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FIRST STRAIN GRADIENT THEORY OF GRANULAR MATERIALS

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1. Introduction

In the present work the first strain gradient theory of granular material is studied. The effects of variation of the solid volume fraction together tith higher strain gradient terms are considered in the present theory. Based on the thermodynamic consideration a set of constitutive equations are derived and the basic equations of motion are obtained and discussed. The example of gravity flow of granular materials, on inclned surface is also investigated. It is observed that the considerations of higher order strain gradient terms modifies the motion and porosity field to an extent.

2. Governing Equatons

Following Goodman and Covin [3, 4] we characterize the granular media by the solid volume distribution funcation ν which is defined as one minus the porosity (void volume) function. If ρ_0 is the granules mass density, then

(2.1) $$\rho = \rho_0 \nu \qquad 0 \leqslant \nu \leqslant 1,$$

where ρ is interpreted as the qulk density of the distributed solid body. The distributed solid body must satisfy the laws of motion of a continuum. Accordingly, the folowing field equations must be satsified for motion of a microstructured granular continuum.

Conservation of Mass

(2.2) $$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underset{\sim}{v}) = 0$$

If ρ_0 be constant, that is the granular material being incompressible, equation (2.2) becomes

(2.3) $$\frac{\partial \nu}{\partial t} + \nabla \cdot (\nu \underset{\sim}{v}) = 0$$

Balance of Linear Momentum

(2.4) $$\tau_{jk,j}+\rho b_k=\rho \dot{v}_k$$

Balance of Angular Momentum

(2.5) $$\mu_{ij,i}+e_{ijk}\tau_{ki}+\rho C_j=0$$

Balance of equilibrated force

(2.6) $$h_{i,i}+\rho l+\hat{g}=\rho K\ddot{\nu}$$

Conservation of equilibrated inertia

(2.7) $$\frac{dK}{dt}-2\dot{\nu}K=0$$

(2.8) $$\rho\dot{e}=\bar{\tau}_{jk}d_{jk}+\mu^D_{ij}P_{ij}+\mu_{ijk}P_{ijk}+h_i\dot{\nu},_i-\hat{g}\dot{\nu}+q_{i,i}+\rho h$$

Entropy Inequality (Clausius-Duhem)

(2.9) $$\rho\dot{\eta}-(q_k/\theta),_k-\rho h/\theta\geqslant 0$$

Throughout this paper theregular cartesian tensor notation is employ ed with superposed dot indicating the material time derivative and indices following a comma denotinng partial differentiations. In equations (2.2) — (2.9) $v_k=\dot{u}_k$ is the velocity vector, u_k is the displacement vector, τ_{jk} is the stress tensor, b_k is the body force per unit mass, μ_{ij} is the couple stress tensor, c_j is the body couple per unit mass, e_{ijk} is the alternating tensor, h_i is the equilibrated stress vector, l is equilibrated force per unit mass, $\hat{g}$ is the internal equilibrated force, K is the equibrated inertia, e is the internal energy density per unit mas, h is the internal heat source per unit mass, θ is the absolute temperature, q_k is the heat fux vector pointinf outward, η, is the entropy density per unit mass, μ^D_{ij} is the deviatoric part of the couple stress tensor, μ_{ijk} is the symmetrized double stress tensor. The stress tensor $\bar{\tau}_{jk}$ is defined by

(2.10) $$\bar{\tau}_{jk}=\tau_{(jk)}+\mu_{ijk,i}$$

where $\tau_{(jk)}$ is the syymmetric part of τ_{jk}. The deformation rate tensor d_{ij} and the microdeformation rate tensors P_{ij} and P_{ijk} are defined by

(2.11) $$d_{ij}=\frac{1}{2}(v_{i,j}+v_{j,i})=d_{ji}$$

(2.12) $$P_{ij}=\frac{1}{2}e_{jlk}v_{kli} \qquad (P_{ii}=0)$$

(2.13) $$P_{ijk}=\frac{1}{3}(v_{k,ij}+v_{i,jk}+v_{j,ki})=P_{jki}=P_{kij}=P_{kji}$$

the equations of balance of linear and angular momentus (2.4) and (2.5) could be combined. Evaluating the antisymmetric part of the stress tensor from equation (2.5) and substitnting the result into equation (2.4), it follows that

$$\tau_{(jk,j)} - \frac{1}{2}\overset{D}{\mu}_{il,ij}\, e_{jkl} + \rho\, b_k - \frac{1}{2}\, e_{jkl}\,(\rho\, C_1),_j - \rho\,\dot{v}_k \tag{2.14}$$

introducing the Helmholtz free energy ψ

$$\psi = e - \eta\theta \tag{2.15}$$

and eliminating ρh between equations (2.8) and (2.9) we find an alternative from of Clausius-Duhem inequality

$$-\rho\,(\dot{\psi} + \theta\dot{\eta}) + \bar{\tau}_{jk}\, d_{jk} + \overset{D}{\mu}_{ij} P_{ij} + \mu_{ijk} P_{ijk} + h_i \dot{\nu},_i + \hat{g}\,\dot{\nu} + \frac{1}{\theta}\, q_k\, \theta,_k \geqslant 0 \tag{2.16}$$

3. Constitutive Bquations

The following set of costitutive equations is now proposed:

$$\psi = \psi\ \rho,\ \theta,\ \nu,\ \nu,_i\ d_{ij},\ P_{ij},\ P_{ijk}),\quad \bar{\tau}_{ij} = \bar{\tau}_{ij}(\dots\dots\dots),\quad \overset{D}{\mu}_{ij} = \overset{D}{\mu}_{ij}(\dots\dots\dots),$$

$$\mu_{ijk} = \mu_{ijk}(\dots\dots\dots),\quad h_i = h_i(\dots\dots\dots),\quad \hat{g} = \hat{g}(\dots\dots\dots),$$

$$q_i = q_i(\dots\dots\dots),\quad \eta = \eta(\dots\dots\dots), \tag{3.1}$$

where the principle of equipresence is employed and the dependent consittutive variables are assumed to be functions of the same set of constitutive independent variables.

Evaluating $\dot{\psi}$ and employing the result in the entropy inequality (2.16), we find

$$-\rho\left(\frac{\partial\psi}{\partial\theta} + \eta\right)\dot{\theta} + \left[\bar{\tau}_{jk} + \pi\delta_{jk} + \rho\frac{\partial\psi}{\partial\nu,_j}\nu,_k\right] d_{jk}$$

$$+\rho\frac{\partial\psi}{\partial\nu,_k}\nu,_j\, \nu_{[j,k]} + \left(h_k - \rho\frac{\partial\psi}{\partial\nu,_k}\right)\dot{\nu},_k - \left(\hat{g} + \rho\frac{\partial\psi}{\partial\nu}\right)\dot{\nu}$$

$$+\overset{D}{\mu}_{ij} P_{ij} + \mu_{ijk} P_{ijk} + \frac{1}{\theta}\, q_k\,\theta,_k - \rho\frac{\partial\psi}{\partial d_{ij}}\dot{d}_{ij} - \rho\frac{\partial\psi}{\partial P_{ij}}\dot{P}_{ij}$$

$$-\rho\frac{\partial\psi}{\partial P_{ijk}}\dot{P}_{ijk} \geqslant \tag{3.2}$$

where $v_{[j,k]}$ is the antisymmetric part of the velocity gradient tensor and the thermodynamic pressure π is defined by

$$\pi = -\frac{\partial \psi}{\partial \rho^{-1}} \tag{3.3}$$

in the derivation of inequality (3.2) the identity

$$\frac{d}{dt} \nu_{,k} = \dot{\nu}_{k} - \nu_{,j} v_{j,k} \tag{3.4}$$

has been employed.

From the requirement that the entropy inequality (3.2) must hold for all independent variations of $\dot{\theta}$, d_{ij}, $v_{[j,k]}$, $\dot{\nu}_{,k}$, $\dot{\nu}$, P_{ij}, P_{ijk}, $\theta_{,k}$, $\dot{d}_{ij}$, $\dot{P}_{ij}$, $\dot{P}_{ijk}$, it follows that $\rho \dfrac{\partial \psi}{\partial \nu_{,k}} \nu_{,j}$ is a symetric tensor and

$$\eta = -\frac{\partial \psi}{\partial \theta} \tag{3,5}$$

$$\frac{\partial \psi}{\partial d_{ij}} = \frac{\partial \psi}{\partial P_{ij}} = \frac{\partial \psi}{\partial P_{ijk}} = 0 \tag{3.6}$$

and (3.2) reduces to

$$_{D}\bar{\tau}_{jk} d_{jk} + {}_{D}h_{k} \nu_{,k} + {}_{D}\overset{D}{\mu}_{ij} P_{ij} + {}_{D}\mu_{ijk} P_{ijk} + \frac{1}{\theta} q_{k} \theta_{,k} \geqslant 0 \tag{3.7}$$

and

$$-\left(\hat{g} + \rho \frac{\partial \psi}{\partial \nu}\right) \dot{\nu} \geqslant 0 \tag{3.8}$$

where the dissipative and elastic parts of stress tensor. couple stress tensor, double stress tensor and equilibrated stress tensor are defind respectively by

$$_{D}\bar{\tau}_{jk} = \bar{\tau}_{jk} - {}_{E}\bar{\tau}_{jk} \tag{3.9}$$

$$_{D}\overset{D}{\mu}_{ij} = \overset{D}{\mu}_{ij} - {}_{E}\mu_{ij} \tag{3.10}$$

$$_{D}\mu_{ijk} = \mu_{ijk} - {}_{E}\mu_{ijk} \tag{3.11}$$

$$_{D}h_{k} = h_{k} - {}_{E}h_{k} \tag{3.12}$$

and

$$_{E}\bar{\tau}_{jk} = -\pi\delta_{jk} - \rho \frac{\partial \psi}{\partial \nu_{,j}} \nu_{,k} \tag{3.13}$$

$$_{E}\overset{D}{\mu}_{ij} = {}_{E}\mu_{ijk} = 0 \tag{3.14}$$

$$_{E}h_{k} = \rho \frac{\partial \psi}{\partial \nu_{,k}} \tag{3.15}$$

For inequality (3.8) to be satisfied for all independent variation of $\dot{\nu}$, it follows that

$$\hat{g} = -\rho \frac{\partial \psi}{\partial \nu} \tag{3.16}$$

For an isotropic medium, the following set of linear constitutive equations for the dissipative part of the stresses and heat flux vector may be considered

$$_D\tau_{kl} = \lambda v_{i,l}\, \delta_{kl} + 2\mu\, d_{kl} \tag{3.17}$$

$$_D\mu^D_{pq} = \mu^D_{pq} = 4\, d_1 P_{pq} + 4\, d_2 P_{qp} + f e_{pqk} P_{kii} \tag{3.18}$$

$$_D\mu_{pqr} = \mu_{pqr} = a_1 (P_{iir}\, \delta_{pq} + P_{iip}\, \delta_{qr} + P_{iiq}\, \delta_{rp}) + 2\, a_2$$

$$P_{pqr} + \frac{1}{3} f P_{ij} (\delta_{pq}\, e_{ijr} + \delta_{qr}\, e_{ijp} + \delta_{rp}\, e_{ijp}) \tag{3.19}$$

$$_D h_k = \eta \dot{\nu}_{,k} \tag{3.20}$$

$$q_k = K T_{,k} \tag{3.21}$$

where λ, η, η, d_1, d_2, f, a_1 and a_2 are coefficients of viscosity. the coefficient of viscosity and heat conductivity are restricted by the entropy inequality (3.7). Accoreingly [1],

$$\mu \geqslant 0, \qquad 3\lambda + 2\mu > 0, \qquad -d_1 < d_2 < d_1$$

$$a_2 \geqslant 0 \qquad 5\, a_1 + 2\, a_2 \geqslant 0 \qquad 5 f^2 < 6\,(d_1 - d_2)\; 5\, a_1 + 2\, a_2)$$

$$n \geqslant 0 \qquad k \geqslant 0 \tag{3.22}$$

Considering (3.1) and (3.6), the free energy fynction is assummed to be given by the following expression

$$\rho\psi = \rho\psi_0 + \alpha(\nu)\, \nu_{,k}\, \nu_{,k} + a_0\, \nu^2 \tag{3.23}$$

the requirement ofpositive definitness of the free energy function inplies that

$$\alpha(\nu) \geqslant 0, \qquad a_0 \geqslant 0 \tag{3.24}$$

Employing the free energy function as given by (3.23), the elastic part of the stresses become

$$_E\bar{\tau}_{jk} = -\pi\delta_{jk} - 2\,\alpha(\nu)\, \nu_{,k}\, \nu_{,1} \tag{3.25}$$

$$_E h_k = 2\,\alpha(\nu)\, \nu_k \tag{3.26}$$

From equation (3.16), the expression for $\hat{g}$ is found to be

$$g = -a_0 \nu - \nu \frac{d}{d\nu}\left(\frac{\alpha}{\nu}\right) \nu_{,k} \nu_{,k} \tag{3.27}$$

Employing constitutive equations (3.17) — (3.19) together with equation (3.25) into equation (2.14) it follows that

$$(\lambda + 2\mu)(1 - l_1^2 \nabla^2) \nabla\nabla \cdot \underset{\sim}{v} - \mu (1 - l_2^2 \nabla^2) \nabla \times \nabla \times \underset{\sim}{v} + \rho \underset{\sim}{b}$$

$$+ \frac{1}{2} \nabla \times (\rho C) - \nabla \pi - 2\pi \cdot (\alpha \nabla \nu \nabla \nu) = \rho \dot{\underset{\sim}{v}} \tag{3.28}$$

The use of constitutive equations (3.26), (3.27) and (3.20) into equation (2.6) yields

$$\nabla \cdot (2\alpha \nabla \nu + \eta \nabla \dot{\nu}) - \nu \frac{d}{d\nu}\left(\frac{\alpha}{\nu}\right) \nabla \nu \cdot \nabla \nu - a_0 \nu + \rho l = \rho k \ddot{\nu} \tag{3.29}$$

where l_1^2 and l_2^2 are defined as follows

$$l_1^2 = (3a_1 + 2a)/(\lambda + 2\mu) \geqslant 0 \tag{3.30}$$

$$l_2^2 = (3d_1 + a_1 + 2a_2 - \overline{f})/3\mu \geqslant 0 \tag{3.31}$$

Equation (3.28) and (3.29) together with equation (2.2) form five equations for the five unknowns $\underset{\sim}{v}$, ν and ρ.

Incompressible Grains

When the grains are incompressible equation (2.3) implies that

$$\dot{\nu} = -v_{i,i} = -\nu d_{ii} \tag{3.32}$$

For such cases, employing (3.32) the terms involving $\dot{\nu}$ in inequality (3.2) may be combined with the coefficients of d_{jk}. Thus, the elastic part of the stress tensor becomes

$${}_E\tau_{jk} = (-\hat{\pi} - \hat{g}\nu)\delta_{jk} - \rho \frac{\partial \psi}{\partial \nu_{,j}} \nu_{,k} \tag{3.33}$$

with the thermodynamic pressure being given by

$$\hat{\pi} = \rho \nu \frac{\partial \psi}{\partial \nu} \tag{3.24}$$

and $\hat{g}$ remains unrestricted.

For this particular case emplying (3.23), we find the expression for thermodynamic pressure

$$\hat{\pi} = a_0 \nu^2 + \nu^2 \frac{d\partial}{d} \left(\frac{\alpha}{\nu}\right) \nu,_k \nu,_k \tag{3.35}$$

Furthemore, assumiug that the equilibrated inertia K, equilibrated force l as well as the dissipative part of the equilibrated stress vector are zero, from equation of balance of equilibrated forces (2.6), it follows that

$$\hat{g} = -h_{i,i} = -2(\alpha \nu,_i),_i \tag{3.36}$$

and the elastic part of the stress tensor becomes

$$_E\bar{\tau}_{jk} = \left[-a\nu^2 - \nu^2 \frac{d}{d\nu}\left(\frac{\alpha}{\nu}\right) \nu,_i \nu,_i + 2\alpha\nu\nu,_{ii} \right] \delta_{jk} - 2\alpha\nu,_j \nu,_k \tag{3.37}$$

Equation (3.37) gives the final form of the constitutive equation for the elastic part of the stress tensor for granular materials with incompressible grains. In particular, if $\nu(\alpha)$ is taken to be a constant, then it follows

$$_E\bar{\tau}_{jk} = [-a_0 \nu^2 + \alpha\nu,_1 \nu,_i + 2\alpha\nu\nu,_{ii}]\, \delta_{jk} - 2\alpha\nu,_j \nu,_k \tag{3.38}$$

Employing constitutive equations (3.17) — (3.19) together with equation (3.38) into equation (2.14) yields

$$(\lambda + 2\mu)(1 - l_1^2 \nabla^2) \nabla\nabla \cdot \underset{\sim}{v} - \mu (1 - l_2^2 \nabla^2) \nabla \times \nabla \times \underset{\sim}{v} + \rho_0 \nu \underset{\sim}{b}$$

$$+ \frac{1}{2} \nabla \times (\rho_0 \underset{\sim}{v} C) - 2a_0 \nu \nabla \nu + 2\alpha\nu\nabla(\nabla^2 \nu) = \rho_0 \nu \bar{v} \tag{3.39}$$

Equations (3.39) and (2.2) form four equations governing the variations of four unknowns, v_i and ν.

For the case in which the couple strees tensor μ_{ij} and the double stress tensor μ_{ijk} are both assumed to be zero and in the absence of body couple the relations (3.38) together with (3.17) for the stress tensor and the equation of moticn (3.39) reduce to those of Coulomb granular materials with constaut ρ_0 [4].

4. Inclined Gravity-Flow Problem

We conclude our presented by solving an example. The problem of steady fully-developed flow of a granular material down an inclined plate was studied by Goodman and Cowin [5]. We employ the present first gradient theory to solve the same problem. The results are then compared and the effects of newly devoloped couple and double stresses are discussed.

Following [5], we consider an infinte slab of incompressible granular material of thicknees l inclined at an angle ξ to the gravity field and having a stress free uper surface while supported below by a flat plate. A cartesian coordinate system fixed to the upper surfaceis employed with x_1 axis oriented down the surface and the x_2 axis normal to the surface. It is assumed that the granular material is characterized by the following assumptions in addition to the constitutive equations (3.38) and (3.17).

(1) steady state, fylly developed flow

(2) $v_1 = v_1(x_2)$, $\quad v_2 = v_3 = 0$

(3) $\nu = \nu(x_1, x_2)$

from above assumptions, the motion is accelerationless and by employing continuty equation (2.3) we conclude that $\nu = \nu(x_2)$, from which it follows that the stress $\bar{\tau}_{jk}$ is a function of the x_2 direction only. Furthemore, employing (2.12) and (2.13), under assumption (2) above, yields the following nonvanishing components of microdeformation rate tensors P_{ij} and P_{ijk}.

$$P_{23} = -\frac{1}{2} v_{1,22} \tag{4.1}$$

$$P_{123} = P_{212} = P_{221} = \frac{1}{3} v_{1,22} \tag{4.2}$$

Employing (4.1) and (4.2) into equations (3.18) and (3.19) yields the nonvanishing components of couple and double stress tensor as follows

$$\mu^{D}_{23} = \left(\frac{1}{3} f - 2 d_1\right) v_{1,22} \tag{4.3}$$

$$\mu^{D}_{32} = -\left(2 d_2 + \frac{1}{3} f\right) v_{1,22} \tag{4.4}$$

$$\mu_{111} = (a_1 - \frac{1}{2} f) v_{1,22} \tag{4.5}$$

$$\mu_{221} = \mu_{212} = \mu_{122} = \left(\frac{1}{3} a_1 + \frac{2}{3} a_2 - \frac{1}{6} f\right) v_{1,22} \tag{4.6}$$

$$\mu_{331} = \mu_{313} = \mu_{133} = \left(\frac{1}{3} a_1 - \frac{1}{6} f\right) v_{1,22} \tag{4.7}$$

since the motion is acccelerationless and $\nabla \cdot v = 0$ by assumption (2) above, the governing equations (2.3) and (3.39) become

$$\dot{\nu} = 0 \tag{4.8}$$

$$2 a_0 \nu\nu_{,2} - 2 \alpha\nu\nu_{,222} = \rho_0 \nu g \sin \xi \tag{4.9}$$

$$-\mu (v_{1,22} - l_2^2 v_{1,2222}) = \rho_0 \nu g \cos \xi \tag{4.10}$$

Equation (4.8) simply states that the volume distribution at a material point does not change as one follows the motion of that point. From (4.9) and (4.10) it is seen that the basic fields uncouple in the sense that (4.9) involves only the volume distribution.

The boundary conditions foı this problem are specified at the boundary surface $x_2=0$ and along the supporting plate $x_2=l$, at $x_2=0$

(4.11) $$\bar{\tau}_{12}=\bar{\tau}_{22}=0$$

(4.12) $$h_2=0$$

(4.13) $$\overset{D}{\mu}_{23}=\overset{D}{\mu}_{32}=0$$

at $x_2=l$

(4.14) $$v_1=v_0$$

(4.15) $$\nu=\nu_0$$

(4.16) $$\overset{D}{\mu}_{23}=\overset{D}{\mu}_{32}=0$$

where v_0 is the slip velocity at the plate.

Eliminating the trivial solution $\nu=0$, then (4.9) yields the general solution

(4.17) $$\nu=A\sin h\,Ly+B\cos h\,Ly+\frac{1}{2}My\sin\xi+C$$

where A, B, and C are arbitrary constants of integration, L, M and Y are the non-dimensional quantities defined by [5].

(4.18) $$L=(a_0/\alpha)^{\frac{1}{2}}\,l$$

(4.19) $$M=\rho_0\,gl/a_0$$

(4.20) $$y=\frac{x_2}{l}$$

Employing (4.17) in (4.10), the velocity field is then obtained in the following form

$$v_1=A_1\sin h\frac{l}{l_2}y+B_1\cos h\frac{l}{l_2}y+AK\Lambda\, l^2/L^2\sin h\,Ly$$

$$+BK\Lambda\frac{l^2}{L^2}\cos h\,Ly-\frac{M}{12}Kl^2\sin\xi\,y^3-\frac{C}{2}Kl^2\,y^2$$

$$-\left(\frac{M}{2}Kl_2^2\sin\xi+DKl^2\right)y-(CKl_2^2+EKl^2)$$

where A_1, B_1, D and E are additional integration constants and K and Λ are defined by

$$\frac{\rho_0 g \cos \xi}{\mu} = K \tag{4.22}$$

$$\frac{1}{(l_2^2/l^2)\, L^2 - 1} = \Lambda \tag{4.23}$$

Evaluating the seven constants of integration from the boundary conditions (4.11) — (4.16) and introducing the non-dimensional parameter

$$M' = M(\sin \xi)/2\nu_0 = \rho_0 g\, l (\sin \xi)/2 a_0 \nu_0 \tag{4.24}$$

Then (4.17) becomes

$$\nu/\nu_0 = -M' \frac{\sin h\, Ly}{L} + \left(1 - M' + M' \frac{\sin L}{L}\right)\left(\frac{1 + \cos h\, Ly}{1 + \cos h\, L}\right) + M'y \tag{4.25}$$

and (4.21) after some manipulations, finally reads

$$\begin{aligned} v_1 - v_0 = {} & \frac{\rho_0 g\, \nu_0 l^2 \cos \xi}{2\mu}\Bigg[2\frac{M'\Lambda}{L^3}(\sin h\, L - \sin Ly) \\ & + \left(1 - M' + M' \frac{\sin hL}{L}\right)\left(\frac{1 - y^2}{1 + \cos h\, L}\right) - 2\frac{\Lambda}{L^2}\Bigg(1 - M' \\ & + M' \frac{\sin h\, L}{L}\Bigg)\left(\frac{\cos h\, L - \cos h\, Ly}{1 + \cos h\, L}\right) + \frac{M'}{3}(1 - y^3) \\ & + 2M' \frac{\Lambda}{L^2}(y - 1)\Bigg] - \frac{\rho_0 g\, \nu_0 l l_2 \cos \xi}{\mu \sin h\, l/l_2}\Bigg[\Bigg(1 - M' \\ & + M' \frac{\sin h\, L}{L}\Bigg)\left(\frac{1 - \Lambda \cos h\, L + (\Lambda - 1)\cos h\, l/l_2}{1 + \cos h\, L}\right) \\ & + M'\left(\frac{\Lambda}{L}\sin h\, L + 1\right)\Bigg](y - 1) + \frac{\rho_0 g\, \nu_0 l_2^2 \cos \xi}{\mu \sin h\, l/l_2}\Bigg\{\Bigg[\Bigg(1 \\ & - M' + M' \frac{\sin h\, L}{L}\Bigg)\left(\frac{1 - \Lambda \cos h\, L + (\Lambda - 1)\cos h\, l/l_2}{1 + \cos h\, L}\right) \\ & + M'\left(1 + \frac{\Lambda}{L}\sin h\, L\right)\Bigg](\sin h\, l/l_2 y - \sin h l/l_2) \\ & + \left(1 - M' + M' \frac{\sin h\, L}{L}\right)\left[\frac{(1 - \Lambda)\sin h l/l_2}{1 + \cos h\, L}\right](\cos h\, l/l_2 y \\ & - \cos h\, l/l_2)\Bigg\} \end{aligned} \tag{4.26}$$

It is easily seen that expressions (4.25) and (4.26) reduces to those when the effects of couple and double stresses are neglected. The expressions (4.25) and (4.26) showsthe stong dependence of solutions on the dimensionless length ratio L and the dimensionless constraining number M'.

Following [5], we now consider the limiting cases in which the length ratio L adproaches zero and infinity, respectively. These two limiting cases form the bounding solutions and, hence, delimit the influence of volume distribution on the material behaviour. Forthe limiting case described by $L=0$, it can be shown that (4.24) and (4.26) become

$$\nu/\nu_0 = 1 \tag{4.27}$$

$$v_1 - v_0 = \frac{\rho_0 g \nu_0 l^2 \cos\xi}{2\mu}\left[\frac{M'}{3}(1-y^3)(1+\Lambda) + \frac{1-y^2}{2}(1+\Lambda) + \frac{l_2^2}{l^2}(1-\Lambda)(\cos h\, l/l_2\, y - \cos h\, l/l_2)\right]$$
$$+ \frac{\rho_0 g \nu_0 l_2 \cos\xi}{\mu \sin l/l_2}[l_2(\sin h\, l/l_2\, y - \sin h\, l/l_2)$$
$$-l(y-1)]\left[M'(1+\Lambda) + \frac{1-\Lambda+(\Lambda-1)\cos h\, l/l_2}{2}\right] \tag{4.28}$$

the upper limiting case represented by L tending to infinity is characterized by

$$\frac{\nu}{\nu_0} = \begin{cases} M' y & 0 \leqslant y < 1 \\ 1 & y = 1 \end{cases} \tag{4.29}$$

and

$$v_1 - v_0 = \frac{\rho_0 g \nu_0 l^2 \cos\xi}{\mu}\left[\frac{M'}{6}(1-y^3) + \frac{l^2 M'}{l \sin h\, l/l_2}(1-y) + \frac{l_2^2 M'}{l^2 \sin h\, l/l_2}(\sin h\, l/l_2 y - \sin h l/l_2)\right] \tag{4.30}$$

these representations indicates that as the length ratio increases, the influence of volume distribution becomes more pronounced. Expression (4.29) shows that for $M' \neq 1$ there exists a discontinuity in the solution at the supporting plate ($y=1$) which is a boundary effect.

Profiles for the volume distribution, dimensionless velocity and the dimensionless mass flux defined by

$$\bar{\nu}=\frac{\nu}{\nu_0};\ \bar{v}_1=\frac{2\,\mu(v_1-v_0)}{\rho_0\,\nu_0\,g\,l^2\cos\xi};$$

(4.31)
$$\overline{\nu v_1}=\frac{2\,\mu\,(v_1-v_0)\,\nu}{\rho_0\,\nu_0^2\,g\,l^2\cos\xi}$$

are shown in figures 1 to 6.

In figure 1 the profile of the solid volume fraction for M' = 1.25 and various L, are shown. The expression for solid volume fraction as given by equation (4.25) is independent of l_2 (i.e independenţ of Λ). For $l_2=0$, the velocity profiles are shown in figuxe 2 for several values of L and for $M'=1.25$. The results shown in figures 1 and 2 are identical with those obtained by Goodman and Cowin [2]. For $L=2$ and various ralues of l_2/l, the velocity profiles of the steady flow of granular materials are shown in figures 3 and 4 for $M'=0.75$ and 1.25, respectively. It is observed that the consideration of first order stain gradient effects modifies the velocity the velocity profiles and in general the velocity decreases with an increase in l_2/l. Furthermore, a relatively flat section appears in the middle of the velocity profile for $0.1>l_2/l>0.05$. The profiles of dimensionless mass fluxed for $L=2$ and $0.1>l_2/l>0.05$ for $M'=0.75$ and 1.25 are shown in figures 5 and 6. Similar flat sections are also observed in those fitures.

Further remakrs. The problem of mechanics of granular materils is investigated in the line of the continuum theory of structured media. The effects of variation of solid volume fraction and first strain gradients are inclued in the theory. The general equations for static and dynamics of granular materials are obtaineg and the example of gravity flow is also considered.

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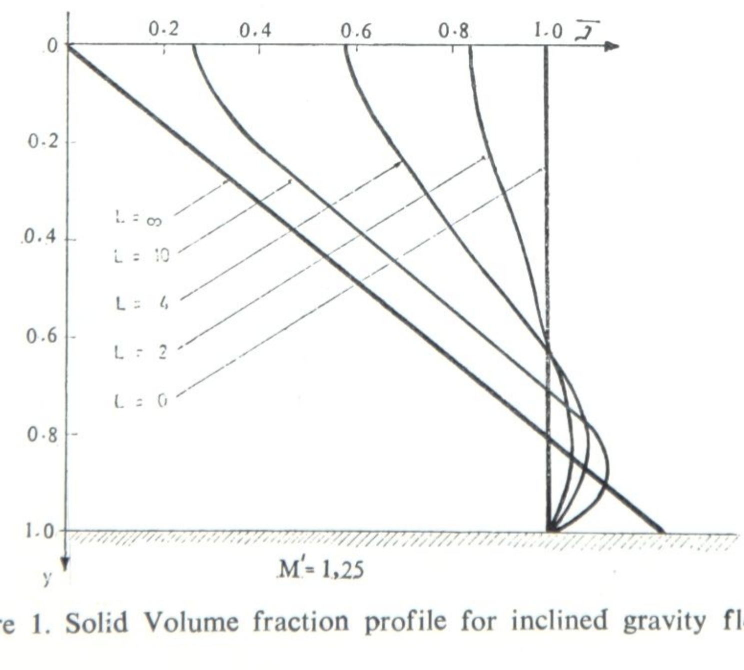


Figure 1. Solid Volume fraction profile for inclined gravity flow.

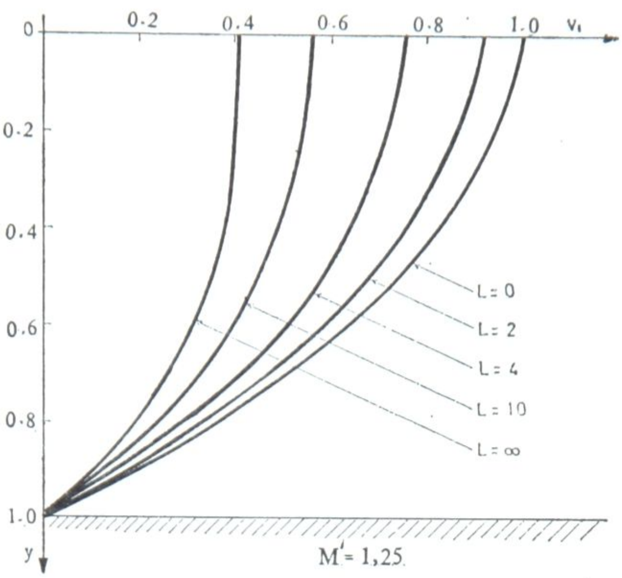


Figure 2. Velocity profile for inclined gravity flow and for $l_2 = 0$.

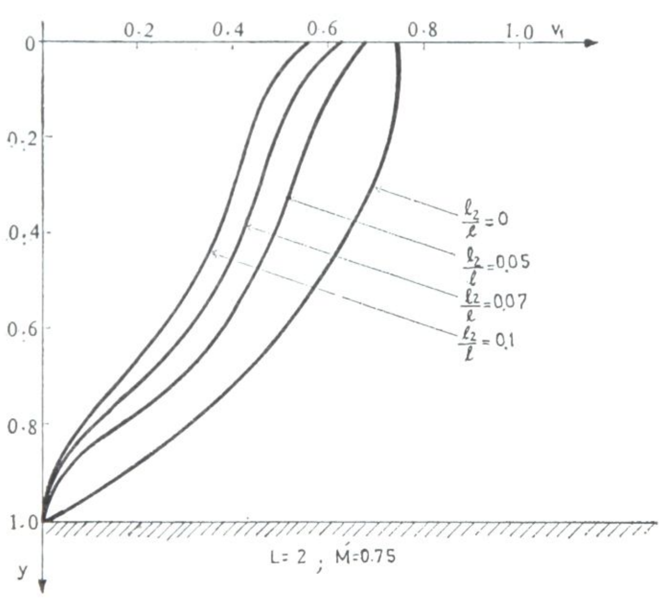


Figure 3. Velocity profile for $L = 2$ $M' = 0.75$ and various values of $\frac{l_2}{l}$.

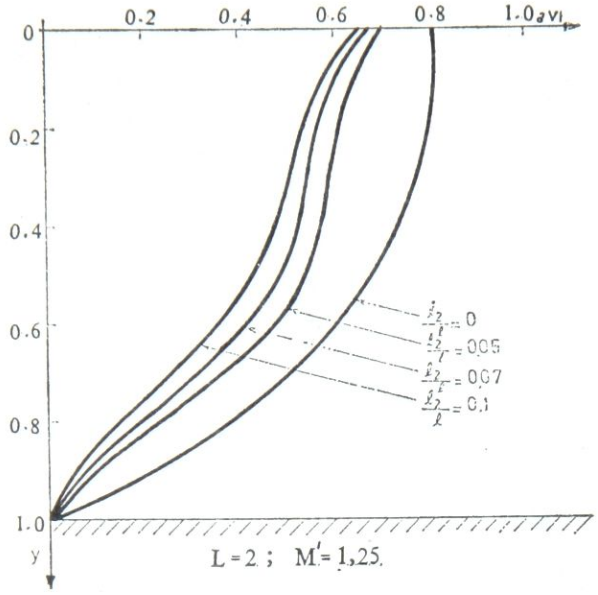
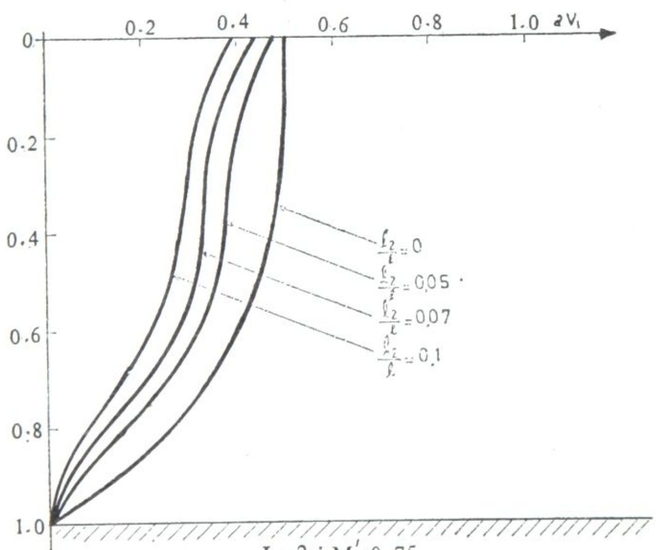


Figure 4 Velocity profile for $L=2$, $M'=1.25$ and valious values of $\frac{l_2}{l}$.



Figurə 5. Mass flux profile for $L=2$, $M'=075$ and various of $\frac{l_2}{l}$.

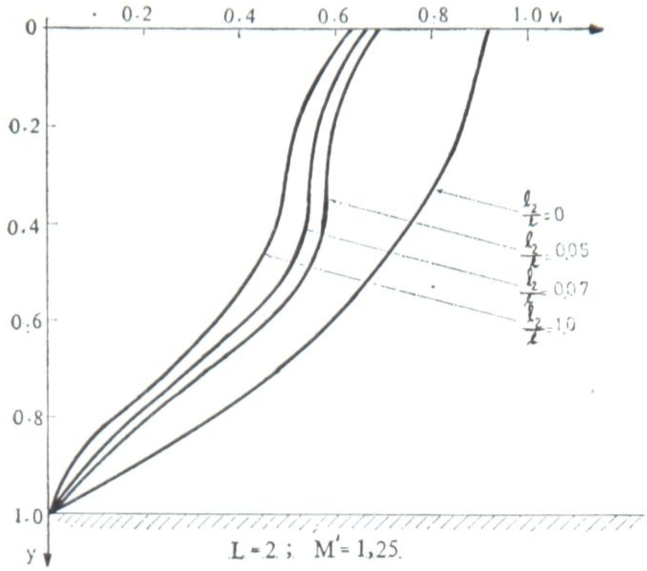


Figure 6. Mass flux profile for $L = 2$, $M' = 125$ and various values of $\frac{l_2}{l}$.

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ТЕОРИЯ ГРАДИЕНТА ДЕФОРМАЦИИ ПЕРВОГО ПОРЯДКА ГРАНУЛИРОВАННЫХ МАТЕРИАЛОВ

K. Firoozbakhsn and *G. Shmadi*

Резюме

Формулируется теория градиента деформации первого порядка в механике гранулираванных материалов. В эту теорию включены эффекты перемены объема трердой части тела и градиента деформации первого порядка. На основании термодинамических рассмотрений выводятея Составлющие уравнения и получаются и обсуждаются основные уравнения движения. Исследуется проблема потока гранулированных материалов тод действием силы тячотения и рассматривахтся эффекты градиента деформации выстего порядка.

TEORIJA GRADIJEOTA DEFORMACIJE PRVOG REDA ZA GRANULARNE MATERIJALE

K. Firoozbakhsn and *G. Ahmadi*

Izvod

U radu je formulisana teorija gradijenta deformacije prvog rada u mehanici granularnih materijala. U ovu teoriju uključeni su efekti promene zapzapremine čvrstog dela posmatranog tela kao i gradijenti deformacije prvog reda. Na osnovu temodinamičkih rasmatranja izvedene su konstitutivne jednačine a dobijene su i analizirane i osnovne jednačine kretanja. Istraživan je problem tečenja granularnih materijala pod dejstvom sile teže kao i uticaj gradijenta deformacije višeg reda.