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PROPERTIES OF DIMENSIONLESS TRANSFORMATION FUNCTIONS OF A COMPOSITE CROSS SECTION

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We consider a cross section of a composite beam in which concrete and various kinds of steel coact. The concrete is treated as a linear viscoelastic material with the property of aging, and the steels as linear elastic materials.

The stress-strain relationship for the concrete is given by the integro-differential equation:

$$\sigma_b(t, \tau^0) = E_b^0 \int_{\tau^0}^{t} K^*(t, \tau) \frac{d\varepsilon(\tau, \tau^0)}{d\tau} d\tau, \tag{1}$$

which by means of a suitable linear integral operator[1] can be written in the form:

$$\sigma_b = E_b^0 \widehat{K}^* \overset{\circ}{\varepsilon}{}^{2)}. \tag{2}$$

The strain ε can be intrerpreted as the input of the transformation K^*, and σ_b as the output function.

Equation (2) can be transformed into a Volterra integral equation of the second kind, with a unique solution which can be represented as an inetgro-differential equation:

$$\varepsilon = \frac{1}{E_b^0} \widehat{R}^* \overset{\circ}{\sigma}_b. \tag{3}$$

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[1] The linear integral operator used here is associated with a twoargument function and is defined as

$$\hat{G} f = \int_{\tau^\circ}^{t} G(t, \tau) f(\tau, \tau^\circ) d\tau \qquad \text{for any } f(\tau, \tau^\circ).$$

[2] $\dot{\varepsilon} = \frac{d}{d\tau} \varepsilon(\tau, \tau^\circ)$.

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The functions $K^* = K^*(t, \tau)$ and $R^* = R^*(t, \tau)$:

$$(4) \qquad K^* = k\,1^* - \Psi^*, \qquad R^* = \frac{1}{k}\,1^* + \Phi^*, \qquad 1^* = H(t-\tau^\circ) = \begin{cases} 1 & \text{for } t > \tau^\circ \\ 0 & \text{for } t \leqslant \tau^\circ \end{cases}$$

with which the operators $\widehat{K}^*$ and $\widehat{R}^*$ are associated, express the properties of the concrete: the variation of the elasticity modulus is described by the dimensionless function:

$$(5) \qquad k = k(t) = \frac{E_b(t)}{E_b^0} = 1 + q(t), \qquad k(t=\tau^\circ) = 1, \qquad q(t) \geqslant 0, \qquad \text{for} \quad t \geqslant \tau^\circ.$$

The viscoelasticity and aging of the concrete are expressed by the function $\Psi^* = \Psi^*(t, \tau)$ $(\tau^\circ \leqslant \tau \leqslant t)$, the relaxation function of the concrete, or by the creep function $\Phi^* = \Phi^*(t, \tau)$ $(\tau^\circ \leqslant \tau \leqslant t)$.

K^* is the dimensionless transformation function for the concrete which, multiplied by the dimensional factor E_b^0, transforms a constant strain into the corresponding stress. R^* is the dimensionless transformation function for the concrete which, multiplied by the dimensional factor $1/E_b^0$, transforms a constant stress into the corresponding strain.

For an arbitrary composite cross section, a basic system of equations can be set up [1] expressing the relationship between the strain parameters (normal strain of the beam axis η, and change in curvature of the beam axis $\varkappa$) and the cross sectional forces (axial force N and bending moment M). This is a system of two inhomogeneous integral equations which, when expressed in terms of linear integral operators, can be formally solved like algebraic equations, thanks to the commutative property of the operators [2].

The solution of these equations yields the dimensionless functions $K_h^* = K_h^*(t, \tau)$ and $R_h^* = R_h^*(t, \tau)$:

$$(6) \qquad K_h^* = k_h\,1^* - \beta_h \Psi^*, \qquad R_h^* = \frac{1}{k_h}\,1^* + \beta_h \Psi_h^*, \qquad h = 1, 2,$$

where $\Psi_h^* = \Psi_h^*(t, \tau) \cdot (\tau^\circ \leqslant \tau \leqslant t)$ $(h = 1, 2)$, and:

$$(7) \qquad k_h = k_h(t) = 1 + \beta_h q(t), \qquad k_h(t=\tau^\circ) = 1, \qquad h = 1, 2.$$

The dimensionless quantities β_h $(h = 1, 2)$ which in the general case satisfy the inequality

$$(8) \qquad 0 < \beta_2 < \beta_1 < 1,$$

are the eigenvalues of the matrix of reduced geometrical characteristics of the cross section considered [2].

The functions K_h^* and R_h^* are mutually related. If $U_h(t, \tau^\circ)$ denotes the input, and $I_h(t, \tau^\circ)$ the output function, so that

$$I_h = \lambda_h \widehat{K_h^*} \mathring{U}_h, \qquad h = 1, 2, \tag{9}$$

where the λ_h are dimensional constants, then the unique solution of equation (9) can be written in the from:

$$U_h = \frac{1}{\lambda_h} \widehat{R_h^*} \mathring{I}_h, \qquad h = 1,2. \tag{10}$$

Equations (9) and (10) show the relationship which exists between K_h^* and R_h^* $(h = 1,2)$.

Dimensionless functions K^*, R^*, K_h^* and R_h^* are identically zero for $t < \tau^\circ$.

By analogy between integro-differential equations (2), (3) and (9), (10) it can be concluded that K_h^* and R_h^* describe the properties of some hypothetical homogeneous material associated with the beam cross section. This material exhibits the property of instantaneous elasticity, whose time dependence is described by the functions k_h (7), and the properties of viscoelasticity and aging, described by the functions $\beta_h \Psi^*$ and $\beta_h \Psi_h^*$ $(h = 1, 2)$ respectively. The properties of this hypothetical homogeneous material depend on the properties of all those materials which go to make up the cross section, and on the cross section geometry.

By analogy with K^* and R^* for the concrete, the functions K_h^* and R_h^* $(h = 1, 2)$ are termed the dimensionless transformation functions for the composite cross section.

Let us compare the physical properties of this hypothetical homogeneous material associated with a cross section with those of concrete.

The operator matrix of the basic system of equations is $\widehat{\mathcal{K}}^* = \|\widehat{K_{h1}^*}\|_{(2,2)}$. Given constant strain parameters, the forces in a cross section of the beam are expressed in terms of the elements of the function matrix $\mathcal{K}^* = \|K_{h1}^*\|_{(2,2)}$, whose eigenvalues are dimensionless transformation functions K_1^* and K_2^* of the composite cross section.

The operator matrix of the solution of the basic system of equations is $\widehat{\mathcal{R}}^* = \|\widehat{R_{h1}^*}\|_{(2,2)}$. Given constant forces in the cross section, the strain parameters in the cross section are expressed in terms of the elements of the function matrix $\mathcal{R}^* = \|R_{h1}^*\|_{(2,2)}$ whose eigenvalues are dimensionless transformation functions R_1^* and R_2^* of the composite cross section.

The following relations can be set up:

$$K_h^* = \alpha_h 1^* + \beta_h K^*, \quad \alpha_h = 1 - \beta_h, \qquad h = 1. 2. \tag{11}$$

They show that the functions K_1^* and K_1^* have the same properties as the dimensionless transformation function for concrete K^*.

It can be shown, for arbitrary cross section geometry, that for $\tau=\text{const.}$ $(\tau^{\circ}\leqslant\tau\leqslant t)$ the following statements are true:

a) the family of curves K_1^*, K_2^* and K^* intersect with the function $1^*(t, \tau^{\circ})$ in a point whose abscissa is $t=t^{\bar{\tau}}$. In particular, for $\tau=\tau^{\circ}=\text{const.}$ the curves of K_1^*, K_2^*, K^* and $1^*(t, \tau^{\circ})$ have a common point of departure with abscissa $t=\tau^{\circ}$. The magnitude of the argument $t^{\bar{\tau}}$ depends on the physical parameters of the concrete and the value chosen for τ°;

b) K_2^* is always the larger eigenvalue of the matrix $\mathcal{K}^*$, except in the interval $\tau^{\circ}<\tau\leqslant t<t^{\bar{\tau}}$, where K_1^* is greater;

c) only in this interval is the dimensionless transformation function for concrete K^* greater than the eigenvalues of the matrix $\mathcal{K}^*$ (Fig. 1).

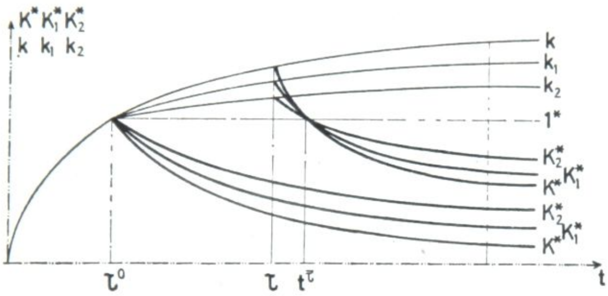

Fig. 1

Since

$$\widehat{\dot{K}_h}\, R_h^* = 1^*, \qquad \widehat{\dot{R}_h}\, K_h^* = 1^*,^{3)} \qquad h=1,\ 2, \quad \text{a) b)} \tag{12}$$

analogous to the relations:

$$\widehat{\dot{K}}\, R^* = 1^*, \qquad \widehat{\dot{R}}\, K^* = 1^*, \qquad \text{a) b)} \tag{13}$$

for the dimensionless transformation functions K^* and R^* of the concrete, and since the $K_h^*\,(h=1, 2)$ have the same properties as K^*, it follows at once that the R_h^* have the same properties as R^*.

Making use of the ralationships betwen the dimensionless transformation functions for concrete and for the composite cross section, it can be shown that for $\tau=\text{const.}$ $(\tau^{\circ}\leqslant\tau\leqslant t)$:

a) if the functions $R_h^*\,(h=1, 2)$ and R^* intersect with $1^*(t, \tau^{\circ})$ at times $t=t^{\tau_h}$ and $t=t^{\tau}$ respectively, $R_h^*\,(h=1, 2)$ with R^* at $t=t^{\tau 0h}$ and R_1^* intersects with R_2^* at

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[3] If $\Gamma^*=g\,1^*+G^*$, $\Gamma^*=\Gamma^*(t, \tau)$, $g=g(t)$, $G^*=G^*(t, \tau)$ then $\hat{\Gamma}=g\,\hat{1}^*+\hat{\dot{G}}$, where $\dot{G}=-\dfrac{\partial}{\partial\tau}G^*(t, \tau)$, and $\hat{1}^*$ is the unit operator.

time $t=t^{\tau_{12}}$, then for arbitrary beam cross section geometry the following inequalities hold:

(14) $$\tau^\circ<\tau<t^{\tau_2}<t^{\tau_1}<t^{\tau}\gtreqless t^{\tau_{12}}<t^{\tau_{02}}<t^{\tau_{01}}.$$

In the case $\tau=\tau^\circ=$ const. all the curves depart from the same point $(\tau^\circ, 1)$ (Fig. 2). The abscissas of these points of intersection depend on the physical proper-

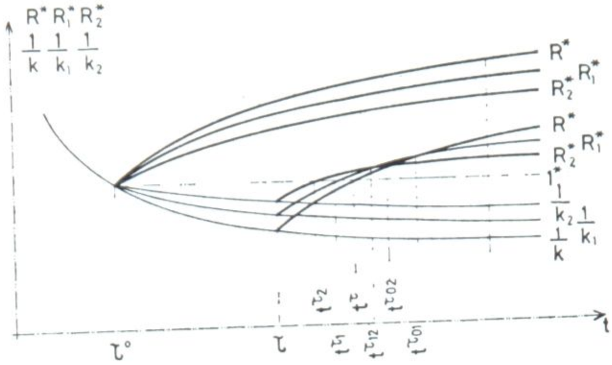

Fig. 2

ties of the materials, the magnitude of τ°, and the cross section geometry, with the exception of t^{τ}, which depends only on the physical properties of the concrete and the value chosen for τ°;

b) R_1^* is always the larger of the eigenvalues of matrix $\mathcal{R}^*$, except in the interval $\tau^\circ<\tau\leqslant t<t^{\tau_{12}}$, in which R_2^* is the larger;

c) the dimensionless transformation function R^* is smaller than the eigenva- of lues $\mathcal{R}^*$ only in the interval $\tau^\circ<\tau\leqslant t<t^{\tau_{01}}$.

The occurrence of intervals in which:

K^* is greater than the eigenvalues of $\mathcal{K}^*$,

R^* is smaller than the eigenvalues of $\mathcal{R}^*$,

K_1^* and R_2^* are the greater eigenvalues of the $\mathcal{K}^*$ and $\mathcal{R}^*$,

is due to the fact that the instantaneous elasticity of the hypothetical homogeneous material associated with the beam cross section is time dependent, this being in fact a consequence of the time dependence of the modulus of elasticity of the concrete. The duration of these intervals depend, as has already been noted, on the choice of τ°, the age of the concrete at the time when the first influence begins to act on the composite beam. The older the concrete, the shorter these intetvals, as is evident from Figs 1 and 2. However, regardless of the value chosen for τ°, the increment of the concrete's modulus of elasticity is less than the decrement of stress in the concrete at constant strain, so that for sufficiently large t these intervals become small relative to the interval $t-\tau$.

If it is assumed that for $t>\tau^\circ$ the concrete's modulus of elasticity is constant, then we have:

(15) $$q(t)\equiv 0,\ k(t)\equiv 1,\ k_h(t)\equiv 1,\quad \text{for } t\geqslant\tau^\circ,\ h=1, 2.$$

Then for any t and $\tau=\text{const.}$ $(\tau^\circ \leqslant \tau < t < \infty)$, K_2^* is the larger eigenvalue of $\mathcal{K}^*$, and K^* is smaller than either eigenvalue of this matrix:

(16) $$K^* \leqslant K_1^* \leqslant K_2^* \leqslant 1^*.$$

Further, R_1^* is the larger eigenvalue of $\mathcal{R}^*$, and R^* is larger than either eigenvalue of this matrix:

(17) $$R^* \geqslant R_1^* \geqslant R_2^* \geqslant 1^*$$

(Fig. 3). The equalities in relationships (16) and (17) correspond to values of the argument $t=\tau$, for constant τ. Families of constant τ curves then have a common point of departure with ordinate 1.

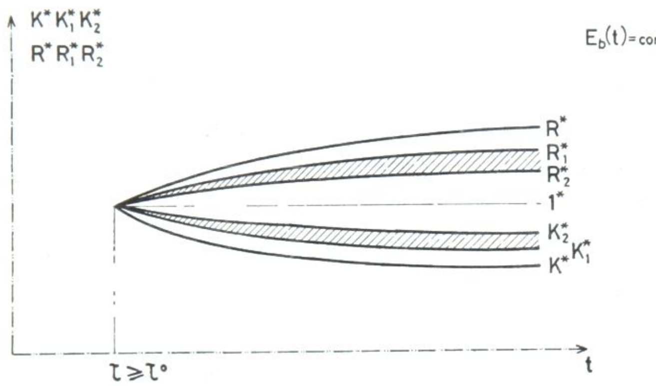

Fig. 3

A similar analysis can be carried out for the dimensionless transformation functions K_h^* and R_h^* of the composite cross section when $t=\text{const.}$, leading to conclusions analogous to those given here for the case of $\tau=\text{const.}$

The foregoing analysis, thus, has shown the following:

a) the dimensionless transformation functions of the composite cross section K_1^* and K_2^*, the eigenvalues of the function matrix $\mathcal{K}^*$ which transforms constant strain into cross sectional forces, have the same properties as the function K^* which transforms constant strain in the concrete into stress;

b) the dimensionless transformation functions of the composite cross section R_1^* and R_2^*, eigenvalues of the function matrix $\mathcal{R}^*$ which transforms constant cross sectional forces into strain parameters, have the same properties as the function R^* which performs this transformation for the concrete alone;

c) the properties of the dimensionless transformation functions of the composite cross section indicate that the instantaneous elasticity and viscoelasticity of the hypothetical homogeneous material associated with the composite cross section are less pronounced than the corresponding properties of the concrete, since K_h^* and R_h^* $(h=1, 2)$ depart less from the function 1^* (t, τ°) than do K^* and R^* (Figs. 1, 2). Since Hooke's law can be expressed in the form $\sigma=E\,1^*\,\varepsilon$, the function 1^* describes the time behavior of an ideally elastic material: the above property of the K_h^* and R_h^* is thus a consequence of the presence of steel in the cross section.

The aging property of the hypothetical homogeneous material is also less pronounced than that of the concrete. This can be immediately seen by considering the difference:

$$[K^*(t,\tau_2)-K^*(t,\tau_1)]-[K^*_h(s,t,\tau_2)-K^*_h(s,t,\tau_1)]=$$

(18) $$=[\Psi^*(t,\tau_1)-\Psi^*(t,\tau_2)][1-\beta_h(s)]>0, \qquad h=1,2,$$

for all $t>\tau_2$, where $\tau^\circ<\tau_1<\tau_2$ and s is the coordinate of the sross section considered.

Let us examine the influence of cross section geometry on the properties of the hypothetical homogeneous material associated with that cross section. Consider the dimensionless transformation functions K^*_{11}, K^*_{22}, R^*_{11}, and R^*_{22}, elements of the principal diagonals of $\mathcal{K}^*$ and $\mathcal{R}^*$, which for $\tau=\text{const.}$ and $\tau^\circ\leqslant\tau\leqslant t<\infty$, lie within the ranges bounded by eigenvalues K^*_h and R^*_h $(h=1,2)$ (Figs. 1, 2). To simplify the discussion, we assume that for $t\geqslant\tau^\circ$ the modulus of elasticity of the concrete is invariant. Then for $\tau=\text{const.}$, K^*_{11}, K^*_{22} and R^*_{11}, R^*_{22} lie within the ranges shown in Fig. 3. These ranges correspond to the general case of arbitrary cross section geometry.

Let us now consider two limiting special cases.

Sections with $I_b=0$ [3] i.e. with small thickness of the concrete slab relative to the overall thickness of the beam. In this case we have:

(19) $$\beta_2=0, \qquad K^*_2\equiv 1^*, \qquad R^*_2\equiv 1^*,$$

so that the ranges within which K^*_{11}, K^*_{22}, and R^*_{11}, R^*_{22} must lie when $\tau=\text{const.}$, are bounded by the functions K^*_1, 1^* and 1^*, R^*_1, respectively (Fig. 4).

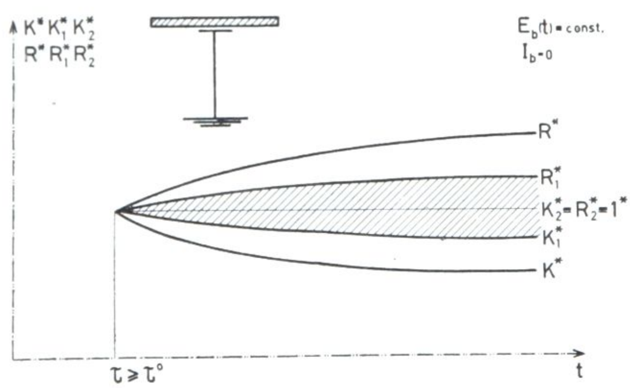

Fig. 4

Sections with $I_a=0$ [3] i.e. prestressed without a steel girder, where the moment of inertia of the steel parts with respect to their own centroidal axis is neglible. For this case it can be shown that:

(20) $$\beta_1=1, \qquad K^*_1\equiv K^*, \qquad R^*_1\equiv R^*.$$

The ranges within which the dimensionless functions K_{11}^*, K_{22} and R_{11}^*, R_{22}^* lie for τ=const., are bounded by the functions K^*, K_2^* and R^*, R_2^*, respectively (Fig. 5).

These special cases eloquently illustrate the influence of cross sestion geometry on the properties of the associated hypothetical homogeneous material: in sections with I_b =0 the properties of the concrete have a minimal influence, so that the ranges of the dimensionless transformation functions of the composite section lie adjacent to the function 1^*, which represents Hooke's law (Fig. 4);

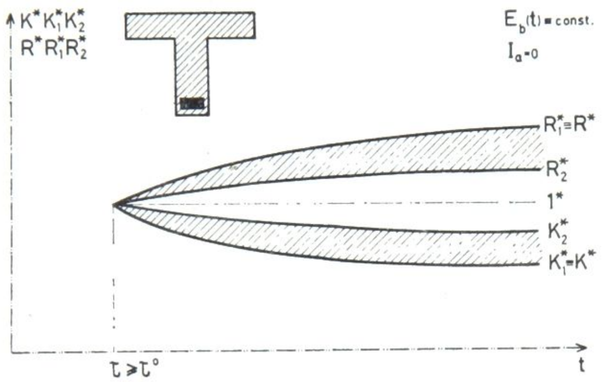

Fig. 5

in sections with $I_a=0$, the influence of the concrete's properties is maximal, and the ranges of the dimensionless transformation functions of the composite cıoss section lie adjacent to the functions for the concrete, R^* and K^* (Fig. 5). In the general case when neither steel nor concrete dominates in the cross section, these ranges lie somewhere between the functions describing Hooke's law and the properties of the concrete (Fig. 3).

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SUR LES FONCTIONS DE TRANSFORMATION DE LA SECTION MIXTE

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Résumé

A l'occasion de la détermination des contraintes et des déplacements de la poutre mixte où le couplage entre le béton, comme le materiau viscoélastique linéaire vieillissant et plusieur types de l'acier parfaitement élastique a lieu, on arrive aux fonction $K_h^*(t, \tau)$ et $R_h^*(t, \tau)$ $(h=1, 2)$, c'est-à-dire aux fonctions de transformation de la section mixte, relatives à la section arbitraire de la poutre.

Les propriétés de ces fonctions ont été étudiées dans le présent article, et sur la base d'elles on a conclu que le matériau homogène, dont les propriétés dépendent de celles des matériaux couplés dans la section et des caractéristiques géométrigues de la section même, peut être associé à la section mixte. Ce matériau a, comme le béton, le modul d'élasticité variable avec le temps, et les propriétés viscoélastiques et vieillissantes, mais ces propriétés sont moin prononcées comparées aux propriétés relatives du béton, ce qui résulte de la présence des parts d'acier de la section.

OSOBINE BEZDIMENZIONALNIH FUNKCIJA TRANSFORMACIJE SPREGNUTOG PRESEKA

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Izvod

Prilikom određivanja napona i pomeranja spregnutog linijskog nosača, u kome sadejstvuje beton kao linearan visokoelastičan materijal sa osobinom starenja i više vrsta čelika, koji su linearni elastični materijali, dolazi se do funkcija $K_h^*(t, \tau)$ i $R_h^*(t, \tau)$ $(h=1, 2)$, tzv. bezdinemzionalnih funkcija transformacije spregnutog preseka, koje se odnose na proizvoljan presek nosača. U radu su ispitane osobine ovih funkcija, na osnovu čega je zaključeno, da se spregnutom preseku može pridružiti homogeni materijal, čije osobine zavise od osobina svih materijala koji sadejstvuju u preseku i od geometrijskih karakterisika toga preseka. Ovaj materijal ima, kao beton, promenljiv modul elastišnosti i osobine viskoelastičnosti i starenja, ali su sve ove osobine manje izražene u poređenju sa odgovarajućim osobinama betona, što je posledica prisustva čeličnih delova preseka.

U opštem slučaju, bezdimenzionalne funkcije transformacije spregnutog preseka leže u oblastima koje su ograničene funkcijama K^* i R^*, koje opisuju osobine betona, i funkcijom 1^*, koja opisuje osobine čelika. U specijalnom slučaju preseka, gde je izrazito prisustvo čeličnih delova, ove funkcije nalaze se neposredno uz funkciju 1^*, koja predstavlja Hukov zakon; u presecima gde je izrazito prisustvo betonskog dela, one se nalaze neposredno uz funkcije K^* i R^*, koje opisuju osobine betona.

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