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CONSTITUTIVE EQUATIONS OF HETEROGENEOUS MICROPOLAR RODS

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1. Introduction

The application of micromorphic theory of mixtures to one-dimensional bodies-rods is presented in paper [1]. The balance laws of the mass, tensor microinertia, momentum and energy, as well as entropic inequalities of α-th constituent of the mixture and the sole mixture are derived.

This paper represents a continuation of paper [1], since using paper one's results constitutive equations of amixture are derived for the case of a rod.

Since the constitutive theory for classical mixture is not sufficiently developed, because the basic difficulty is the choice of variables which figure in constitutive assumption, we adopted the procedure for the three dimensional body as used by R. J. Twiss and A. C. Eringen [2], adapting it to the rod case (one-dimensional body). In absence of chemical reactions as in paper [3], we derived constitutive equations. In this paper we also ignored interaction betwen constituents, believing that their omission can be justified in the case of a special model, i.e, a one-dimensional body for which this omission can be done. Thus, during constitutive equation derivation the part relating to the interaction was equalized with zero. Even so, the constitutive theory derived here is more general in character than the existing theories, since it assumes that the mixture is made up of any number of constituents inslead of only two.

Since in paper [1] balance laws of α-th constituent of the mixture and the sole mixture are derived in detail, here we shall state only those details that we shall use for the derivation of constitutive equations:

mass balance law of the α-th constituent of the mixture

$$(1.1) \qquad \dot{\rho}_{(\alpha)} + \rho_{(\alpha)} \frac{\partial \dot{\varphi}_{(\alpha)}}{\partial \mathscr{S}} = \rho \hat{\beta}_{(\alpha)};$$

microinertia tensor balance law of the α-th constituent of the mixture

$$(1.2) \qquad \rho_{(\alpha)} \overline{\dot{i}_{(\alpha)}^{l_1 \cdots l_p}} - \sum_s \rho_{(\alpha)} \nu_{(\alpha)k}^{l_s} \dot{i}_{(\alpha)}^{l_1 \cdots k \cdots l_p} = \rho \left(\hat{\beta}_{(\alpha)}^{l_1 \cdots l_p} - \hat{\beta}_{(\alpha)} \dot{i}_{(\alpha)}^{l_1 \cdots l_p} \right);$$

balance law for movement quantity of the α-th constituent of the mixture

$$\frac{\partial \tilde{t}_{(\alpha)}}{\partial \mathcal{S}} + \rho_{(\alpha)} (\underset{\sim}{f}_{(\alpha)} - \underset{\sim}{\dot{v}}_{(\alpha)}) = \rho \hat{\beta}_{(\alpha)} \underset{\sim}{v}_{(\alpha)}; \tag{1.3}$$

balance law of the momentum of the α-th constituent of the mixture

$$\frac{\partial \tilde{t}^{l}_{(\alpha)}}{\partial \mathcal{S}} + \lambda^l \underset{\sim}{t}_{(\alpha)} - \underset{\sim}{\bar{t}}^{l}_{(\alpha)} + \rho_{(\alpha)} (\underset{\sim}{f}^{l}_{(\alpha)} - \overset{l}{i}_{(\alpha)} \underset{\sim}{\dot{v}}_{(\alpha)}) - \rho_{(\alpha)} \overset{.kl}{i}_{(\alpha)} (\underset{\sim}{\dot{\nu}}_{(\alpha)k} + \underset{\sim}{\nu}_{(\alpha)r} \overset{r}{\nu}_{(\alpha)k}) = \tag{1.4}$$

$$= \rho (\hat{\beta}^{l}_{(\alpha)} \underset{\sim}{v}_{(\alpha)} + \hat{\beta}^{kl}_{(\alpha)} \underset{\sim}{\nu}_{(\alpha)k})$$

and energy balance law for the mixture as a whole

$$-\rho \dot{\varepsilon} + \underset{\sim}{t} \frac{\partial \tilde{v}}{\partial \mathcal{S}} + \underset{\sim}{t}^{r} \frac{\partial \tilde{\nu}_r}{\partial \varphi} + (\underset{\sim}{\bar{t}}^{r} - \lambda^r \underset{\sim}{t}^{r}) \underset{\sim}{\nu}_r + \frac{\partial q}{\partial \mathcal{S}} + \rho h = 0. \tag{1.5}$$

2. Micropolar theory of rod

Micropolar theory is a special case of micromorphic theory, as micropolar materials are a special class of micromorphic materials, who indicate microrotacion efects, which means that microdisplacement is made up of rotation.

In this case the gyratio tensor is asymetric, i.e.

$$\nu_{(\alpha)kl} = -\nu_{(\alpha)lk}. \tag{2.1}$$

The vector

$$\nu_{(\alpha)m} = \frac{1}{2} \varepsilon_{mkr} \overset{kr}{\nu}_{(\alpha)}, \qquad \overset{kr}{\nu}_{(\alpha)} = -\varepsilon^{krm} \nu_{(\alpha)m} \tag{2.2}$$

determines the rotation speed of material point of the micropolar body.

If we assume that

$$\overset{.k}{i}_{(\alpha)} = 0 \tag{2.3}$$

which means that the centre of the macroelement mass lies at the same point, thus, we have that (see [1] eq. (4.12))

$$\underset{\sim}{\dot{\sigma}}^{l}_{(\alpha)} = \rho_{(\alpha)} \overset{.kl}{i}_{(\alpha)} (\underset{\sim}{\dot{\nu}}_{(\alpha)k} + \underset{\sim}{\nu}_{(\alpha)k} \overset{r}{\nu}_{(\alpha)k}) + \rho (\hat{\beta}^{l}_{(\alpha)} \underset{\sim}{v}_{(\alpha)} + \hat{\beta}^{kl}_{(\alpha)} \underset{\sim}{\nu}_{(\alpha)r}), \tag{2.4}$$

or in component from

$$\dot{\sigma}^{lm}_{(\alpha)} = \rho_{(\alpha)} \overline{\overset{.kl}{i}_{(\alpha)} \overset{m}{\nu}_{(\alpha)k}}^{\cdot} - \rho_{(\alpha)} \overset{.rk}{i}_{(\alpha)} \overset{l}{\nu}_{(\alpha)k} \overset{m}{\nu}_{(\alpha)r} + \rho (\hat{\beta}^{l}_{(\alpha)} \overset{k}{v}_{(\alpha)} + \hat{\beta}^{kl}_{(\alpha)} \nu_{(\alpha)rk}), \tag{2.5}$$

From this, it follows that

(2.6) $$\varepsilon_{lmk}\dot{\sigma}^{lm}_{(\alpha)}=\rho_{(\alpha)}\overline{(i^{p}_{(\alpha)p}\,q_{kl}-i_{(\alpha)kl})\,\nu^{l}_{(\alpha)}}^{\cdot}+\rho\,\varepsilon_{lmk}(\hat{\beta}^{l}_{(\alpha)}v^{k}_{(\alpha)}+\hat{\beta}^{kl}_{(\alpha)}\nu_{(\alpha)rk}),$$

where ε_{lmk} is an alternation tensor, and where we used (2.2).

If we take that

(2.7) $$j_{(\alpha)kl}=i^{p}_{(\alpha)p}\,g_{kl}-i_{(\alpha)kl}$$

we shall have

(2.8) $$j_{(\alpha)kl}\,\nu^{l}_{(\alpha)}=\sigma_{(\alpha)k},$$

i.e.

(2.9) $$\rho_{(\alpha)}\dot{\sigma}_{(\alpha)k}=\rho_{(\alpha)}\overline{j_{(\alpha)kl}\,\nu^{l}_{(\alpha)}}^{\cdot}=\rho_{(\alpha)}(i^{p}_{(\alpha)p}\,\nu_{(\alpha)k}-i^{p}_{(\alpha)k}\,\nu_{(\alpha)p})=$$
$$=\varepsilon_{lmk}\dot{\sigma}^{lm}_{(\alpha)}-\rho\,\varepsilon_{lmk}(\hat{\beta}^{l}_{(\alpha)}v^{k}_{(\alpha)}+\hat{\beta}^{kl}_{(\alpha)}\nu_{(\alpha)rk}).$$

Henceforward we shall use the following relationship

(2.10) $$\frac{\partial \underset{\sim}{t}^{l}_{(\alpha)}}{\partial\mathscr{S}}=t^{l}_{(\alpha),k}\frac{\partial x^{k}_{(\alpha)}}{\partial\mathscr{S}}=(t^{lm}_{(\alpha)}\,\underset{\sim}{g}_{m})_{,k}\frac{\partial x^{k}_{(\alpha)}}{\partial\mathscr{S}}=$$
$$=t^{lm}_{(\alpha),k}\,\underset{\sim}{g}_{m}\frac{\partial x^{k}_{(\alpha)}}{\partial\mathscr{S}}=\frac{\delta t^{lm}_{(\alpha)}}{\delta\mathscr{S}}\underset{\sim}{g}_{m}.$$

If in the micromorphic rod theory of the first degree [1] we use (2.3), we shall obtain the following micropolar theory equations:

mass conservation of the α-th constituent of the mixture

(2.11) $$\dot{\rho}_{(\alpha)}+\rho_{(\alpha)}\frac{\partial\dot{\varphi}_{(\alpha)}}{\partial\mathscr{S}}=\rho\hat{\beta}_{(\alpha)}:$$

tensor's conservation of microinertia of the α-the constituent of the mixture

(2.11) $$\rho_{(\alpha)}(\dot{i}^{kl}_{(\alpha)}-\nu^{k}_{(\alpha)r}\,i^{rl}_{(\alpha)}-\nu^{l}_{(\alpha)r}\,i^{kr}_{(\alpha)})=\rho\,(\hat{\beta}^{kr}_{(\alpha)}-\hat{\beta}_{(\alpha)}\,i^{kr}_{(\alpha)});$$

aalance of the momentum of the α-th constituent of the mixture

(2.13) $$\frac{\partial \underset{\sim}{t}_{(\alpha)}}{\partial\mathscr{S}}+\rho_{(\alpha)}(\underset{\sim}{f}_{(\alpha)}-\dot{\underset{\sim}{v}}_{(\alpha)})=\rho\hat{\beta}_{(\alpha)}\underset{\sim}{v}_{(\alpha)};$$

energy balance of the α-th constituent of the mixture

(2.14) $$\rho_{(\alpha)}\dot{\varepsilon}_{(\alpha)}-\underset{\sim}{t}_{(\alpha)}\left(\frac{\partial\underset{\sim}{v}_{(\alpha)}}{\partial\mathscr{S}}-\underset{\sim}{\lambda}\times\underset{\sim}{\nu}\right)-\underset{\sim}{t}^{r}_{(\alpha)}\frac{\partial\underset{\sim}{\nu}_{(\alpha)r}}{\partial\mathscr{S}}-\frac{\partial q_{(\alpha)}}{\partial\mathscr{S}}-\rho_{(\alpha)}h_{(\alpha)}=$$
$$=\rho\hat{\beta}_{(\alpha)}\left(\frac{1}{2}\underset{\sim}{v}_{(\alpha)}\underset{\sim}{v}_{(\alpha)}-\dot{\varepsilon}_{(\alpha)}\right)+\frac{1}{2}\rho\hat{\beta}^{rs}_{(\alpha)}\underset{\sim}{\nu}_{(\alpha)r}\underset{\sim}{\nu}_{(\alpha)s},$$

where

$$\lambda^r \underset{\sim}{\nu}_{(\alpha)r} = \lambda_r \nu^{kr}_{(\alpha)} \underset{\sim}{g}_k = -\underset{\sim}{\lambda} \times \underset{\sim}{\nu}_{(\alpha)}$$

$$\underset{\sim}{\bar{t}}^r_{(\alpha)} \underset{\sim}{\nu}_{(\alpha)r} = \bar{t}^{rm}_{(\alpha)} \nu_{(\alpha)rm} = 0,$$

consedering (2.1), (2.2) and

$$(2.15) \qquad t'^{kl}_{(\alpha)} = t'^{lk}_{(\alpha)} \Rightarrow \bar{t}^{kl}_{(\alpha)} dv = \int_{dv} t'^{kl}_{(\alpha)} dv' = \bar{t}^{lk}_{(\alpha)}.$$

Conservation of momentum on obtains that (1.4), and using (2.10) in the component form

$$(2.16) \qquad \frac{\delta t^{lm}_{(\alpha)}}{\delta \mathscr{S}} + \lambda^l t^m_{(\alpha)} - \bar{t}^{lm}_{(\alpha)} + \rho_{(\alpha)} f^{lm}_{(\alpha)} - \grave{\sigma}^{lm}_{(\alpha)} = 0$$

where

$$(2.17) \qquad \grave{\sigma}^{lm}_{(\alpha)} = \rho_{(\alpha)} \overline{\overset{\grave{}}{i^{kl}_{(\alpha)} \nu^m_{(\alpha)k}}} - \rho_{(\alpha)} \dot{i}^{rk}_{(\alpha)} \nu^l_{(\alpha)k} \nu^m_{(\alpha)r} + \rho \, (\hat{\beta}^l_{(\alpha)} v^k_{(\alpha)} + \hat{\beta}^{kl}_{(\alpha)} \nu_{(\alpha)rk}).$$

If Equation (2.16) is multiplied by ε_{lmk}, which in effect is equivalent to the operation of determining its symmetric part, and using

$$(2.18) \qquad \mathfrak{M}_{(\alpha)k} = \varepsilon_{lmk} t^{lm}_{(\alpha)}, \qquad l_{(\alpha)k} = \varepsilon_{lmk} f^{lm}_{(\alpha)}$$

and (2.9), we shall obtain the balance of the momentum in the form of

$$(2.19) \qquad \frac{\delta \mathfrak{M}_{(\alpha)k}}{\delta \mathscr{S}} + \varepsilon_{lmk} \lambda^l t^m_{(\alpha)} + \rho_{(\alpha)} l_{(\alpha)k} = \rho_{(\alpha)} \grave{\sigma}_{(\alpha)k} + \rho \, \varepsilon_{lmk} (\hat{\beta}^l_{(\alpha)} v^k_{(\alpha)} + \hat{\beta}^{kl}_{(\alpha)} \nu_{(\alpha)rk}).$$

Considering that part $\underset{\sim}{t}^r_{(\alpha)} \frac{\partial \underset{\sim}{\nu}_{(\alpha)r}}{\partial \varphi}$ in (2.14) we can rewrite it in the following form

$$(2.20) \qquad \begin{aligned} \underset{\sim}{t}^r_{(\alpha)} \frac{\partial \underset{\sim}{\nu}_{(\alpha)r}}{\partial \varphi} &= \underset{\sim}{t}^r_{(\alpha)} \frac{\delta \nu_{(\alpha)rm}}{\delta \mathscr{S}} \underset{\sim}{g}^m = t^{rm}_{(\alpha)} \frac{\delta \nu_{(\alpha)mr}}{\delta \mathscr{S}} = -\varepsilon_{mrk} t^{rm}_{(\alpha)} \frac{\delta \nu^k_{(\alpha)}}{\delta \mathscr{S}} = \\ &= \varepsilon_{rmk} t^{rm}_{(\alpha)} \frac{\delta \nu^k_{(\alpha)}}{\delta \mathscr{S}} = \mathfrak{M}_{(\alpha)k} \frac{\delta \nu^k_{(\alpha)}}{\delta \varphi} = \underset{\sim}{\mathfrak{M}}_{(\alpha)} \frac{\partial \underset{\sim}{\nu}_{(\alpha)}}{\partial \mathscr{S}}, \end{aligned}$$

where (2.2) and (2.18) were used, thus, the new equation for energu balance of the α-th constituent of the mixture becomes

$$(2.21) \qquad \begin{aligned} \rho_{(\alpha)} \grave{\varepsilon}_{(\alpha)} - \underset{\sim}{t}_{(\alpha)} \left(\frac{\partial \underset{\sim}{v}_{(\alpha)}}{\partial \varphi} - \underset{\sim}{\lambda} \times \underset{\sim}{\nu}_{(\alpha)} \right) - \underset{\sim}{\mathfrak{M}}_{(\alpha)} \frac{\partial \underset{\sim}{\nu}_{(\alpha)}}{\partial \mathscr{S}} - \frac{\partial q_{(\alpha)}}{\partial \mathscr{S}} - \rho_{(\alpha)} h_{(\alpha)} &= \\ = \rho \hat{\beta}_{(\alpha)} \left(\frac{1}{2} \underset{\sim}{v}_{(\alpha)} \underset{\sim}{v}_{(\alpha)} - \varepsilon_{(\alpha)} \right) + \frac{1}{2} \rho \hat{\beta}^{rs}_{(\alpha)} \underset{\sim}{\nu}_{(\alpha)r} \underset{\sim}{\nu}_{(\alpha)s}. \end{aligned}$$

3. Constitutive epuations

In order to have a specific system of equations, the number of equations should be the same as the number of variables. In micromorphic balance equations of the α-th constituent the variables are

$$
(3.1)\qquad \begin{cases} \overset{kl}{t}_{(\alpha)}, & \overset{k}{t}_{(\alpha)}, & \overset{k}{q}_{(\alpha)}, & \psi_{(\alpha)}, & \eta_{(\alpha)} \\ x_{(\alpha)k}, & \chi_{(\alpha)kK}, & \theta & & \\ \overset{kl}{i}_{(\alpha)}, & \rho_{(\alpha)} & & & \\ \overset{lm}{\nu}_{(\alpha)}, & \overset{l}{v}_{(\alpha)}, & \varepsilon_{(\alpha)}, & \overset{\cdot}{\mathcal{S}}_{(\alpha)}, & \overset{\cdot l}{v}_{(\alpha)} \\ \overset{l}{f}_{(\alpha)}, & h_{(\alpha)}, & t. & & \end{cases}
$$

It is assumed that variables in $(3.1)_5$ are predefned, whilst in $(3.1)_4$ they are determined on the basis of the remaining variables. Variables in $(3.1)_3$ are given by (1.1) and (1.2) for $p=2$. It is assumed that motion and temperature are independant variables, as they are not dependent on the process. Remaining variables in $(3.1)_1$ are indepenent variables.

There are $(12n+1)$ variables in $(3.1)_2$ where $\alpha=1, 2, \ldots, n$ and more $17n$ variables in $(3.1)_1$, giving a total of $(29n+1)$ variables. Using the balance equation for movement quantity of the α-th constituent (1.3), the balance equation of momentum of the α-th constituent (1.4) for $p=1$, as energy balance equation for mixture (1.5), we shall have $(12n+1)$ equations. If we write constitutive equation for each dependent variable as a function of independent variables, we shall have additional $17n$ equations, i.e. a total of $(29n+1)$ equations for $(29n+1)$ variables, thus defining the equation system.

For micromorphic theory of mixtures the constitutive equation is

$$
(3.2)\qquad \begin{gathered} \Psi_{(\alpha)}(S_{(\alpha)}, t) = \psi_{(\alpha)}\left\{u_{(\alpha-\beta)k}, \frac{\delta x_{(\gamma)k}}{\delta S_{(\gamma)}}, \chi_{(\gamma)kK}, \frac{\delta \chi_{(\gamma)kK}}{\delta S_{(\gamma)}}, \Lambda_{(\alpha)K}, \theta, S_{(\alpha)}\right\} \\ \alpha=1, 2, \ldots, n, \qquad \beta=1, 2, \ldots) \alpha(\ldots, n. \end{gathered}
$$

Objective principle ipmoses limitationss on constitutive equations, requiring constitutive function to have an invariant form within an appropriate group of orthogonal transformations of spacial coordinate system, i.e.

$$
(3.3)\qquad \begin{aligned} &\psi_{(\alpha)}\left\{u_{(\alpha-\beta)k}, \frac{\delta x_{(\gamma)k}}{\delta S_{(\gamma)}}, \chi_{(\gamma)kK}, \frac{\delta \chi_{(\gamma)kK}}{\delta S_{(\gamma)}}, \Lambda_{(\alpha)K}, \theta, S_{(\alpha)}\right\} = \\ &= \psi_{(\alpha)}\left\{Q_{kl}\, u_{(\alpha-\beta)k}, Q_{kl}\frac{\delta x_{(\gamma)k}}{\delta S_{(\gamma)}}, Q_{kl}\,\chi_{(\gamma)kK}, Q_{kl}\frac{\delta \chi_{(\gamma)kK}}{\delta S_{(\gamma)}}, \Lambda_{(\alpha)K}\ \theta, S_{(\alpha)}\right\} \end{aligned}
$$

where $u_{(\alpha-\beta)k}$ is a microchange relative vector given by

$$
(3.4)\qquad u_{(\alpha-\beta)k} = u_{(\alpha)k} - u_{(\beta)k}
$$

where according to definition

$$u_{(\alpha)k} = x_{(\alpha)k} - X_{(\alpha)K}\,\delta_{Kk}$$

(3.5)

$$u_{(\beta)k} = x_{(\beta)k} - X_{(\beta)K}\,\delta_{Kk}.$$

Condition (3.3) for scalar value of the vectors variables function $\Psi_{(\alpha)} b_k^{(r)})$, $r = 1, 2, \ldots, r_0$, $k = 1, 2, 3$, assumes that

$$\Psi_{(\alpha)}(b_k^{(r)}) = \Psi_{(\alpha)}(Q_{kl}\, b_k^{(r)}) = \Psi\,(B^{(pr)}) \tag{3.6}$$

$$p = 1, 2, 3, \qquad r = 1, 2, \ldots, r_0 \geqslant 3$$

where $B^{(pr)}$ has an independent set of all possible scalar multiples derived from vestor $b_k^{(r)}$, i.e.

$$B^{(pr)} \equiv b_k^{(p)}\, b_k^{(r)}, \qquad p = 1, 2, 3, \qquad r = 1, 2, \ldots, r_0 \geqslant 3 \tag{3.7}$$

and since

$$b_k^{(r)} = \sum_p B^{(pr)}\, b^{-1(p)}, \qquad b^{(s)} = \sum_q B^{(qs)}\, b_k^{-1(q)}$$

remaining scalar products shall be

$$b_k^{(r)}\, b_k^{(s)} = \sum_{p,q=1}^{3} B^{(pr)}\, B^{(qs)}\, B^{-1(pq)}, \qquad r, s, = 1, 2, \ldots, r_0. \tag{3.8}$$

Entropic inequality also constrains the constitutive equations. Due to equation system's specific nature, only energy equation for a mixture can be used, thus, we have to use entropic inequality of a mixture in a developed form (see eq. (3.4.60)), i.e.,

$$\sum_\alpha \Big\{ -\rho_{(\alpha)}(\dot{\Psi} + \dot{\theta}\,\eta_{(\alpha)}) + t_{(\alpha)}^{k}\left(\frac{\delta\, v_{(\alpha)k}}{\delta \mathscr{S}} + \nu_{(\alpha)rk}\,\lambda^r\right) + t_{(\alpha)}^{rm}\,\frac{\delta \nu_{(\alpha)mr}}{\delta \mathscr{S}} +$$

$$+ (\bar{t}_{(\alpha)}^{rk} - 2\,\lambda^r\, t_{(\alpha)}^{k})\,\nu_{(\alpha)(kr)} + \frac{q_{(\alpha)}}{\theta}\,\frac{\partial \theta}{\partial \mathscr{S}} + \rho\,\beta_{(\alpha)}\left(\frac{1}{2}\,v_{(\alpha)}^{2} + \Psi_{(\alpha)}\right) + \tag{3.9}$$

$$+ \frac{1}{2}\,\rho\,\hat{\beta}_{(\alpha)}^{rs}\,\nu_{(\alpha)kr}\,\nu_{(\alpha)ks}\Big\} \geqslant 0.$$

During the derivation of equations, we shall consider micropolar mixture case, as a special micromorphic mixture class, excluding the existence of chemical reaction. Then,

$$\nu_{(\alpha)(kr)} = 0, \qquad \hat{\beta}_{(\alpha)} = 0, \qquad \hat{\beta}_{(\alpha)}^{rs} = 0. \tag{3.10}$$

Since the tensor $\chi_{(\alpha)kK}$ is now orthogonal, which means that it has three mutually indeqendent coordinates, the number of variaqles in $(3.1)_2$ being reduced, it now has $(6n+1)$ variables which together with $17n$ variables in $(3.1)_1$ mahe a total of $(23n+1)$ variables. Using balance equations for movement quantity (2.13), the balance of the momentum (2.19) and balance equation for mixtures energy, we shall have $(6n+1)$ equations, which with remaining $17n$ equations gives a total of $(23n+1)$ equations.

Using (3.10), from (3.9) it follows that

$$\sum_{\alpha}\Big\{-\rho_{(\alpha)}(\dot{\psi}_{(\alpha)}+\dot{\theta}\eta_{(\alpha)})+t^{k}_{(\alpha)}\Big(\frac{\delta v_{(\alpha)k}}{\delta\mathcal{S}}+\nu_{(\alpha)rk}\lambda^{r}\Big)+ \tag{3.11}$$
$$+t^{rm}_{(\alpha)}\frac{\delta\nu_{(\alpha)mr}}{\delta\mathcal{S}}+\frac{q_{(\alpha)}}{\theta}\frac{\partial\theta}{\partial\mathcal{S}}\Big\}\geqslant 0,$$

i.e.

$$-\sum_{\alpha}\rho_{(\alpha)}(\dot{\psi}_{(\alpha)}+\dot{\theta}\eta_{(\alpha)})+\sum_{\gamma}\Big\{t^{k}_{(\gamma)}\Big(\frac{\delta v_{(\gamma)k}}{\delta\mathcal{S}}+\nu_{(\gamma)rk}\lambda^{r}\Big)+ \tag{3.12}$$
$$+t^{rm}_{(\gamma)}\frac{\delta\nu_{(\gamma)mr}}{\delta\mathcal{S}}+\frac{q_{(\gamma)}}{\theta}\frac{\partial\theta}{\partial\mathcal{S}}\Big\}\geqslant 0.$$

If in (3.12) we introduce (3.2) and use a rule related differentiation, we shall have

$$-\sum_{\alpha}\rho_{(\alpha)}\Big(\frac{\partial\psi_{(\alpha)}}{\partial\theta}+\eta_{(\alpha)}\Big)\dot{\theta}+\sum_{\alpha}\Big\{\frac{q_{(\alpha)}}{\theta}-\rho_{(\alpha)}\Big(\frac{\partial\psi_{(\alpha)}}{\partial\theta}+\eta_{(\alpha)}\Big)u_{(\alpha)}\Big\}\frac{\partial\theta}{\partial\mathcal{S}}+$$
$$+\sum_{\gamma}\Big(t^{k}_{(\gamma)}-\sum_{\alpha}\rho_{(\alpha)}\frac{\partial\psi_{(\alpha)}}{\partial x_{(\gamma)k,S}}J_{(\gamma)}\Big)\frac{\delta v_{(\gamma)k}}{\delta\mathcal{S}}+$$
$$+\sum_{\gamma}\Big(t^{rk}_{(\alpha)}-\sum_{\alpha}\rho_{(\alpha)}\frac{\partial\psi_{(\alpha)}}{\partial\chi_{(\gamma)kK,S}}\chi^{l}_{(\gamma)\cdot K}J_{(\gamma)}\Big)\frac{\delta\nu_{(\gamma)kl}}{\delta\mathcal{S}}-$$
$$-\sum_{\gamma}\Big[\sum_{\alpha}\rho_{(\alpha)}\Big(\frac{\partial\psi_{(\alpha)}}{\partial\chi_{(\gamma)kK,S}}\chi^{l}_{(\gamma)\cdot K,S}+\frac{\partial\psi}{\partial\chi_{(\gamma)kK}}\chi^{l}_{(\gamma)\cdot K}\Big)-t^{l}_{(\gamma)}\lambda^{k}\Big]\nu_{(\gamma)kl}- \tag{3.13}$$
$$-\sum_{\alpha}\sum_{\gamma}\rho_{(\alpha)}J^{-1}_{(\gamma)}w_{(\alpha-\gamma)}\Big(\frac{\partial\psi_{(\alpha)}}{\partial x_{(\gamma)k,S}}x_{(\gamma)k,S^2}+\frac{\partial\psi_{(\alpha)}}{\partial\chi_{(\gamma)kK,S}}\chi_{(\gamma)kK,S^2}+$$
$$+\frac{\partial\psi_{(\alpha)}}{\partial\chi_{(\gamma)kK}}\chi_{(\gamma)kK,S}+\frac{\partial\psi_{(\alpha)}}{\partial u_{(\alpha-\gamma)k}}\Lambda_{(\alpha)K}\delta_{Kl}\Big)\geqslant 0,$$

$$\sum_{\gamma}\Big[\sum_{\alpha}\rho_{(\alpha)}\Big(\frac{\partial\psi_{(\alpha)}}{\partial\chi_{(\gamma)kK,S}}\chi^{l}_{(\gamma)\cdot K,S}+\frac{\partial\psi_{(\alpha)}}{\partial\chi_{(\gamma)kK}}\chi^{l}_{(\gamma)\cdot K}\Big)+t^{k}_{(\gamma)}\lambda^{l}\Big]_{[kl]}=0,$$

where we used the following expression

$$J_{(\gamma)} \equiv \frac{\partial \mathcal{S}}{\partial S_{(\gamma)}} (S_{(\gamma)}, t)$$

$$J_{(\gamma)}^{-1} = \frac{\partial S_{(\gamma)}}{\partial \mathcal{S}}$$

$$\lambda^k = - J_{(\gamma)}^{-1} \frac{\partial x^k}{\partial S_{(\gamma)}}$$

$$\frac{D^{(\alpha)}}{Dt} \mathcal{S}_{(\gamma)} = \overset{\grave{}}{\overline{\mathcal{S}_{(\alpha)}}} = w_{(\alpha-\gamma)}$$

$$\frac{D^{(\alpha)}}{Dt} x_{(\beta) i} = v_{(\alpha) l} - v_{(\beta) l} = \lambda_l w_{(\alpha-\beta)}$$

$$\dot{\psi}_{(\alpha)} (S_{(\alpha)}, t) = \sum_{i, \gamma} \frac{\partial \psi_{(\alpha)}}{\partial \Phi_{(\gamma, i)}} \left\{ \frac{D^{(\gamma)} \Phi_{(\gamma, i)}}{Dt} + \Phi_{(\gamma, i), S} J_{(\gamma)}^{-1} w_{(\alpha-\gamma)} \right\} \tag{3.14}$$

$$\dot{\theta} = \dot{\theta} + \frac{\partial \theta}{\partial \mathcal{S}} u_{(\alpha)}$$

$$\overset{\grave{}}{\overline{x_{(\gamma) k, S}}} = v_{(\gamma) k, S} + x_{(\gamma) k, S^2} J_{(\gamma)}^{-1} w_{(\alpha-\gamma)}$$

$$\overset{\grave{}}{\overline{\chi_{(\gamma) k K}}} = \nu_{(\gamma) kl} \chi^{l}_{(\gamma) \cdot K} + \chi_{(\gamma) k K, S} J_{(\gamma)}^{-1} w_{(\alpha-\gamma)}$$

$$\overset{\grave{}}{\overline{\chi_{(\gamma) k K, S}}} = \nu_{(\gamma) kl, S} \chi^{l}_{(\gamma) \cdot K} + \nu_{(\gamma) kl} \chi^{l}_{(\gamma) \cdot K, S} + \chi_{(\gamma) k K, S^2} J_{(\gamma)}^{-1} w_{(\alpha-\gamma)}$$

$$\overset{\grave{}}{\overline{u_{(\alpha-\gamma) k}}} = \Lambda_{(\gamma) L} \delta_{Ll} J_{(\gamma)}^{-1} w_{(\alpha-\gamma)}.$$

Since $\dot{\theta}$, $\frac{\partial \theta}{\partial \mathcal{S}}$, $\frac{\delta v_{(\gamma) k}}{\delta \mathcal{S}}$, $\nu_{(\gamma) kl}$, $\frac{\delta \nu_{(\gamma) kl}}{\delta \mathcal{S}}$ and $\chi_{(\gamma) k K, S^2}$ are independent variables, then the multipliers along with variables must disappear, and that gives the necessary and sufficient conditions for constitutive equations of the type (3.2) to be thermodynamically acceptable, i.e.

$$\eta_{(\alpha)} = \frac{-\partial \psi_{(\alpha)}}{\partial \theta}$$

$$\sum_{\alpha} q_{(\alpha)} = 0$$

$$t^{k}_{(\gamma)} = \sum_{\alpha} \rho_{(\alpha)} \frac{\partial \psi_{(\alpha)}}{\partial x_{(\gamma) k, S}} J_{(\gamma)}$$

$$t^{lk}_{(\gamma)} = \rho_{(\gamma)} \frac{\partial \psi_{(\gamma)}}{\partial \chi_{(\gamma) k K, S}} \chi^{l}_{(\gamma) \cdot K} J_{(\gamma)}$$

(3.15)
$$t^{r}_{(\gamma)}\lambda^{l}=\sum_{\alpha}\rho_{(\alpha)}\frac{\partial\psi_{(\alpha)}}{\partial x_{(\gamma)k,S}}J_{(\gamma)}\lambda^{l}=\sum_{\alpha}\rho_{(\alpha)}\frac{\partial\psi_{(\alpha)}}{\partial x_{(\gamma)k,S}}\overset{l}{x}_{(\gamma),S}$$

$$-\sum_{\alpha}\sum_{\gamma}\rho_{(\alpha)}J^{-1}_{(\gamma)}w_{(\alpha-\gamma)}\left(\frac{\partial\psi_{(\alpha)}}{\partial x_{(\gamma)k,S}}x_{(\gamma)k,S^2}+\frac{\partial\psi_{(\alpha)}}{\partial\chi_{(\gamma)kK,S}}\chi_{(\gamma)kK,S^2}+\right.$$

$$\left.+\frac{\partial\psi}{\partial\chi_{(\gamma)kK}}\chi_{(\gamma)kK,S}+\frac{\partial\psi_{(\alpha)}}{\partial u_{(\alpha-\gamma)k}}\Lambda_{(\gamma)K}\delta_{Kl}\right)=0$$

$$\sum_{\gamma}\left[\sum_{\alpha}\rho_{(\alpha)}\left(\frac{\partial\psi_{(\alpha)}}{\partial\chi_{(\gamma)kK,S}}\overset{l}{\chi}_{(\gamma)\cdot K,S}+\frac{\partial\psi_{(\alpha)}}{\partial\chi_{(\gamma)kK}}\overset{l}{\chi}_{(\gamma)\cdot K}\right)+\overset{k}{t}_{(\gamma)}\lambda^{l}\right]_{[kl]}=0.$$

If we accept the following definition

(3.16)
$$\rho\psi^{I}=\sum_{\alpha}\rho_{(\alpha)}\psi_{(\alpha)},$$

then using $(3.15)_5$, $(3.15)_7$ becomes

(3.17)
$$\sum_{\gamma}\left(\frac{\partial\psi^{I}}{\partial\chi_{(\gamma)kK,S}}\overset{l}{\chi}_{(\gamma)\cdot K,S}+\frac{\partial\psi^{I}}{\partial\chi_{(\gamma)kK}}\overset{l}{\chi}_{(\gamma)\cdot K}+\frac{\partial\psi^{I}}{\partial x_{(\gamma)k,S}}\overset{l}{x}_{(\gamma),S}\right)_{[kl]}=0.$$

Integrals which satisfy the objectivity condition for micropolar mixture are

(3.18)
$$\mathcal{F}_{(\alpha)K}=\overset{k}{x}_{(\alpha),S}\chi_{(\alpha)kK}$$
$$\mathcal{F}_{(\alpha-\mu)K}=\overset{k}{x}_{(\alpha),S}\chi_{(\mu)kK}$$
$$\Gamma_{(\alpha)KL}=\overset{k}{\chi}_{(\alpha)\cdot K,S}\chi_{(\alpha)kL}$$
$$\Gamma_{(\alpha-\mu)KL}=\overset{k}{\chi}_{(\alpha)\cdot K,S}\chi_{(\mu)kL}$$
$$H_{(\alpha-\mu)KL}=\chi_{(\alpha)kK}\overset{k}{\chi}_{(\mu)\cdot L}.$$

Multiplying $(3.18)_5$ by $\overset{lK}{\chi}_{(\alpha)}$, we shall have

$$\chi_{(\alpha)kK}\overset{k}{\chi}_{(\mu)\cdot L}\overset{lK}{\chi}_{(\alpha)}=H_{(\alpha-\mu)KL}\overset{lK}{\chi}_{(\alpha)}.$$

Since

(3.19)
$$\overset{k}{\chi}_{(\mu)\cdot L}=H_{(\alpha-\mu)KL}\overset{kK}{\chi}_{(\alpha)}$$
$$\chi_{(\mu)kL}=H_{(\alpha-\mu)KL}\overset{K}{\chi}_{(\alpha)k}$$

then

$$\chi_{(\mu)kK}\overset{k}{\chi}_{(\mu)L}=(H_{(\alpha-\mu)MK}\overset{M}{\chi}_{(\alpha)K})\,(H_{(\alpha-\mu)NL}\overset{kN}{\chi}_{(\alpha)})$$

i.e.

(3.20) $$\delta_{KL} = H_{(\alpha-\mu)MK} H_{(\alpha-\mu)NL} \chi_{(\alpha)}^{kN} \chi_{(\alpha)k}^{M} = \delta^{MN} H_{(\alpha-\mu)MK} H_{(\alpha-\mu)NL}$$

from which it can be seen that the symetrical part $H_{(\alpha-\mu)MK}$ as a function of its antisymetric part, i.e.

(3.21) $$H_{(\alpha-\mu)(MK)} = H_{(\alpha-\mu)(MK)} (H_{(\alpha-\mu)L})$$

since

$$H_{(\alpha-\mu)M} = \frac{1}{2} \varepsilon_{MKL} H_{(\alpha-\mu)KL} = \frac{1}{2} \mathrm{E}_{MKL} \chi_{(\alpha)kK} \chi_{(\mu)\cdot L}^{k}$$

i.e.

(3.22) $$H_{(\alpha-\mu)[KL]} = \varepsilon_{KLM} H_{(\alpha-\mu)M}.$$

In the following way it can be shown that $(3.18)_2$ and $(3.18)_4$ can be expressed like antisymentric part $H_{(\alpha-\mu)KL}$, i.e.

(3.23) $$\mathcal{F}_{(\alpha-\mu)N} = \mathcal{F}_{(\alpha-\mu)N} (\mathcal{F}_{(\mu)K}, H_{(\gamma-\mu)K}),$$
$$\Gamma_{(\alpha-\mu)NL} = \Gamma_{(\alpha-\mu)NL} (\Gamma_{(\mu)M}, H_{(\alpha-\mu)M}).$$

(3.24) $$\psi^I = \psi^I (u_{(\alpha-\mu)k}, \mathcal{F}_{(\gamma)k}, \Gamma_{(\gamma)K}, H_{(\alpha-\mu)K}, \Lambda_{(\gamma)K}, \theta)$$
$$\gamma = 1, 2, \dots, n, \qquad \mu = 1, 2, \dots)\alpha(\dots, n.$$

By demanding that mixtures free energy function ψ^I and free energy function of the α-th constituent $\psi_{(\alpha)}$ should satisfy the objectivy principle, we obtain further constraints on constitutive equations. Aconstraint of type (3.3) can be represented by a partial differential equation

(3.25) $$\varepsilon_{klm} \sum_{\gamma} \left(\frac{\partial \psi^I}{\partial u_{(\alpha-\gamma)l}} u_{(\alpha-\gamma)m} + \frac{\partial \psi^I}{\partial x_{(\gamma)l,s}} x_{(\gamma)m,s} + \frac{\partial \psi^I}{\partial \chi_{(\gamma)lk}} \chi_{(\gamma)mk} + \right.$$
$$\left. + \frac{\partial \psi^I}{\partial \chi_{(\gamma)lk,s}} \chi_{(\gamma)mk,s} \right) = 0$$

and with a same equation for $\psi_{(\alpha)}$. Subtracting (3.17) from (3.25), we shall have

$$\varepsilon_{klm} \sum_{\gamma} \frac{\partial \psi^I}{\partial u_{(\alpha-\mu)l}} u_{(\alpha-\gamma)m} = 0$$

and in a similar way (3.26)

$$\varepsilon_{klm} \sum_{\gamma} \sum_{\mu} \rho_{(\alpha)} \frac{\partial \psi_{(\gamma)}}{\partial u_{(\alpha-\mu)l}} u_{(\alpha-\mu)m} = 0.$$

If we use (3.7) and (3.8), the solution of above equations can be written in this way

(3.27) $$\psi^I = \psi^I (y^{(\delta\mu)}, \mathcal{F}_{(\gamma)K}, \Gamma_{(\gamma)K}, H_{(\beta-\mu)K}, \Lambda_{(\gamma)K}, \theta)$$

$$\gamma = 1, 2, \ldots, n, \qquad \mu = 1, 2, \ldots) \ \beta(\ldots, n$$

$$\alpha = 1, 2, \ldots, n$$

(3.28) $$\psi_{(\alpha)} = \psi_{(\gamma)} (\mathcal{F}_{(\mu)K}, \mathcal{F}_{(\gamma)K}, \Gamma_{(\alpha)K}, H_{(\beta-\mu)K}, \Lambda_{(\alpha)K}, \theta)$$

where we used (3.21), (3.23), (3.24) and definitions

(3.29) $$\mathcal{F}_{(\alpha, \alpha-\mu)K} = x_{(\alpha)kK}\, u_{(\alpha-\mu)k} \stackrel{\text{def}}{=\!=} \mathcal{F}_{(\mu)K}$$

(3.30) $$y^{(\delta\mu)} = u_l^{(\alpha-\delta)}\, u_l^{(\alpha-\mu)} = \mathcal{F}_K^{(\delta)}\, \mathcal{F}_K^{(\mu)}.$$

Here the sign δ takes up three values in a cyclic set α.

When (3.28) is replaced in (3.15) we have appropriate constitutive equations

(3.31) $$t^k_{(\gamma)} = \sum_\alpha \rho_{(\alpha)} \frac{\partial \psi_{(\alpha)}}{\partial x_{(\gamma)k,S}} = J_{(\gamma)}\, \chi^k_{(\gamma)\cdot K} \sum_\alpha \rho_{(\alpha)} \frac{\partial \psi_{(\alpha)}}{\partial \mathcal{F}_{(\gamma)K}} = \rho\, J_{(\gamma)}\, \chi^k_{(\gamma)\cdot K} \frac{\partial \psi^I}{\partial \mathcal{F}_{(\gamma)K}}$$

(3.32) $$t^{lk}_{(\gamma)} = \frac{1}{2} \rho_{(\gamma)o}\, \chi_{(\gamma)iK}\, \chi_{(\gamma)kR}\, \varepsilon_{KRM} \frac{\partial \psi_{(\gamma)}}{\partial \Gamma_{(\gamma)M}}$$

(3.33) $$M_{(\gamma)m} = \rho_{(\gamma)o}\, \chi_{(\gamma)mM} \frac{\partial \psi_{(\gamma)}}{\partial \Gamma_{(\gamma)M}}$$

(3.34) $$p_{(\gamma)} = \sum (\delta_{\alpha\sigma}\delta_{\gamma\rho} - \delta_{\gamma\sigma}\delta_{\alpha\sigma})\, J^{-1}_{(\rho)}\, \rho_{(\sigma)} \left(\frac{\partial \psi_{(\sigma)}}{\partial x_{(\rho)k,S}}\, x_{(\rho)k,S^2} + \right.$$
$$\left. + \frac{\partial \psi_{(\sigma)}}{\partial \chi_{(\rho)kK}}\, \chi_{(\rho)kK,S} + \frac{\partial \psi_{(\sigma)}}{\partial u_{(\sigma-\rho)k}}\, \Lambda_{(\rho)K}\, \delta_{Kl} \right) = 0.$$

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КОПСТИТУТИВНЫЕ УРАВНЕНИЯ ГЕТЕРОГЕНЫХ МИКРОПОЛЯРНЫХ СТЕРЖНЕЙ

П. Цветкович

Р е з ю м е

Пользуясь законамы баланса масси, тензора микроинерции, количества движения и энергии микроморфной теорий смеси, применненой к теорий стержней, в статье выведенны конститутивные уравнения смеси для случая микрополарной теории стержней.

KONSTITUTIVNE JEDNAČINE HETEROGENIH MIKROPOLARNIH ŠTAPOVA

Predrag Cvetković

I z v o d

U radu [1] izložena je mikromorfna teorija heterogenih tela (mešavine) primenjena za slučaj jednodimenzionalnog tela, odnosno štapova, i izvedeni zakoni balanca kako za jedan sastojak tako i za heterogeno telo (mešavinu) u celini. Koristeći zakone balansa mase, tenzora mikroinercije, količine kretanja i energije za slučaj mikropolarne teorije štapa, izvedene su konstitutivne jednačine za tenzore napona prvog i drugog reda i naponskog sprega. Konstitutivne jednačine su izvedene u odsustvu hemijskih reakcija kao u radu [3], a pošto se radi o štapu kao jednodimenzionalnom telu, to su zanemarene interakcije između sastojaka. I pored ovih zanemarivanja konstitutivne jednačine su opštijeg karaktera od postojećih, jer se pretpostavlja da se heterogeno telo sastoji od proizvoljnog broja sastojaka, a ne samo iz dva.

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