемо значење услова класичне теореме о потпуној интеграбилности Хамилтонових једначина као и да истакнемо улогу тензорских инваријанти.

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(Received 10.01.2025)
(Revised 29.05.2025)
(Available online 10.06.2025)