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THE GENERALIZED BOUNDARY VALUE PROBLEMS IN THEORY OF PLATES

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In Kirchhoff's theory of thin plates the components of stress tensor σ_x, σ_y, τ_{xy} are linear functions of variable Z, but τ_{xz} and τ_{yz} are the polynomials of degree two of the variable Z. Better results are given by Stevenson's theory of moderately thick plates [5]. Components σ_x, σ_y, τ_{xy} are the polynomials of degree five. Also the remaining stress components and also the displacements, the forces and moments are likewise more exactly expressed in Stevenson's theory. In Kirchhoff's theory the mentioned quantities can be represented by two analytic functions, which can be found bu solving boundary value problem given in Lehnicki form [2]. For representation Stevenson used five functions of which one (P) is defined by transverse load q. In this paper we define the generalized boundary value problems which make it possible to determine the remaining four functions. Our attention will be concentrated on elastic,, homogeneous and isotropic plates. We will suppose there are no body-forces and that linear Hook's law holds.

2. Complex representation of fundamental mechanical quantities

Stevenson's results [5] are given in the Theorems 1. 1 and 1.2.

Theorem 1.1. *At supposition*

$$\sigma_z = \frac{q_{(x,y)}}{4h^3}(Z^3 - 3h^2 Z - 2h^3 = \frac{1}{8h^3} Z^3 - 3h^2 Z - 2h^3)\,[P^{II}(z) + \overline{P^{II}(z)}] \tag{1.1}$$

the quantities

$$A = \sigma_x + \sigma_y, \quad B = \sigma_y - \sigma_x + 2i\tau_{xy}, \quad C = \tau_{xy} + i\tau_{yx} \tag{1.2}$$

are expressed in a form:

$$\begin{aligned} A &= a_0 + a_1 Z + a_3 Z^3 \\ B &= b_0 + b_1 Z + b_2 Z^2 + b_3 Z^3 + b_5 Z^5 \\ C &= (Z^2 - h^2)(c_0 + c_2 Z^2) \end{aligned} \tag{1.3}$$

where

$$a_0=\frac{1}{2}[\varphi^I(z)+\overline{\varphi^I(z)}]+\frac{1}{4}[P^{II}(z)+\overline{P^{II}(z)}],$$

(1.4) $$a_1=\frac{1+\nu}{2}[\Omega^I(z)+\overline{\Omega^I(z)}]+\frac{3(1+\nu)}{16h^3}[\bar{z}P^I(z)+\overline{P^I(z)}]+\frac{3}{8h}[P^{II}(z)+\overline{P^{II}(z)}],$$

$$a_3=-\frac{2+\nu}{8h^3}[P^{II}(z)+\overline{P^{II}(z)}],$$

$$b_0=\frac{1}{2}[\bar{z}\varphi^{II}(z)+\psi^I(z)]+\frac{1}{4}\bar{z}P^{III}(z),$$

$$b_1=-\frac{1-\nu}{2}[\bar{z}\Omega^{II}(z)+\omega^{II}(z)]-\frac{3(1-\nu)}{16h^3}[\overline{P(z)}+\frac{1}{2}\bar{z}^2P^{II}(z)]-\frac{3}{8h}\bar{z}P^{III}(z),$$

(1.5) $$b_2=-\frac{\nu}{1+\nu}\varphi^{III}(z)-\frac{1}{2}P^{IV}(z),$$

$$b_3=\frac{2-\nu}{3}\Omega^{III}(z)+\frac{2-\nu}{8h^3}\bar{z}P^{III}(z)+\frac{1}{4h}P^{IV}(z),$$

$$b_5=-\frac{3-\nu}{40h^3}P^{IV}(z),$$

$$c_0=-\frac{1}{2}\overline{\Omega^{II}(z)}-\frac{3}{16h^3}[P^I(z)+z\overline{P^{II}(z)}]-\frac{1}{4h}\overline{P^{III}(z)},$$

(1.6) $$c_2=\frac{1}{8h^3}\overline{P^{III}(z)},$$

Results of this theorem satisfy Equilibrium equations, Conditions of compatibility of Beltrami — Michell and the following boundary conditions:

$$\sigma_z(Z=h)=-q,\qquad \sigma_z(Z=-h)=0$$

(1.7) $$\tau_{xz}(Z=\pm h)=0,\qquad \tau_{yz}(Z=\pm h)=0$$

$$\frac{\partial\sigma_z}{\partial Z}(Z=\pm h)=0.$$

Theorem 1.2.

(1.8) $$D = u + iv = D_0 + D_1 Z + D_2 Z^2 + D_3 Z^3 + D_5 Z^5$$

(û.9) $$w = w_0 + w_1 Z + w_2 Z^2 + w_4 Z^4$$

$$8\,\mu\,D_0 = (\varkappa\varphi(z) - z\,\overline{\varphi^I(z)} - \overline{\psi(z)}) + \frac{3}{2} P^I(z) - \frac{1}{2} z\,\overline{P^{II}(z)}$$

$$8\,\mu\,D_1 = (1-\nu)\,[\Omega(z) + z\,\overline{\Omega^I(z)} + \overline{\omega^I(z)}] + \frac{3}{4\,h}[P^I(z) + z\,\overline{P^{II}(z)}] +$$

$$+ \frac{3\,(1-\nu)}{8\,h^3}\left[\bar{z}\,P(z) + \frac{1}{2} z^2\,\overline{P^I(z)}\right]$$

$$8\,\mu\,D_2 = \frac{2\,\nu}{1+\nu}\,\overline{\varphi^{II}(z)} + \overline{P^{III}(z)}$$

$$8\,\mu\,D_3 = -\frac{2\,(2-\nu)}{3}\,\overline{\Omega^{II}(z)} - \frac{2-\nu}{4\,h^3}\,[P^I(z) + \overline{P^{II}(z)}] - \frac{1}{2\,h}\,\overline{P^{III}(z)}$$

$$8\,\mu\,D_5 = \frac{3-\nu}{20\,h^3}\,\overline{P^{III}(z)}$$

$$2\,\mu\,w_0 = -\frac{1-\nu}{8}\,[\bar{z}\,\Omega(z) + z\overline{\Omega(z)} + \omega(z) + \overline{\omega(z)}] + \frac{h^2}{2}\,[\Omega^I(z) + \overline{\Omega^I(z)}] +$$

$$+ \frac{3}{32\,h}\,[\bar{z}\,P^I(z) + z\,\overline{P^I(z)}] + \frac{h}{4}\,[P^{II}(z) + \overline{P^{II}(z)}] - \frac{3\,(1-\nu)}{128\,h^3}\,[z^2\,\overline{P(z)} + \bar{z}^2\,P(z)]$$

$$2\,\mu\,w_1 = -\frac{\nu}{2\,(1+\nu)}\,[\varphi^I(z) + \overline{\varphi^I(z)}] - \frac{1}{4}\,[P^{II}(z) + \overline{P^{II}(z)}]$$

$$2\,\mu\,w_2 = -\frac{\nu}{4}\,[\Omega^I(z) + \overline{\Omega^I(z)}] - \frac{3}{16\,h}\,[P^{II}(z) + \overline{P^{II}(z)}] - \frac{3\,\nu}{32\,h^2}\,[\bar{z}\,P^I(z) + z\,\overline{P^I(z)}]$$

$$2\,\mu\,w_4 = \frac{1+\nu}{32\,h^3}\,[P^{II}(z) + \overline{P^{II}(z)}]$$

$$\left(\varkappa = \frac{3-\nu}{1+\nu}\right)$$

Besides unit forces N_x and N_y which are contained in the theory of thin plates in the theory of moderately thick plates the following unit forces are also found:

(1.10) $$\Sigma_x = \int_{-h}^{+h} \sigma_x\,dZ, \qquad \Sigma_y = \int_{-h}^{+h} \sigma_y\,dZ, \qquad T_{xy} = T_{yx} = \int_{-h}^{+h} \tau_{yx}\,dZ$$

(In Kirchhoff's theory holds $\sum_x \equiv \sum_y \equiv T_{xy} \equiv 0$)
Considering theorem 1.1 and definitions of unit forces and unit moment we find:

Theorem 1.3.

$$\sum_x + \sum_y = h\,[\varphi^I(z) + \overline{\varphi^I(z)}] + \frac{h}{2}\,[P^{II}(z) + \overline{P^{II}(z)}]$$

$$\sum_y - \sum_x + 2\,i\,T_{xy} = h\,[\bar{z}\,\varphi^{II}(z) + \psi^I(z)] - \frac{2\,\sigma\,h^3}{3}\,\varphi^{III}(z) + \frac{h}{2}\,\bar{z}\,P^{III}(z) - \frac{h^3}{3}\,P^{IV}(z) \tag{1.11}$$

$$N_x + i\,N_y = \frac{2}{3}\,h^3\,\overline{\Omega^{II}(z)} + \frac{1}{4}\,[P^I(z) + z\,\overline{P^{II}(z)}] + \frac{3}{10}\,h^2\,\overline{P^{III}(z)}$$

$$G_x + G_y = \frac{1+\nu}{3}\,h^3\,[\Omega^I(z) + \overline{\Omega^I(z)}] + \frac{1+\nu}{8}\,[\bar{z}\,P^I(z) + z\,\overline{P^I(z)}] + \frac{3-\nu}{20}\,h^2\,[P^{II}(z) + \overline{P^{II}(h)}] \tag{1.12}$$

$$G_y - G_x + 2\,i\,H_{xy} = -\frac{1-\nu}{3}\,h^3\,[\bar{z}\,\Omega^{II}(z) + \omega^{II}(z)] + \frac{2(2-\nu)}{15}\,h^5\,\Omega^{III}(z) - \frac{1-\nu}{8}\,[\overline{P(z)} + \frac{1}{2}\,\bar{z}^2\,P^{II}(z)] - \frac{3+\nu}{20}\,h^2\,\bar{z}\,P^{III}(z) + \frac{11+\nu}{140}\,h^4\,P^{IV}(z)$$

$$\left(\sigma = \frac{\nu}{1+\nu}\right)$$

Theorem 1.4. Bending moment and generalized shear force

$$G_n = G_x \cos^2\alpha + G_y \sin^2\alpha + H_{xy} \sin 2\,\alpha$$

$$N_n + \frac{d}{ds}\,H_{nt} = N_x \cos\alpha + N_y \sin\alpha + \frac{d}{ds}\left[H_{xy} \cos 2\,\alpha + \frac{1}{2}\,(G_y - G_x) \sin 2\,\alpha\right] \tag{1.13}$$

at $P(z)\equiv 0$ we can express in the form

$$G_n=\frac{1}{12}h^3 G(\tilde{w}),\qquad N_n+\frac{d}{ds}H_{nt}=\frac{1}{12}h^3 H(\tilde{w}) \tag{1.14}$$

where

$$\tilde{w}=\alpha_0 w_0+\beta_0 \Delta w_0;\ \alpha_0=-\frac{16\mu}{1-\nu},\qquad \beta_0=-\frac{8\mu(8+\nu)}{5(1-\nu)^2}h^2 \tag{1.15}$$

and

$$G(\tilde{w})=\nu\Delta\tilde{w}+(1-\nu)\left[\frac{\partial^2\tilde{w}}{\partial x^2}\cos^2\alpha+\frac{\partial^2\tilde{w}}{\partial y^2}\sin^2\alpha+\frac{\partial^2\tilde{w}}{\partial x\partial y}\sin 2\alpha\right] \tag{1.16}$$

$$H(\tilde{w})=\frac{d\Delta\tilde{w}}{dn}+(1-\nu)\frac{d}{ds}\left[\frac{\partial^2\tilde{w}}{\partial x\partial y}\cos 2\alpha+\frac{1}{2}\left(\frac{\partial^2\tilde{w}}{\partial y^2}-\frac{\partial^2\tilde{w}}{\partial x^2}\right)\sin 2\alpha\right]$$

Theorem 1.5. a. Main force

$$[\vec{F}]_{\widehat{AB}}=\frac{1}{2h}\int_{\widehat{AB}}\int_{-h}^{+h}(X_n, Y_n, Z_n)\,dZ\,ds \tag{1.17}$$

can be expressed in the following way

$$[F_x+iF_y]_{\widehat{AB}}=\frac{i}{2}\left[-\frac{1}{2}\left(\varphi(z)+z\overline{\varphi^I(z)}+\overline{\psi(z)}\right)+\frac{\sigma h^2}{3}\overline{\varphi^{II}(z)}\right]_A^B-\frac{i}{8}\left[z\overline{P^{II}(z)}+\right.$$

$$\left.+P^I(z)-\frac{2}{3}h^2P^{III}(z)\right]_A^B$$

$$[F_z]_{\widehat{AB}}=\frac{ih^2}{6}\left[\overline{\Omega^I(z)}-\Omega^I(z)\right]_A^B+\frac{i}{80h}\left[5\left(z\overline{P^I(z)}-\bar{z}P^I(z)\right)+6h^2\left(\overline{P^{II}(z)}-P^{II}(z)\right)\right]_A^B$$

$$+\frac{i}{8h}\int_{\widehat{AB}}\left[P^I(z)\,d\bar{z}-\overline{P^I(z)}\,dz\right] \tag{1.18}$$

In proving, we find the following intermediate results

$$[F_x+iF_y]_{\widehat{AB}}=-\frac{i}{4h}\int_{\widehat{AB}}(\Sigma_y-\Sigma_x-2iT_{xy})\,d\bar{z}+(\Sigma_x+\Sigma_y)\,dz$$

$$[F_z]_{\widehat{AB}}=\frac{i}{4h}\int_{\widehat{AB}}(N_x+iN_y)\,d\bar{z}-(N_x-iN_y)\,dz \tag{1.19}$$

Theorem 1.5. b. Main moment

$$[\overline{M}]_{\widehat{AB}} = \frac{1}{2h} \int\limits_{\widehat{AB}} \int\limits_{-h}^{+h} (Z_n y - Y_n Z,\ X_n Z - Z_n x,\ Y_n x - X_n y)\, dZ\, ds \tag{1.20}$$

can be expressed in the following way

$$[M_x + i M_y]_{\widehat{AB}} = \frac{h^2}{12}\Big[(3+\nu)\,\Omega(z) - (1-\nu)\,(z\,\overline{\Omega^I(z)} + \overline{\omega^I(z)}) + 2\,z\,\overline{\Omega^I(z)} -$$

$$- \Omega^I(z)) + \frac{2(2-\nu)}{5}\,h^2\,\overline{\Omega^{II}(z)}\Big]_A^B + \frac{1}{4h}\Big[-\frac{1-\nu}{8}\,\bar{z}\,P(z) + \frac{1}{4}\,z\,\bar{z}\,P^I(z) +$$

$$+ \frac{9-\nu}{20}\,h^2\,P^I(z) + \frac{3+\nu}{16}\,z^2\,\overline{P^I(z)}\Big] - \frac{3+\nu}{20}\,h^2\,z\,P^{II}(z) +$$

$$+ \frac{11+\nu}{140}\,h^4\,\overline{P^{III}(z)}\Big]_A^B - \frac{1}{8h}\int\limits_{\widehat{AB}} z\,[\overline{P^I(z)} + \bar{z}\,P^{II}(z)]\,dz \tag{1.21}$$

$$[M_z]_{\widehat{AB}} = \frac{1}{8}\,[\chi(z) + \overline{\chi(z)} - z\,\bar{z}\,(\varphi^I(z) + \overline{\varphi^I(z)}) - z\,\psi(z) - \bar{z}\,\overline{\psi(z)}]_A^B -$$

$$- \frac{\sigma h^2}{12}[\varphi^I(z) + \overline{\varphi^I(z)} - z\,\varphi^{II}(z) - \overline{z\,\varphi^{II}(z)}]_A^B - \frac{1}{16}\,z\,\bar{z}\,[P^{II}(z) + \overline{P^{II}(z)}]_A^B +$$

$$+ \frac{1}{24}\,h^2\,z\,[P^{III}(z) + \overline{\bar{z}\,P^{III}(z)} - P^{II}(z) - \overline{P^{II}(z)}]_A^B \tag{1.22}$$

Some intermediate results:

$$[M_x + i M_{y\widehat{AB}} = \frac{1}{4h}\int\limits_{\widehat{AB}} [z\,(N_x + i\,N_y) + (G_y - G_x - 2\,i\,H_{xy})]\,d\bar{z} +$$

$$+ [-z\,(N\ - i\,N_y) + (G_x + G_y)]\,dz$$

$$[M_z]_{\widehat{AB}} = -\frac{1}{8h}\int\limits_{\widehat{AB}} [\bar{z}\,(\Sigma_y - \Sigma_x - 2\,i\,T_{xy}) + z\,(\Sigma_x + \Sigma_y)]\,d\bar{z}$$

$$+ [z\,(\Sigma_y - \Sigma_x + 2\,i\,T_{xy}) + \bar{z}\,(\Sigma_x + \Sigma_y)]\,dz$$

2. Stress functions

Theorem 2.1. Components of stress tensor, unit forces and moments are not altered, if one replaces

$$\varphi(z) \;\; by \;\; \varphi_1(z) = \varphi(z) + iCz + \gamma$$

$$\varphi^I(z) \;\; by \;\; \varphi_1^I(z) = \varphi^I(z) + iCz$$

$$\chi(z) \;\; by \;\; \chi_1(z) = \chi(z) + \gamma^I z_I + \gamma^{II} \tag{2.1}$$

$$\psi(z) \;\; by \;\; \psi_1(z) = \psi(z) + \gamma^I$$

$$\Omega(z) \;\; by \;\; \Omega_1(z) = \Omega(z) + iC_1 z + \gamma_1$$

$$\Omega^I(z) \; by \;\; \Omega_1^I(z) = \Omega^I(z) + iC_1 \tag{2.1'}$$

$$\omega(z) \;\; by \;\; \omega_1(z) = \omega(z) + \gamma_1^I z + \gamma_1^{II}$$

$$\omega^I(z) \; by \;\; \omega_1^I(z) = \omega^I(z) + \gamma_1^I$$

where C, C_1 are real and $\gamma, \gamma^I, \ldots \gamma_1^{II}$ are arbitrary complex constants. Substitutions of the form (2.1) and (2.2) will affect the displacements, unless

$$C = 0, \quad \varkappa\gamma - \overline{\gamma}^I = 0$$

$$\gamma_1 + \overline{\gamma}_1^I = 0, \quad \gamma_1^{II} + \overline{\gamma}_1^{II} = 0 \tag{2.2}$$

Theorem 2.2. In the case of finite multiply connected regions the stress functions have the following form:

$$\Phi(z) = \sum_{k=1}^{m} A_k \ln(z - z_k) + \Phi^*(z)$$

$$\varphi(z) = z\sum_{k=1}^{m} A_k \ln(z - z_k) + \sum_{k=1}^{m} \gamma_k \ln(z - z_k) + \varphi^*(z)$$

$$\psi(z) = \sum_{k=1}^{m} \gamma_k^I \ln(z - z_k) + \psi^*(z) \tag{2.3}$$

$$\chi(z) = z\sum_{k=1}^{m} \gamma_k^I \ln(z - z_k) + \sum_{k=1}^{m} \gamma_k^{II} \ln(z - z_k) + \chi^*(z)$$

$$\Omega^I(z) = \sum_{k=1}^{m} A_{k,1} \ln(z - (z_k) + \tilde{\Omega}(z)$$

$$\Omega(z) = z\sum_{k=1}^{m} A_{k,1} \ln(z - z_k) + \sum_{k=1}^{m} \gamma_{k,1} \ln(z - z_k) + \Omega^*(z) \tag{2.4}$$

$$\omega^I(z) = \sum_{k=1}^{m} \gamma_{k,1}^{I} \ln(z - z_k) + \tilde{\omega}(z)$$

$$\omega(z) = z \sum_{k=1}^{m} \gamma_{k,1}^{I} \ln(z - z_k) + \sum_{k=1}^{m} \gamma_{k,1}^{II} \ln(z - z_k) + \omega^*(z)$$

$z_k\,(k=1, \ldots, m)$ is fixed point, arbitrarily chosen inside the contour c_k; A_k, $A_{k,1}$ $(k=1, \ldots, m)$ are real constants, $\gamma_k, \ldots, \gamma_{k,1}^{II}$ are complex constants, $\Phi^*, \varphi^*, \ldots, \omega^*, \tilde{\Omega}, \tilde{\omega}$ are holomorphic functions.

For the single — valuedness of displacements it is necessary and sufficient that in the formulae (2.3) and (2.4):

$$A_k = 0, \quad \varkappa\gamma_k + \overline{\gamma}_k^{I} = 0$$

$$\gamma_{k,1} - \overline{\gamma}_{k,1}^{I} = 0, \quad \gamma_{k,1}^{II} = \overline{\gamma}_{k,1}^{II} = \alpha_{k,1}^{II} \in \mathcal{R} \tag{2.5}$$

$$(k = 1, 2, \ldots, m)$$

Constants have the following physical meaning

$$\gamma_k = -\frac{2}{(1+\varkappa)\pi}\left[(F_x + i\,F_y) - (F_x^0 + i\,F_y^0)\right]_k$$

$$A_{k,1} = \frac{-3}{2\,h^2\,\pi}\left[F_z - F_z^0\right]_k$$

$$\gamma_{k,1} = \frac{3\,i}{2\,h^2\,\pi}\left[(M_x + i\,M_y) - (M_x^0 + i\,M_y^0)\right]_k \tag{2.6}$$

$$\beta_k^{II} = Im\,\gamma_k^{II} = \frac{2}{\pi}\left[M_z - M_z^0\right]_k$$

Index k means that we must take a main force (moment) which is acting the contour c_k.

Theorem 2.3. In case of Infinite plate with holes it is necessary and sufficient for the boundedness of un·t forces and moments that stress functions should for large $|z|$, have this form:

$$\varphi(z) = B_1 \ln z + \Gamma_1 z + \varphi_0(z); \qquad \varphi_0(z) = a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots$$

$$\psi(z) = -\varkappa\,\overline{B}_1 \ln z + \Gamma_1^I z + \psi_0(z); \qquad \psi_0(z) = a_0^I + \frac{a_1^I}{z} + \frac{a_2^I}{z^2} + \cdots$$

$$\Omega(z)=B_2 \ln z+\Gamma_2 z+\Omega_0(z); \qquad \Omega_0(z)=b_0+\frac{b_1}{z}+\frac{b_2}{z^2}+\cdots$$
$$\omega^I(z)=\overline{B}_2 \ln z+\Gamma_2^I z+\omega_0^I(z); \qquad \omega_0^I(z)=b_0^I+\frac{b_1^I}{z}+\frac{b_2^I}{z^2}+\cdots \tag{2.8}$$

Constants have the following mechanical meaning

$$B_1=-\frac{2}{(1+\varkappa)\pi}\sum_{k=1}^{m}[(F_x+i\,F_y)-(F_x^0+i\,F_y^0)]_k$$
$$B_2=\frac{3\,i}{4\,h^2\,\pi}\sum_{k=1}^{m}[(M_x+i\,M_y)-(M_x^0+i\,M_y^0)]_k \tag{2.9}$$

$$R_e\,\Gamma_1=\frac{1}{2\,h}(\Sigma_x^\infty+\Sigma_y^\infty-\Sigma_x^{0,\infty}-\Sigma_y^{0,\infty}); \quad \mathrm{Im}\,\Gamma_1=0$$
$$R_e\,\Gamma_2=\frac{3}{2(1+\nu)\,h^3}(G_x^\infty+G_y^\infty-G_x^{0,\infty}-G_y^{0,\infty}); \quad \mathrm{Im}\,\Gamma_2=0 \tag{2.10}$$

$$\Gamma_1^I=\frac{1}{h}(\Sigma_y^\infty-\Sigma_x^\infty+2\,i\,T_{xy}^\infty-\Sigma_y^{0,\infty}-\Sigma_x^{0,\infty}-2\,i\,H_{xy}^{0,\infty})$$
$$\Gamma_2^I=-\frac{3}{(1-\nu)\,h^3}(G_y^\infty-G_x^\infty+2\,i\,H_{xy}^\infty-G_y^{0,\infty}+G_x^{0,\infty}-2\,i\,H_{xy}^{0,\infty}) \tag{2.11}$$

$\mathrm{Im}\,\Gamma_1$ and $\mathrm{Im}\,\Gamma_2$ are arbitrary, therefore we choose $\mathrm{Im}\,\Gamma_1=\mathrm{Im}\,\Gamma_2=0$. Functions with index zero are holomorphic for $|z|>R$ and we can take $a_0=a_0^I=b_0=b_0^I=0$. For boundedness of displacements at $z=\infty$ it is necessary and sufficient that

$$B_1=0, \quad \Gamma_1=0, \quad \Gamma_1^I=0$$
$$B_2=0, \quad \Gamma_2=0, \quad \Gamma_2^I=0, \quad b_0=b_0^I=b_1^I=0 \tag{2.12}$$

In this case

$$\sum_{k=1}^{m}[F_x-F_x^0]_k=\sum_{k=1}^{m}[F_y-F_y^0]_k=\sum_{k=1}^{m}[F_z-F_z^0]=0$$
$$\sum_{k=1}^{m}[M_x-M_x^0]_k=\sum_{k=1}^{m}[M_y-M_y^0]_k=0 \tag{1.13}$$

$\sum_{k=1}^{m}[M_z-M_z^0]_h$ has no influence on displacements.

$$\sum_x^{\infty} - \sum_x^{0,\infty} = \sum_y^{\infty} - \sum_y^{0,\infty} = T_{xy}^{\infty} - T_{xy}^{0,\infty}$$

$$N_x^{\infty} - N_x^{0,\infty} = N_y^{\infty} - N_y^{0,\infty} = 0 \tag{2.14}$$

$$G_x^{\infty} - G_x^{0,\infty} = G_y^{\infty} - G_y^{0,\infty} = H_{xy}^{\infty} - H_{xy}^{0,\infty} = 0$$

For stress functions by large $|z|$ we have

$$\varphi(z) = 0(1), \quad \psi(z) = 0(1)$$

$$\Omega(z) = 0(z^{-1}), \quad \omega(z) = 0(1), \quad \omega^I(z) = 0(z^{-2}) \tag{2.15}$$

Notes. (1) Index zero means the part of convenient quantity which by function P is expressed. So i.e. from (1.11) we find

$$\sum_x^0 + \sum_y^0 = \frac{h}{2}\left[P^{II}(z) + \overline{P^{II}(z)}\right]$$

(2) Index ∞ has the meaning which the following definition shows

$$\sum_x^{\infty} = \lim_{|z|\to\infty} \sum_x, \qquad \sum_x^{0,\infty} = \lim_{|z|\to\infty} \sum_x^0$$

(3) All theorems of this chapter can be proved in a way analogous to that used in [1], [4], etc.

3. Generalized boundary value problems

The first fundamental boundary value problem is usually defined in this way:

Find the elastic equilibrium of a body, if the external stresses acting on its boundaries are given.

In case of thin plates holds $\Sigma_x \equiv \Sigma_y \equiv T_{xy} \equiv 0$, but not in our case. Therefore our discussion is an essential generalization of the classic one. We shall show that the first generalized problem consists of two classic boundary value problems:

— the first is the boundary value problem of a wall to which we come from known Σ_x, Σ_y, T_{xy}

— the second is the boundary value problem of moderately thick plates to which we come from known N_x, N_y, G_x, G_y, H_{xy} on boundary.

Definition 3.1. (definition of the first generalized boundary value problem for analytic functions). Find the functions φ, ψ, Ω, ω holomorphic in region S so that these functions and the derivatives φ^I, φ^{II}, Ω^I, Ω^{II} are continuous in S up to the boundary of S and so that on every boundary curve c_k ($k = 0, 1, \ldots, m$) it holds:

$$\varphi(t) + t\,\overline{\varphi^I(t)} + \overline{\psi(t)} - \frac{2}{3}\,\sigma\, h^2\, \overline{\varphi^{II}(t)} = f(s(t)) + Q_1(t) + \alpha_k \tag{3.1}$$

$$\varkappa^I\, \Omega(t) - t\,\overline{\Omega^I(t)} - \overline{\omega^I(t)} + \varkappa^{II}\, h^2\, \overline{\Omega^{II}(t)} = \tilde{f}(s(t)) + \tilde{Q}_1(t) + i r_k\, t + \beta_k \tag{3.2}$$

Here is

$$\sigma = \frac{\nu}{1+\nu}, \qquad \varkappa^I = \frac{3+\nu}{1-\nu}, \qquad \varkappa^{II} = \frac{2(2-\nu)}{5(1-\nu)} \tag{3.3}$$

$$f(s) = \frac{2i}{h} \int_{0\,c}^{s} \int_{-h}^{+h} (X_n + i Y_n)\, dZ\, ds$$

$$\tilde{f}(s) = \frac{6}{(1-\nu)h^3} \int_0^s [m(s) + i\, r(s)]\, dt, \qquad r(s) = \int_0^s p(s)\, ds, \tag{3.4}$$

where $m(s)$ and $p(s)$ are functions defined on the boundary which have the following physical meaning

$$G_n = m(s), \qquad N_n + \frac{d}{dt} H_{nt} = p(s) \tag{3.5}$$

From Theorem 1.1 we find

$$Q_1(t) = -\frac{1}{2}\left[t\,\overline{P^{II}(t)} + P^I(t) - \frac{2}{3} h^2 P^{III}(t)\right] \tag{3.6}$$

$$\tilde{Q}_1(t) = \frac{3}{8h^3}\,\bar{t}\,P(t) + \frac{3}{16h^3}\,t^2\,\overline{P^I(t)} + \frac{3(3+\nu)}{20(1-\nu)h}\,t\,\overline{P^{II}(t)} -$$

$$-\frac{3(9-\nu)}{20(1-\nu)h} P^I(t) - \frac{3(11+\nu)}{140(1-\nu)}\, h\,\overline{P^{III}(t)} -$$

$$-\frac{3}{2(1-\nu)h^3} \int_0^s \left[\bar{t}\,P^I(t) - \int_0^{s_0} \left(P^I(\tau)\, d\bar{\tau} - \overline{P^I(\tau)}\right) d\tau\right] dt \tag{3.7}$$

The equation (3.2) is obtained in an analogous manner as Lehnicki [8] or Muršič [3] obtained it. The equation (3.1) is found if we calculate $[F_x + i F_y]_{z=t}$ from equations (1.19) and (1.11).

The second fundamental boundary value ptoblem is usually defined in this way:

Find the elastic equilibrim of a body, if the displacements of the points of its boundary are given.

In the theory of thin plates we suppose the displacements u_0 and v_0 in the middle plane are $= 0$. This supposition is not necessary in our discussion.

We shall suppose that on the boundary u_0, v_0, w_0, $\frac{dw_0}{dn}$ are given.

Definition 3.2. (definition of the second generalized boundary value problem). Find the elastic equilibrium of a plate if $u_0(s)$, $v_0(s)$, $w_0(s)$ $\frac{dw_0}{dn}$ on the boundary are given:

$$u_0 = g_1(t), \quad v_0 = g_2(t) \quad (t \in c_k, k = 0, \ldots, m) \tag{3.8}$$

$$w_0 = h(t), \quad \frac{dw_0}{dn} = \tilde{g}_2(t) \quad (t \in c_k, k = 0, 1, \ldots, m) \tag{3.9}$$

$g_1, g_2, h, \tilde{g}_2$ are (smooth enough) functions defined on the boundary c of the region S.

The problem was called generalized because it is bu (3.8) richer than the classic one. Also this problem will be transacted into a complex function theory.

Definition 3.3. (definition of the second generalized boundary value problem is a complex function theory).

Find the functions φ, ψ, Ω, ω holomorphic in S so that these functions and derivatives φ^I, Ω^I, ω^{II} are continuous in S up to the boundary c and so that on every curve c_k $(k = 0, 1, \ldots, m)$:

$$\varkappa \varphi(t) - t\,\overline{\varphi(t)} - \overline{\psi(t)} = g(t) + Q_2(t) \tag{3.10}$$

$$\Omega(t) + t\,\overline{\Omega^I(t)} + \overline{\omega^I(t)} - \frac{4}{1-\nu} h^2\,\overline{\Omega^{II}(t)} = \tilde{g}(t) + \tilde{Q}_2(t)$$

$$w_0 = h(s)$$

Here is

$$g(t) = 8\mu(u_0 + i v) = 8\mu[g_1(t) + i g_2(t)]$$

$$\tilde{g}_1(s) = \frac{d w_0}{ds} = h^I(s)$$

$$\tilde{g}(s) = -\frac{8\mu}{1-\nu}\left(\frac{dw_0}{dn} + i\frac{dw_0}{ds}\right) = -\frac{8\mu}{1-\nu}[\tilde{g}_2(s) + i\tilde{g}_1(s)] \tag{3.12}$$

$$Q_2(t) = -\frac{3}{2}P^I(t) + \frac{1}{2}t\,\overline{P^{II}(t)}$$

$$\tilde{Q}_2(t) = \frac{3}{4(1-\nu)h}[P^I(t) + t\,\overline{P^{II}(t)}] + \frac{2h}{1-\nu}\overline{P^{III}(t)} - $$

$$- \frac{3}{16h^3}[t^2\,\overline{P^I(t)} + 2\bar{t}\,P(t)]$$

The definition 3.3 is equivalent to the following

Definition 3.4. Find the functions φ_0, ψ_0, Ω_0, ω_0^I holomorphic in region S, real constants $A_{k,1}$ $(k=1, \dots, m)$, complex constants γ_k, $\gamma_{k,1}$ $(k=1, \dots, m)$ so that

(1) φ_0, φ_0^I, ψ_0, Ω_0, Ω_0^I, Ω_0^{II}, ω_0^I, G_0, $\tilde{G}_0$ are functions continuous up to the boundary. We denote

$$G_0(z) = \varkappa \varphi_0(z) - z\overline{\varphi_0^I(z)} - \overline{\psi_0(z)} \quad (z \in \overline{S} = S \cup c), \tag{3.13}$$

$$\tilde{G}_0(z) = \Omega_0(z) + z\overline{\Omega_0^I(z)} + \overline{\omega_0^I(z)} - \frac{4h^2}{1-\nu}\overline{\Omega_0^{II}(z)} \quad (z \in \overline{S}). \tag{3.14}$$

(2) On boundary

$$G_0(t) = g_0(t) \qquad (\forall t \in c)$$

where

$$\tilde{G}_0(t) = \tilde{g}_0(t) \qquad (\forall t \in c) \tag{3.15}$$

$$g_0(t) = g(t) + Q_2(t) - 2\varkappa \sum_{k=1}^{m}\left(\gamma_k \ln|t-z_k| + \frac{1}{2\varkappa}\frac{\overline{\gamma}_k t}{\overline{t} - \overline{z}_k}\right) \tag{3.16}$$

$$\tilde{g}_0(t) = \tilde{g}(t) + \tilde{Q}_2(t) - \sum_{k=1}^{m}\left[2(A_{k,1}t + \gamma_{k,1})\ln|t-z_k| + \right]A_{k,1}t\overline{t} +$$

$$+ \overline{\gamma}_{k,1}t - \frac{8h^2 A_{k,1}}{1-\gamma}\Bigg)\cdot\frac{1}{\overline{t}-\overline{z}_k} - \frac{A_{k,1}\overline{t} + \overline{\gamma}_{k,1}}{(\overline{t}-\overline{z}_k)^2}\Bigg] \tag{3.17}$$

$(t \in c)$

(3) $$w_0(t) = h(s(t)) \qquad (\forall t \in c) \tag{3.18}$$

Uniqueness of a solution for functions φ and ψ we prove in the same way as Babuška [1] did. For functions Ω and ω these problems are formed in real form and the following formula is used:

$$\oint_C \left[w H(w) - \frac{dw}{dn} G(w)\right] dt = -\iint_S [\nu(\Delta w)^2 +$$

$$+ (1-\nu)(w_{xx}^2 + w_{yy}^2 + 2w_{xy}^2)]\, dx\, dy \tag{3.19}$$

In she well known way we introduce boundary problems for infinite plates. So we find the following results:

Definition 3.5. The region S is bounded with curves $c_1, \dots, c_m$. On each c_k there is given a loading determined by functions $m_k(s)$, $p_k(s)$, $f_k(s)$ with the following physical meaning

$$G_n = m_k(s), \qquad N_n + \frac{dH_{nt}}{ds} = p_k(s), \tag{3.20}$$

$$f_k(s)=\frac{2i}{h}\int_{0\,c_k}^{s}\int_{-h}^{+h}(X_n+iY_n)\,dZ\,ds, \tag{3.21}$$

$$0\leqslant s<l_k; \qquad k=1,2,\ldots,m$$

The first generalized boundary value problem is now:

(1) Find the functions φ, ψ, Ω, ω^I holomorphic in S so that these functions and derivatives φ^I, φ^{II}, Ω^I, Ω^{II} are continuous in S up to the boudary c, so that on every boundary-curve c_k $(k=1,\ldots,m)$:

$$\varphi(t)+t\overline{\varphi^I(t)}+\overline{\psi(t)}-\frac{2}{3}\sigma h^2\overline{\varphi^{II}(t)}=f_k(s(t))+Q_1(t)+\alpha_k$$

$$\varkappa^I\Omega(t)-t\overline{\Omega^I(t)}-\overline{\omega^I(t)}+\varkappa^{II}h^2\overline{\Omega^{II}(t)}=$$

$$\tilde{f}_k(s(t))+\tilde{Q}_1(t)+i\,r_k\,t+\beta_k$$

All quantities on the right hand side have a meaning analogous to that in the definition 3.1.

(2) Stresses at $z=\infty$ which must be bounded are given. Hence the conditions of the theorem 2.3 are realized and the Γ_1, Γ_2, Γ_1^I, Γ_2^I are given.

The second generalized problem for infinite plates is def ned in this way:

Definition 3.6. Find the functions φ_0, ψ_0, Ω_0, ω_0 holomorphic in S, real constans $A_{k,1}$ $(k=1,\ldots,m)$, complex constants γ_κ, $\gamma_{k,1}$ $(k=1,\ldots,m)$ so that: (1) φ_0, φ_0^I, Ω_0, Ω_0^I, Ω_0^{II}, G_0 and $\tilde{G}_0$ are cantinuous in S up to the boundary (G_0, $\tilde{G}_0$ are defined by (3.13) and (3.14)); (2) on the boundary there is $G_0(t)=g_0(t)$, $\tilde{G}_0(t)=\tilde{g}_0(t)$ $(\forall\, t\in c)$ (g_0, $\tilde{g}_0$ are by (3.16) and (3.13)); (3) The equations (2.12) must be fulfilled.

Finally we have to fulfill the original boundary condition $w_0(t)=h(s\,t))$, $(t\in c_k,\ k=1,\ldots,m)$.

4. Solution of the second generalized problem for a circular plate

As a simple case of our theory we analyse a rigidly restrained circular plate with the load'ng $g(x,y=\frac{q_0}{2R}x+\frac{q_0}{2}=\frac{q_0}{4R}(z+\bar{z})+\frac{q_0}{2}$, $q_0=\text{const}$, $R=$ radius of a plate.

For the function P we can take $P(z)=\frac{q_0}{12R}z^3+\frac{q_0}{4}z^2$

and from the last two equations (3.12)

$$Q_2(t) = -\frac{1}{8} q_0 R (3t^2 + 4t - 2),$$

$$\tilde{Q}_2(t) = \left(\frac{3q_0R}{16(1-\nu)h} - \frac{q_0R^3}{32h^3}\right)t^2 + \left(\frac{3q_0R}{4(1-\nu)h} - \frac{3q_0R^3}{16h^3}\right)t +$$

$$+\frac{3q_0R}{8(1-\nu)h} + \frac{q_0h}{R(1-\nu)} - \frac{3q_0R^3}{64h^3},$$

$$t = R e^{i\vartheta}.$$

The physical boundary conditions $u_0(r=R)=0$, $v_0(r=R)=0$, $w_0(r=L)=0$, $\frac{dw_0}{dn}(r=R)=0$ can be put $\varkappa\varphi(t) - t\overline{\varphi^I(t)} - \overline{\psi(t)} = Q_2(t)$,

$$\Omega(t) + t\,\overline{\Omega^I(t)} + \overline{\omega^I(t)} - \frac{4}{1-\nu} h^2\,\overline{\Omega^{II}(t)} = \tilde{Q}_2(t),$$

$$w_0(r=R)=0, \quad (g(t)=0, \ \tilde{g}(t)=0)$$

Now we can solve this problem by Mushelišvili's method ([14], §81) and we find [6]:

$$\varphi(z) = C_1 z^2 + C_2 z, \quad \psi(z) = C_3, \quad \Omega(z) = C_4 z^2 + C_5 z, \quad \omega(z) = C_6 z + \alpha,$$

where following notations are used

$$C_1 = -\frac{3(1+\nu)q_0}{8R(3-\nu)}, \qquad C_2 = -\frac{(1+\nu)q_0}{4(1-\nu)}$$

$$C_3 = \frac{\nu R q_0}{3-\nu}, \qquad C_4 = \frac{3q_0}{16Rh(1-\nu)} - \frac{q_0R}{32h^3}$$

$$C_5 = \frac{3q_0}{8h(1-\nu)} - \frac{3q_0R^2}{32h^3}$$

$$C_6 = \frac{q_0R^3}{64h^3} - \frac{q_0R}{4h(1-\nu)} + \frac{q_0h(5+2\nu)}{2R(1-\nu^2)}$$

$$Re\,\alpha_1 = \frac{3q_0R^4}{64h^3} - \frac{3q_0R^2}{8h(1-\nu)} + \frac{q_0h(5-2\nu)}{2(1-\nu)^2}$$

From known φ, ψ, Ω, ω the stresses, displacements, unit forces and moments can be trivially calculated by using theorems 1.1 and 1.2.

For example we put down only the bending w:

$$w = w_0 + w_1 Z + w_2 Z^2 + w_4 Z^4 \qquad \left(\tilde{D} = \frac{2 E h^3}{3(1-\nu^2)}\right)$$

$$w_0 = -\frac{q_0}{384 R \tilde{D}} (x + 3 R) [(x^2 + y^2)(x^2 + y^2 - 2 R^2) + R^4]$$

$$w_1 = -\frac{q_0 h^3}{64 R \tilde{D} (1-\nu)^2 (3-\nu)} [(1 - 2\nu)(3-\nu) R - (3 - 4\nu)(1-\nu) x]$$

$$w_2 = -\frac{q_0 \nu}{32 R \tilde{D} (1-\nu)} (x^2 + y^2)(x + 2 R) +$$

$$+ \frac{q_0}{96 R \tilde{D} (1-\nu^2)} [\nu (1-\nu)(2 R^2 x + 3 R^3) - 12 h^2 (x + R)]$$

$$w_4 = \frac{q_0 (1+\nu)}{48 R \tilde{D} (1-\nu)} (x + R)$$

If we were to calculate by the theory of thin plates we should have found the same w_0, but quite another w, because in this theory $w_1 = w_2 = w_4 = 0$.

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VERALLGEMEINERTE RANDWERTPROBLEME IN DER THEORIE DER PLATTEN

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Zusammenfassung

Die Grundresultate der Kirchhoffschen Theorie der dünnen Platten ($\sigma_x = \alpha Z$, $\sigma_y = \beta Z, \ldots, w_0 = w_0(x, y)$) sind nur eine Näherung des tatsächlchen Zustandes. Viel bessere Aproximationen wurden im Jahre 1942 von Stevenson vorgeschlagen. Er hat die-Komponenten des Spannungstensors und Verschiebungen mit den Polynomen bis zum fünften Grade approximiert, bei denen die Koeffiziente mit fünf analitischen Funktionen Ω, ω, φ, ψ, P deffiniert werden. Die Funktion P ist mit der Querbelastung q deffiniert, die andere vier Funktionen aber mit den Randwertbedingungen. In der vorliegenden Arbeit zeigen wir, wie konkrete Probleme, wenn an dem Rande die Spannungen (Erster Randwertproblem) oder die Veschiebungen (Zweiter Randwertp.) gegeben werden, transformiert werden können in die Randwertprobleme der analytyschen Funktionen (vergl. Deff. 3.2., 3.5., 3.6., 3.7., 3.8) Es ist auch die eindeutige Lösbarkeit dieser Problemen bewiesen. Ahnliche Resultate, die in der vorliegenden Arbeit fur die mitteldicke Platten gegeben werden, wurde im Jahre 1938 von Lehnickij [2] für dünne Platten und im Jahre 1965 von Muršič [3] für mitteldicke Platten für $q = 0$ erzielt. Im beiden Beispiele treten nur Funktionen Ω und ω vor, die die Biegung der mittleren Ebene beschreiben. Inn unserem Beispiel werden noch Funktionen φ und ψ hinzugefügt, die zeigen, dass die Plattenbiegung simultan mit dem Problem der Wand gebunden ist. Der ganze Beitrag geht aus den Stevensonschen Sätzen aus und die Gänze eine Neuigkeit in der Plattentheorie darstellt.

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POSPLOŠENI ROBNI PROBLEMI V TEORIJI PLOŠČ

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Povzetek

Osnovni rezultati Kirchhoffove teorije tankih plošč ($\sigma_x = \alpha Z$, $\sigma_y = \beta Z$, $\ldots, w_0 = w_0(x, y)$) so seveda le približki dejanskega stanja. Dosti boljše aproksimacije je leta 1943 predlagal Stevenson [5], saj je napetostne komponente in pomike aproksimiral celo s polinomi do pete stopnje, pri čemer so koeficienti določeni s petimi analitičnimi funkcijami Ω, ω, φ, ψ, P. Funkcija P ie določna s prečno obtežbo q, ostale štiri z robnimi pogoji. V članku pokažemo

kako konkretni problem, ko so na robu plošče dane napetosti (prvi robni problem) ali pomiki (drugi robni problem), prevedemo na robne probleme analitičnih funkcij (gl. definicije 3.2., 3.5., 3.6., 3.7., 3.8.). Dokazana je tudi enolična rešljivost teh problemov. Podobne rezultate, ki jih dobimo v tem delu za srednje debele plošče, je l. 1938 dal Lehnickij [2] za tanke plošče in l. 1965 Marušič [3] za srednje debele plošče za $q=0$. V obeh primerih nastopata samo funkciji Ω in ω, ki opisujeta upogib srednje ravnine. V našem primeru sta dodani še funkciji φ in ψ, ki kažeta, da je upogib plošče simultano povezan s problemom stene.

Celoten prispevek izhaja iz Stevensonovih izrekov (Teorem 1.1 in 1.2) in kot celota predstavlja novost v teoriji plošč.